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# Wavelet Galerkin method for solving the Fredholm's integral equation of the second kind

**Abstract.** Wavelets are very discussed today in many fields of mathematics. For this reason we want to show their application to solving the Fredholm's integral equation of the second kind. The basis of discretisation of such an equation will be called Galerkin method. We introduce semi-orthogonal wavelets use them for solving Fredholm's integral equation.

## 1. Multi-resolution analysis and wavelets

In this section, we briefly present basic wavelet principles that are used to construct and facilitate the wavelets.
A multiresolution analysis of $L^2(\mathbb{R})$ is defined as a nested sequence of closed subspaces $V_j$ of $L^2(\mathbb{R}); j \in \mathbb{R}$, where $0 \leftarrow \ldots \subset V_{-1} \subset V_0 \subset V_1 \ldots \rightarrow L^2$ with the following properties:

$$f(x) \in V_j \leftrightarrow f(2x) \in V_{j+1},$$

$$f(x) \in V_0 \leftrightarrow f(x+1) \in V_0$$

and a scaling function $\phi \in V_0$, with a non-vanishing integral, exists such that the collection $\{\phi(x-k)k \in \mathbb{Z}\}$ is a Riesz basis of $V_0$. This scaling function satisfies the dilation equation, namely

$$\phi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} h_k \phi(2x-k).$$

The subspaces $V_j$ are generated by $\phi_{j,k} = 2^{\frac{j}{2}}\phi(2^j x - k)$. For each scale $j$, since $V_j \subset V_{j+1}$, there exists a unique orthogonal complementary subspace $V_{j+1} = V_j \oplus W_j$ and holds $\oplus_{j=-\infty}^{\infty} W_j = L^2(\mathbb{R})$. This subspace $W_j$ is called "wavelet subspace" and is generated by $\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$ where function $\psi(x)$ is called the "wavelet"; the collection of functions $\{\psi(x-k)|k \in \mathbb{Z}\}$ forms a Riesz basis of $W_0$. Since the wavelet $\psi$ is an element of $V_1$, a sequence $\{g_k\} \in l^2(\mathbb{R})$ exists such that

$$\psi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} g_k \phi(2x-k)$$

and for coefficient $g_k$ holds the relation $g_k = (-1)^{k-1} h_{-(k-1)}$.
An important property of wavelets is vanishing moment. A wavelet is said to have a vanishing moment of order $m$ if

$$\int_{-\infty}^{\infty} x^p \psi(x) dx = 0, \quad p = 0, ...., m-1.$$

All wavelets satisfy the above condition for $p = 0$.
The wavelets $\{\psi_{j,k}\}$ form an orthonormal basis if

$$\langle \psi_{j,k}; \psi_{i,l} \rangle = \delta_{j,i} \delta_{k,l}, \quad i, j, k, l \in \mathbb{Z}.$$

The wavelets $\{\psi_{j,k}\}$ form a semi-orthogonal basis if

$$\langle \psi_{j,k}; \psi_{i,l} \rangle = 0, \quad i \neq j, \quad i, j, k, l \in \mathbb{Z}.$$

## 2. Wavelets on a Bounded Interval

In the previous subsection we described wavelets and scaling functions defined on the real line. For many applications it is necessary, or at least more natural, to work on a subset of the real line. Many of these cases can be dealt with by introducing periodicized scaling function and wavelets, which we define as follows:

$$\tilde{\phi}_{j,k}(x) = \sum_{n \in \mathbb{Z}} \phi_{j,k}(x+n),$$

$$\tilde{\psi}_{j,k}(x) = \sum_{n \in \mathbb{Z}} \psi_{j,k}(x+n)$$

and it can be shown that $\{\tilde{\phi}_{0,0}\} \cup \{\tilde{\psi}_{j,k}\}$, $j \in \mathbb{Z}^+ = \{0, 1, ....\}, k = \{0, 1, \cdot, 2^J - 1\}$ generates $L^2\langle 0, 1 \rangle$.
A different approach is to start from cardinal B-spline, which generates a multiresolution analysis. Cardinal B-spline of order $n$ regarding the set of points $\{x_i, x_{i+1}, ..., x_{i+n+1}\}$. $i = -n, ..., N-1$ is defined as

