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Implicit function in the calculus infinitesimalis

Abstract. The theorem about implicit function with respect to non-standard analysis in the special meaning of the Calculus Infinitesimalis is proved (see [1]).

THEOREM 1

Let $F(x,y)$ be a continuous function on the interval I containing point $[x_0,y_0]$. Let $F(x_0,y_0) = 0$ and let function $F(x_0,y)$ of one variable y be increasing at point y_0. Then $F(x_0+\alpha,y_0+\gamma)$ is for all α and all positive infinitely small γ greater than 0 and $F(x_0+\alpha,y_0+\gamma)$ is for all α and all negative infinitely small γ less than 0.

Proof. The proof will be done only for positive γ. Since function $F(x_0,y)$ of one variable y is at point y_0 increasing it is clear that for every $\gamma > 0$ is valid $F(x_0,y_0+\gamma) > 0$. Hence for every $n \in$ IN (IN means the class of infinitely great natural numbers) has to be $F(x_0,y_0+\frac{1}{n}) > 0$ where $[x_0,y_0+\frac{1}{n}] \in I$. According to Law of Back Projection (see [1]), the smallest natural number fulfilling both these conditions belongs to $\mathcal{N}$, that is there exists $m \in$ FN (FN means the class of finitely great natural numbers) such that $F(x_0,y_0+\frac{1}{m}) > 0$. The number $F(x_0,y_0+\frac{1}{m})$ belongs to $\mathcal{N}$ and function $F(x,y)$ is at point $[x_0,y_0+\frac{1}{m}]$ continuous, i.e. for all α we get $F(x_0+\alpha,y_0+\frac{1}{m}) \doteq F(x_0,y_0+\frac{1}{m}) > 0$. Thus for every γ is $F(x_0+\alpha,y_0+\gamma) > 0$.

THEOREM 2

Let $F(x,y)$ be a function with continuous derivatives till the n-th order on interval I containing point $[x_0,y_0]$. Let $F(x_0,y_0) = 0$ and $\frac{\partial F(x_0,y_0)}{\partial y} \neq 0$. Then

1. *To every infinitely small α there exists one and only one infinitely small β such that $F(x_0+\alpha,y_0+\beta) = 0$. Thus at point $[x_0,y_0]$ is valid*

$$y = y_0+\beta = f(x_0+\alpha) = f(x) \Leftrightarrow F(x_0+\alpha,y_0+\beta) = F(x,y) = 0.$$

2. *The function $f(x)$ has a continuous derivative till the n-th order at point x_0.*

Proof. Without loss of generality we can assume that $\frac{\partial F(x_0,y_0)}{\partial y} > 0$ (for $\frac{\partial F(x_0,y_0)}{\partial y} < 0$, we should investigate function $-F$ instead of function F because equations $F = 0$ and $-F = 0$ are equivalent). Hence $F(x_0, y)$ of one variable y is at point y_0 increasing, therefore for all $\gamma > 0$ is $F(x_0, y_0 + \gamma) > 0$ and $F(x_0, y_0 - \gamma) < 0$. The function $F(x, y_0 - \gamma)$ of one variable x is thus both continuous at point x_0 and negative and the function $F(x, y_0 + \gamma)$ is both continuous at x_0 and positive. According to the former theorem we get $F(x_0 + \alpha, y_0 + \gamma) > 0$ and $F(x_0 + \alpha, y_0 - \gamma) < 0$ for all α. According to the Bolzano theorem about intermediate value for a function of one variable y there exists such β that $-\gamma < \beta < \gamma$ and $F(x_0 + \alpha, y_0 + \beta) = 0$. Since $F(x_0 + \alpha, y)$ is increasing at y_0 and therefore one-to-one function, there exists one and only one β. The first part of the proof is over.

