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# Non-Riemannian geometry and the theory of lattice imperfections

**Abstract.** Creation and development of continuum theory of imperfections of a crystal structure (dislocations and disclinations) is closely associated with ideas and methods of non-Euclidean and non-Riemannian geometry. In the geometrical interpretation of the continuum theory of lattice defects Kondo [1] and Bilbi *et al.* [2] identified the Cartan torsion tensor with the dislocation density, Anthony [3] used the Riemann-Christoffel curvature tensor to describe disclination. The above-mentioned authors considered static imperfections. To give a differential-geometrical interpretation of the imperfection kinematics it is necessary to take time into consideration. We discuss a three-dimensional space of affine connection with time as a parameter and introduce the material time derivative using the "time-connection" tensor.

In non-Euclidean space the connection $\nabla\mathbf{T}$ is defined according to a certain axiomatic [4] and allows us to introduce the tensor

$$\nabla\mathbf{T} = \nabla_i T^{k}_{.m} \mathrm{d}\xi^i \otimes \frac{\partial}{\partial \xi^k} \otimes \mathrm{d}\xi^m, \tag{1}$$

for the tensor $\mathbf{T}$ with typical arrangement of indices, where the covariant derivative

$$\nabla_i T^{k}_{.m} = \partial_i T^{k}_{.m} + T^{p}_{.m}\Gamma^{k}_{ip} - T^{k}_{.p}\Gamma^{p}_{im} \tag{2}$$

is calculated using the coefficients of connection $\Gamma^k_{ip}$; $\dfrac{\partial}{\partial \xi^k}$ and $\mathrm{d}\xi^m$ are the basic vectors in tangent and cotangent spaces, respectively.

The torsion tensor S and the curvature tensor $\mathbf{R}$ are defined by the following relations [4]:

$$2\mathrm{S}(\mathbf{X},\mathbf{Y}) = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}], \tag{3}$$

$$\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z} = \nabla_{\mathbf{X}}(\nabla_{\mathbf{Y}}\mathbf{Z}) - \nabla_{\mathbf{Y}}(\nabla_{\mathbf{X}}\mathbf{Z}) - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}, \tag{4}$$

where $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ are vector fields, $[\mathbf{X},\mathbf{Y}]$ is the Lie derivative.

Tensors S and $\mathbf{R}$ satisfy the Bianki–Padova relations

$$\nabla_{[i} S_{jk]}^{\cdot\cdot m} = \frac{1}{2} R_{[ijk]}^{\cdot\cdot\cdot m} + 2S_{[ij}^{\cdot\cdot p} S_{k]p}^{\cdot\cdot m}, \tag{5}$$

$$\nabla_{[i}R_{jk]m}^{\cdot\cdot\cdot r} = 2S_{[ij}^{\cdot\cdot p}R_{k]pm}^{\cdot\cdot\cdot r} \tag{6}$$

with [ ] denoting antisymmetric part of a tensor with respect to the indices enclosed.

The torsion tensor **S** and the curvature tensor **R** have the following geometrical interpretation [5]. At some point of a space of affine connection we draw an infinitesimal contour $C$ bounding an area with bivector $\mathbf{d\Sigma} = \mathbf{n}\cdot\boldsymbol{\epsilon}\mathrm{d}\Sigma$, where $\boldsymbol{\epsilon}$ is the anti-symmetric Levi–Civita tensor. Using the procedure of rolling we map the contour to the tangent space at the considered point. The image of the contour $C$ in the tangent space will be, in general, a non-closed line. If we take a closed contour in the tangent space, the pre-image may be non-closed. The discrepancy vector **db** can be expressed in terms of the torsion tensor **S**:

$$\mathbf{db} = -\mathbf{n}\cdot\boldsymbol{\epsilon} : \mathbf{S}\,\mathrm{d}\Sigma. \tag{7}$$

The curvature tensor at the given point of the space determines the deviation from the original value of vector **A** parallel–transported along the closed contour $C$:

$$\mathbf{dA} = \frac{1}{2}\mathbf{A}\cdot\mathbf{R} : \boldsymbol{\epsilon}\,\mathrm{d}\Sigma. \tag{8}$$

At the same time we consider the density $\mathbf{d\Omega}$ of vector **dA** defined by the relation

$$\mathbf{dA} = \mathbf{A}\times\mathbf{d\Omega}. \tag{9}$$

From formulae (8) and (9) we obtain the geometrical interpretation of the curvature tensor **R**

$$\mathbf{d\Omega} = \frac{1}{4}\mathbf{n}\cdot\boldsymbol{\epsilon} : \mathbf{R} : \boldsymbol{\epsilon}\,\mathrm{d}\Sigma. \tag{10}$$

The internal logic of the development of the mechanics of continuum, as well as the growing field of applications of the theory, has led to the study of media with motion determined by the displacement field **u** and by the rotation field $\boldsymbol{\omega}$ independent of it, which causes the appearance of a couple of stresses $\boldsymbol{\mu}$ alongside with the usual stresses $\boldsymbol{\sigma}$. Such media are known as Cosserat continua. The continuous theory of dislocations and disclinations can be considered as the theory of incompatible deformation of a Cosserat continuum [6].

