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Slow divergence of the harmonic series

Abstract. It is well known that the harmonic series diverges because its sequence of partial sums is not bounded. The aim of this article is to show how slow this divergence is. For this purpose, we use a simple theorem that all undergraduate students are familiar with.

Let us start with the theorem used very often to check various series for convergence.

THEOREM 1
If $(a_n)_{n=1}^{\infty}$ is a decreasing sequence of positive real numbers, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

Proof. Let s_n denotes the n-th partial sum of the series $\sum_{n=1}^{\infty} a_n$, i.e.

$$s_n = a_1 + a_2 + \ldots + a_n$$

and let t_n denotes the n-th partial sum of the series $\sum_{n=1}^{\infty} 2^n a_{2^n}$, i.e.

$$t_n = 2^0 a_{2^0} + 2^1 a_{2^1} + \ldots + 2^n a_{2^n}.$$

Since $a_n > 0$ for all n both the partial sums s_n and the partial sums t_n form an increasing sequence. From

$$\begin{aligned} s_{2^k} - s_{2^{k-1}} &= a_1 + a_2 + \ldots + a_{2^{k-1}} + a_{2^{k-1}+1} + \ldots + a_{2^k} \\ &\quad - (a_1 + a_2 + \ldots + a_{2^{k-1}}) \\ &= a_{2^{k-1}} + a_{2^{k-1}+1} + \ldots + a_{2^k} \end{aligned}$$

we infer

$$2^{k-1} a_{2^k} \leqslant s_{2^k} - s_{2^{k-1}} \leqslant 2^{k-1} a_{2^{k-1}}$$

because $(a_n)_{n=1}^{\infty}$ is a decreasing sequence by the hypothesis. Adding, we get

$$\frac{1}{2} \sum_{k=1}^{n} 2^k a_{2^k} \leqslant s_{2^n} - s_1 \leqslant \sum_{k=1}^{n} 2^{k-1} a_{2^{k-1}},$$

since $s_1 = a_1 = 2^0 a_{2^0}$, we have found

$$\frac{1}{2}\sum_{k=0}^{n} 2^k a_{2^k} \leqslant s_{2^n} \leqslant \sum_{k=1}^{n} 2^{k-1} a_{2^{k-1}} \tag{1}$$

i.e.

$$\frac{1}{2} t_n \leqslant s_{2^n} \leqslant a_1 + t_{n-1}.$$

Thus the sequence (s_n) is bounded if and only if the sequence (t_n) is bounded and our theorem is proved.

Now by means of the above theorem, we show how slow the divergence of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is. The inequality (1) for $a_n = \frac{1}{n}$ gives

$$\frac{1}{2}\sum_{k=0}^{n} 2^k \frac{1}{2^k} \leqslant 1 + \frac{1}{2} + \ldots + \frac{1}{2^n} \leqslant 1 + \sum_{k=1}^{n} 2^{k-1} \frac{1}{2^{k-1}},$$

$$\frac{1}{2}\underbrace{(1 + 1 + \ldots + 1)}_{n+1} \leqslant 1 + \frac{1}{2} + \ldots + \frac{1}{2^n} \leqslant 1 + \underbrace{(1 + 1 + \ldots 1)}_{n},$$

$$\frac{1}{2}(n+1) \leqslant 1 + \frac{1}{2} + \ldots + \frac{1}{2^n} \leqslant n + 1. \tag{2}$$

Let us use this inequality (2) to estimate partial sums of this divergent harmonic series.

Suppose that the summation started with $s_1 = a_1 = 1$ thirteen billion years ago (estimated age of the universe) and that a new term has been added every second since then. The following question arises: How large is this sum today? (Bear in mind that the harmonic series diverges, that is the sum diverges to infinity $+\infty$).

Using arithmetics we easily get the following:

The number of seconds since the beginning of the universe until now : $13.10^9.365.24.60.60 \doteq 4.10^{17}$. The number of the terms of the sum : 4.10^{17} as well. (Since we have supposed that a new term has been added every second).

It means that $2^n = 4.10^{17}$ and we'd like to know n:

$$n = \frac{\ln 4.10^{17}}{\ln 2} \doteq 59.$$

For the inequality (2) we now get that our sum is surely smaller than sixty

$$s_{2^n} = s_{4.10^{17}} \leqslant n + 1 = 60$$

(in fact even less than that, as we can infer by another way).

Who would have expected the sum to be so small? Amazing, is it not?

References

[1] A. Howard, *Calculus with Analytic Geometry*, John Willey inc., New York 1992.

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