$$B_n^i(x) = (n+1) \sum_{p=1}^{i+n+1} \frac{(x_p - x)_+^n}{\omega'_{n+1,i}(x_p)}$$

where

$$(x_p - x)_+^n = \begin{cases} (x_p - x)^n, & x \leqslant x_p \\ 0, & \text{otherwise} \end{cases}$$

and

$$\omega_{k,i}(x) = \prod_{j=i}^{i+k} (x - x_j).$$

Now, we descibe the compactly supported SO-spline, which are specially constructed for the bounded interval $\langle 0,1\rangle$. The second-order B-spline (scaling function) is given by

$$\phi_{j,k} = \begin{cases} 2^j x - k, & x \in \langle \frac{k}{2^j}; \frac{k+1}{2^j}\rangle, \\ 2 - (2^j x - k), & x \in \langle \frac{k+1}{2^j}; \frac{k+2}{2^j}\rangle \end{cases}$$

for

$$k = 0, 1, ..., 2^j - 2$$

with the respective left- and right-side boundary scaling functions

$$\phi_{j,-1} = 3 - 2^j x, \quad x \in \left\langle 0, \frac{1}{2^j}\right\rangle,$$

$$\phi_{j,2^j-1} = 2^j x - 2^j + 1, \quad x \in \left\langle \frac{2^j - 1}{2^j}, 1\right\rangle.$$

The second-order B-spline Wavelets are given by

$$\psi_{j,k} = \frac{1}{6}\begin{cases} 2^j x - k, & x \in \langle \frac{k}{2^j}; \frac{k+0.5}{2^j}\rangle, \\ 4 - 7(2^j x - k), & x \in \langle \frac{k+0.5}{2^j}; \frac{k+1}{2^j}\rangle, \\ -19 + 16(2^j x - k), & x \in \langle \frac{k+1}{2^j}; \frac{k+1.5}{2^j}\rangle, \\ 29 - 16(2^j x - k), & x \in \langle \frac{k+1.5}{2^j}; \frac{k+2}{2^j}\rangle, \\ -17 + 7(2^j x - k), & x \in \langle \frac{k+2}{2^j}; \frac{k+2.5}{2^j}\rangle, \\ 3 - (2^j x - k), & x \in \langle \frac{k+2.5}{2^j}; \frac{k+3}{2^j}\rangle \end{cases}$$

for

$$k = 0, ...., 2^j - 3$$

with boundary wavelets

$$\psi_{j,k} = \frac{1}{6}\begin{cases} -6 + 23(2^j x), & x \in \langle 0; \frac{1}{2^{j+1}}\rangle, \\ 14 - 17(2^j x), & x \in \langle \frac{1}{2^{j+1}}; \frac{1}{2^j}\rangle, \\ -10 + 7(2^j x), & x \in \langle \frac{1}{2^j}; \frac{1.5}{2^j}\rangle, \\ 2 - 2^j x, & x \in \langle \frac{1.5}{2^j}; \frac{2}{2^j}\rangle \end{cases}$$

for the left-side boundary and

$$\psi_{j,k} = \frac{1}{6}\begin{cases} 2 - (k + 2 - 2^j x), & x \in \langle \frac{k}{2^j}; \frac{k+0.5}{2^j}\rangle, \\ -10 + 7(k + 2 - 2^j x), & x \in \langle \frac{k+0.5}{2^j}; \frac{k+1}{2^j}\rangle, \\ 14 - 17(k + 2 - 2^j x), & x \in \langle \frac{k+1}{2^j}; \frac{k+1.5}{2^j}\rangle, \\ -6 + 23(k + 2 - 2^j x), & x \in \langle \frac{k+1.5}{2^j}; 1\rangle \end{cases}$$

where $k = 2^j - 2$, for the right-side boundary.