The surface density of the Bürgers vector **db** is connected with the dislocation density tensor $\boldsymbol{\alpha}$ by the relation

$$\mathbf{db} = \mathbf{n}\cdot\boldsymbol{\alpha}\,\mathrm{d}\Sigma. \tag{11}$$

The similar relation between the surface density of the Frank vector $\mathbf{d\Omega}$ and the dislocation density tensor $\boldsymbol{\vartheta}$ reads

$$\mathbf{d\Omega} = \mathbf{n}\cdot\boldsymbol{\vartheta}\,\mathrm{d}\Sigma. \tag{12}$$

Identification of Bürgers vector db in equation (11) with the corresponding discrepancy vector in formula (7) allows us to obtain the physical interpretation of the torsion tensor in terms of the dislocation density [1, 2]

$$\boldsymbol{\alpha} = -\boldsymbol{\epsilon} : \mathbf{S}. \tag{13}$$

Comparison of formulae (10) and (12) leads to an interpretation of the curvature tensor in terms of the disclination density [3]

$$\boldsymbol{\vartheta} = \frac{1}{4}\boldsymbol{\epsilon} : \mathbf{R} : \boldsymbol{\epsilon}. \tag{14}$$

The material time derivative of tensor $\mathbf{T}$ with respect to the material (Lagrangean) basis of the three-dimensional material Euclidean continuum can be written in the form [7]

$$\left(\frac{\partial \mathbf{T}}{\partial \tau}\right)_{\xi} = \overset{\tau}{\nabla} T^{k}_{.m}\mathbf{a}_k \otimes \mathbf{a}^m, \tag{15}$$

where

$$\overset{\tau}{\nabla} T^{k}_{.m}\mathbf{a}_k \otimes \mathbf{a}^m = \left(\frac{\partial T^{k}_{.m}}{\partial \tau}\right)_{\xi} + T^{p}_{.m}\nabla_p v^k - T^{k}_{.p}\nabla_m v^p, \tag{16}$$

and $v^k$ are the components of the velocity vector $\mathbf{v}$.

Following [8], we consider a three-dimensional space of affine connection with properties depending on time. On the basis of an axiomatic analogous to that used by introducing the connection $\boldsymbol{\nabla}$, we introduce a time connection $\overset{\tau}{\boldsymbol{\nabla}}$, which assigns to an arbitrary tensor field the material time derivative

$$\overset{\tau}{\boldsymbol{\nabla}}\mathbf{T} = \overset{\tau}{\nabla} T^{k}_{.m}\frac{\partial}{\partial \xi^k} \otimes \mathrm{d}\xi^m, \tag{17}$$

where the covariant time derivative

$$\overset{\tau}{\nabla} T^{k}_{.m} = \left(\frac{\partial T^{k}_{.m}}{\partial \tau}\right)_{\xi} + T^{p}_{.m}\gamma_p^{\cdot k} - T^{k}_{.p}\gamma_m^{\cdot p} \tag{18}$$

is calculated using the components of tensor $\boldsymbol{\gamma}$, which play the role of coefficients of time connection.

The connections $\overset{\tau}{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$ do not commute, and the following formula holds

$$\mathbf{P}(\mathbf{Y})\mathbf{Z} = \overset{\tau}{\boldsymbol{\nabla}}\,\boldsymbol{\nabla}_{\mathbf{Y}}\mathbf{Z} - \boldsymbol{\nabla}_{\mathbf{Y}}\overset{\tau}{\boldsymbol{\nabla}}\,\mathbf{Z} - \boldsymbol{\nabla}_{\overset{\tau}{\mathbf{Y}}}\mathbf{Z}, \tag{19}$$

where

$$\overset{\tau}{\mathbf{Y}} = \left(\frac{\partial Y^k}{\partial \tau}\right)_{\xi}\frac{\partial}{\partial \xi^k}. \tag{20}$$

The tensors $\gamma$ and $\mathbf{P}$ are additional characteristics of a space of affine connection with properties depending on time.

In the Euclidean space the tensor $\gamma$ equals the gradient of the velocity vector

$$\gamma = \nabla \mathbf{v}, \tag{21}$$

and in the non-Euclidean space the deviation of $\gamma$ from $\nabla \mathbf{v}$ is interpreted as the dislocation current tensor:

$$\mathbf{J} = \nabla \mathbf{v} - \gamma, \tag{22}$$

while the tensor $\mathbf{P}$ is connected with the disclination current tensor:

$$\mathbf{I} = \frac{1}{2}\mathbf{P} : \epsilon. \tag{23}$$

From geometrical equations for tensors $\mathbf{S}$, $\mathbf{R}$, $\gamma$ and $\mathbf{P}$ we obtain their physical counterparts describing the kinematics of dislocations and disclinations.

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