To proof the second proposition we start with the following statement: *The function $f(x)$ has got the derivative at x_0*

$$f'(x_0) = -\frac{\frac{\partial F(x_0,y_0)}{\partial x}}{\frac{\partial F(x_0,y_0)}{\partial y}}, \tag{1}$$

where $y_0 = f(x_0)$.

Proof. Since $F(x, y)$ has at point $[x_0, y_0]$ continuous derivatives till the n-th order we get

$$F(x_0 + \alpha, y_0 + \beta) - F(x_0, y_0) = \frac{\partial F(x_0, y_0)}{\partial x}\alpha + \frac{\partial F(x_0, y_0)}{\partial y}\beta + \alpha\mu + \beta\nu.$$

Let $\beta = f(x_0 + \alpha) - f(x_0)$. To put this β to the above equation we get

$$= \frac{\partial F(x_0, y_0)}{\partial x}\alpha + \frac{\partial F(x_0, y_0)}{\partial y}\left(f(x_0 + \alpha) - f(x_0)\right) + \alpha\mu + \left(f(x_0 + \alpha) - f(x_0)\right)\nu.$$

Because $F(x_0 + \alpha, y_0 + \beta) = 0$ and $F(x_0, y_0) = 0$ the former term equals 0. Thus the following is valid

$$\frac{\partial F(x_0, y_0)}{\partial x} + \frac{\partial F(x_0, y_0)}{\partial y}\frac{(f(x_0 + \alpha) - f(x_0))}{\alpha} + \mu + \frac{(f(x_0 + \alpha) - f(x_0))}{\alpha}\nu = 0,$$

$$\frac{\partial F(x_0, y_0)}{\partial x} + \mu = -\frac{f(x_0 + \alpha) - f(x)}{\alpha}\left(\frac{\partial F(x_0, y_0)}{\partial y} + \nu\right).$$

Since according to the hypothesis $\frac{\partial F(x_0,y_0)}{\partial y} \neq 0$ and therefore even $\frac{\partial F(x_0,y_0)}{\partial y} + \nu \neq 0$ we can divide by this term the former one and we get

$$\frac{f(x_0 + \alpha) - f(x)}{\alpha} = -\frac{\frac{\partial F(x_0,y_0)}{\partial x} + \mu}{\frac{\partial F(x_0,y_0)}{\partial y} + \nu},$$

thus

$$\frac{f(x_0+\alpha)-f(x)}{\alpha} \doteq -\frac{\frac{\partial F(x_0,y_0)}{\partial x}}{\frac{\partial F(x_0,y_0)}{\partial y}},$$

what is the fact we wanted to prove.

We have just finished the proof of the second proposition for $n = 1$. For the general n we use the following statement.

PROPOSITION $\mathbf{C_k}$:

Under assumption of the above theorem there exists a continuous derivative $f^{(k)}(x)$ *at* $[x_0, y_0]$ $(1 \leqslant k \leqslant n, k \in N)$ *fulfilling*

$$f^{(k)}(x) = \left(\frac{\partial F(x,f(x))}{\partial y}\right)^{-k} \cdot V_k(x), \tag{2}$$

where $V_k(x)$ *is the sum of the finite number of terms of the form*

$$cF_1(x,f(x))F_2(x,f(x))\dots F_r(x,f(x))Y_1(x)Y_2(x)\dots Y_s(x) \tag{3}$$

where $F_j(x,y)$ *denotes a partial derivative of the function* F *till the* k*-th order,* Y_j *a derivative of function* $f(x)$ *till the order* $k-1$*.*

Proof. For $n = k = 1$ the proposition has been proved above. It remains to prove the following. Let propositions $C_1, C_2, \dots, C_k$ are true $(1 \leqslant k < n)$ then the proposition C_{k+1} is valid.