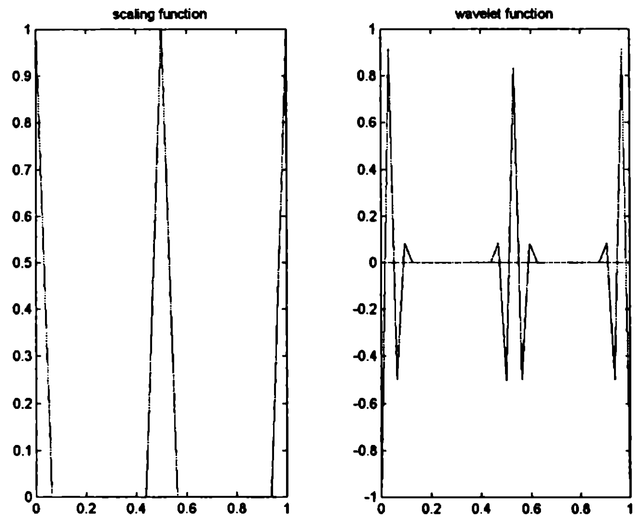


**Figure 1.**

The advantage of SO-wavelets against ON-wavelets is their compact support, explicit form and symmetric graph simultaneously.

## 3. Integral Equation and Wavelet Expansion

We consider Fredholm's integral equation of the second kind in the form

$$f(x) - \int_0^1 K(x,y)f(y)dy = g(x) \tag{1}$$

where $g(x)$ is a known function, $K(x,y)$ is the kernel of integral equation and $f(x)$ is an unknown function from $L^2_{\langle 0,1\rangle}$.
The weak formulation of the problem is to seek $f \in L^2_{\langle 0,1\rangle}$ such that

$$((\mathcal{I} - \mathcal{K})f, h)_{L^2_{\langle 0,1\rangle}} = (g,h)_{L^2_{\langle 0,1\rangle}}$$

where $\mathcal{I}$ is identical operator and operator $\mathcal{K}$ has the following form

$$(\mathcal{K}f)(x) = \int_0^1 K(x,y)f(y)dy.$$

In Galerkin method we replace the space $L^2_{\langle 0,1\rangle}$ by spaces $V_n = \overline{\mathcal{L}\{\phi_{j,k}\}_{k\in Z}}$ and within the integration domain $\langle 0,1\rangle$ we can expand the function $f(x)$ in the integral equation in the terms of scaling functions at the highest level $J$ on bounded interval

$$f(x) = \sum_{k=-1}^{2^{J+1}-1} c_{J,k}\phi_{J,k}(x) = \sum_{j=j_0}^{J} \sum_{k=-1}^{2^{j_0}} d_{j,k}\psi_{j,k}(x) + \sum_{k=-1}^{2^{j_0}-1} c_{j_0,k}\phi_{j_0,k}. \quad (2)$$

For $n$-order B-spline must be satisfied the following condition for the lowest level $j_0$

$$2^{j_0} \geqslant 2n-1.$$

Then for the second order B-spline the lowest level is determined as $j_0 = 2$. Substituting the second expansion (2) of function $f(x)$ to the considered integral equation (1) and taking the tested function $h(x)$ as function $\{\phi_{j_0,k}\}_{k=-1}^{2^{j_0}-1}$ and $\{\psi_{j,k}\}_{k=-1,j=j_0}^{2^{j_0}-1,J}$ we obtain the following system of linear equations:

$$\left\{\begin{bmatrix} X_{\phi,\phi} & X_{\phi,\psi} \\ X_{\psi,\phi} & X_{\psi,\psi} \end{bmatrix} + \begin{bmatrix} Y_{\phi,\phi} & Y_{\phi,\psi} \\ Y_{\psi,\phi} & Y_{\psi,\psi} \end{bmatrix}\right\} * \begin{bmatrix} c_{j_0,k} \\ d_{j,k} \end{bmatrix} = \begin{bmatrix} G_\phi \\ G_\psi \end{bmatrix}$$

where the elements of matrices $X$ and $Y$ are

$$X_{\phi,\phi} = \int_0^1 \phi_{i_0,l}(x)\phi_{j_0,k}(x)dx, \quad X_{\phi,\psi} = \int_0^1 \phi_{i_0,l}(x)\psi_{j,k'}(x)dx,$$

$$X_{\psi,\phi} = \int_0^1 \psi_{i,l'}(x)\phi_{j_0,k}(x)dx, \quad X_{\psi,\psi} = \int_0^1 \psi_{i,l'}(x)\psi_{j,k'}(x)dx,$$

$$Y_{\phi,\phi} = \int_0^1\int_0^1 \phi_{i_0,l}(x)\phi_{j_0,k}(y)K(x,y)dydx,$$

$$Y_{\phi,\psi} = \int_0^1\int_0^1 \phi_{i_0,l}(x)\psi_{j,k'}(y)K(x,y)dydx,$$

$$Y_{\psi,\phi} = \int_0^1\int_0^1 \psi_{i,l'}(x)\phi_{j_0,k}(y)K(x,y)dydx,$$

$$Y_{\psi,\psi} = \int_0^1\int_0^1 \psi_{i,l'}(x)\psi_{j,k'}(y)K(x,y)dydx,$$

and the subscripts $k,l,k',l',i,j$, are given as $k,l = -1,...,2^{j_0}-1; k',l' = -1,...,2^{j}-1$ and $j_0 \leqslant j \leqslant J; i_0 \leqslant i \leqslant I$. The elements of vector $G_\phi$ are integrals $\int_0^1 g(x)\phi_{i_0,l}(x)dx$ and the elements $G_\psi$ are integrals $\int_0^1 g(x)\psi_{i,l}(x)dx$. The total number of unknowns (N) in this system of linear equations is $N = 2^{J+1}+1$.

If we remember the properties of SO-wavelets we know that matrix $X_{\phi,\phi}$ is three-diagonal matrix, $X_{\phi,\psi}$, $X_{\psi,\phi}$ are zero matrices, and $X_{\psi,\psi}$ is a block-diagonal matrix. Let us look at details of the elements of matrix $Y$. Even though the limits of integration in every element of matrix $Y$ range from zero to one, the actual integration limits are much smaller because of the finite supports of SO-scaling function and SO-wavelets. Matrix $Y_{\phi,\phi}$ is a dense matrix with not very small elements. But this matrix occupies very little, $(5 \times 5)$, of matrix $Y$. The matrices $Y_{\phi,\psi}, Y_{\psi,\phi}$ and $Y_{\psi,\psi}$ are dense too. But because of local supports and vanishing moment properties of wavelets many elements of these matrices are very small compared to the largest element. And hence they can be dropped without significantly affect the solution. So the elements whose magnitudes are smaller then $\epsilon * A_{max}$ can be set zero where $\epsilon$ $(0 \leqslant \epsilon < 1)$ is called threshold parameter and $A_{max}$ is the largest element of the matrix. We can evaluate the percentage scarcity $S_\epsilon$ of matrix $(X - Y)$ as

$$S_\epsilon = \frac{N^2 - N_\epsilon}{N^2} \times 100$$

where $N$ is the number of unknowns and $N_\epsilon$ is the number of nonzero elements after thresholding.
Sparcity of the matrix of the system of linear equations is very important for decreasing memory capacity and computation time in inverting the matrix. Semi-orthogonal wavelets are a very good choice in Galerkin method for solving Fredholm's integral equations of the second kind.

## References


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