According to theorems concerning the derivative of a composite, of a product, and of a power of functions the derivative of the right hand side of equations (2) equals

$$-k\left(\frac{\partial F(x,f(x))}{\partial y}\right)^{-k-1} \cdot \frac{d}{dx}\left(\frac{\partial F(x,f(x))}{\partial y}\right) V_k(x) + \left(\frac{\partial F(x,f(x))}{\partial y}\right)^{-k} \frac{dV_k(x)}{dx},$$

if the derivatives

$$\frac{d}{dx}\left(\frac{\partial F(x,f(x))}{\partial y}\right), \ \frac{dV_k(x)}{dx},$$

exist. However, the first of these derivatives exists according to theorem about the derivative of a composite (see [3]) (it is the function $\frac{\partial F(u,v)}{\partial v}$, where $u = x$, $v = f(x)$ here) and equals to

$$\frac{d}{dx}\left(\frac{\partial F(x,f(x))}{\partial y}\right) = \frac{\partial^2 F(x,y)}{\partial y \partial x} + \frac{\partial^2 F(x,y)}{\partial y^2} f'(x), \tag{4}$$

where $f(x)$ is substituted for y.

The derivative of $V_k(x)$ is equal to the sum of derivatives of separate products (3) and the derivative of every such product equals to the sum where all but one factors are left without any changes and that one is derived (if such derivative exists). We therefore dealt with differentiation of the following functions of variable x: $F_j(x, f(x))$ and $Y_j(x)$. The function $F_j(u,v)$ is a partial derivative of the highest order $k < n$, thus it has yet continuous partial derivatives and since $f'(x)$ exists according C_1, we get

$$\frac{dF_j(x, f(x))}{dx} = \frac{\partial F_j(x, f(x))}{\partial x} + \frac{\partial F_j(x, f(x))}{\partial y} \cdot f'(x),$$

where partial derivatives on the right hand side are any partial derivatives of F of the highest order $k+1$. Further, Y_j is the derivative of function $f(x)$ of the highest order $k-1$, e.g. $Y_j(x) = f^{(l)}(x)$, where $l \leqslant k-1$. Hence according to propositions $C_1, \dots, C_k$ $\frac{dY_j(x)}{dx} = f^{(l+1)}(x)$ exists, $l+1 \leqslant k$. Therefore $\frac{dV_k(x)}{dx}$ exists and is equal to the sum of terms (5), where F_j could be derivatives of orders until $k+1$, Y_j derivatives of the k-th order. One can see that right hand sides of equations (2) exist and therefore left hand sides exist as well and are valid:

$$\begin{aligned} f^{(k+1)}(x) &= \left(\frac{\partial F(x, f(x))}{\partial y}\right)^{-k-1} \left\{\frac{\partial F(x, f(x))}{\partial y} \cdot \frac{dV_k(x)}{dx} \right. \\ &\left. - k \left(\frac{\partial^2 F(x, f(x))}{\partial y \partial x} + \frac{\partial^2 F(x, f(x))}{\partial y^2} f'(x)\right) \cdot V_k(x)\right\}. \end{aligned}$$

If we denote the term within brackets as $V_{k+1}(x)$ we can see that $V_{k+1}(x)$ is again a sum of terms (3). Here $F_j(u,v)$ can mean the partial derivative of F till order $k+1$, and $Y_j(x)$ the derivative of f till order k. Continuity of functions $f^{(k+1)}$ follows then from the continuity of functions $F_j(u,v)$ and $Y_j(x)$. By this the assertion C_{k+1} was proved and we have finished the proof of the whole theorem (2).

References

[1] P. Vopěnka, *Calculus Infinitesimalis*, Práh, Praha, 1996.

[2] P. Rys, T. Zdráhal, *Generalization of the infinitesimal difference of the higher orders in the Calculus Infinitesimalis*, Czech – Polish mathematical school 99, ACTA UNIVERSITATIS PURKYNINAE 42, Ústí nad Labem, 1999, 29–33.

[3] P. Rys, T. Zdráhal, *Calculus Infinitesimalis funkcí dvou proměnných*, ERGO 3) 01, Ústí nad Labem, 2001.

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