Annales Academiae Paedagogicae Cracoviensis

Folia 16 Studia Mathematica III (2003)

Marie Tichá

On the role of translations between modes of representation – case of fractions

Abstract. We believe that utilization of various modes of representation is important for concept development and deepens understanding. Many didacticians stress that the level of understanding is related to the continuos enrichment of the set of various modes of representation and emphasize the development of student's capability of translationing between modes of representation. But it is also possible to utilize the work with representations for recognition (diagnosis) of the level of understanding, also for searching for misunderstandings and obstacles in understanding, and further as a tool for re-education.

This contribution focuses on the study of student's formulations of word problems to the given formula. (Pose a word problem which can be solved by the calculation $\frac{2}{3}+\frac{1}{4}$.) or to the given visual representation. Our present investigation shows that this way we can get a lot of didactically interesting information.

In the contribution some examples of problems formulated by students will be shown. Teacher's reaction to these problems will be mentioned. I will concentrate especially on those problems that contain the most frequent misunderstandings and I will outline suggestions how to re-educate these misunderstandings.

The results of many investigations confirm great significance of the use of various modes of representation for the development and deepening of understanding. Many authors (e.g. [1], [2], [3], [5], [6]) stress that the level of understanding is related to the continuous enrichment of a set of various modes of representation and emphasize the development of a student's capability of translationing between modes of representation; a student's understanding can be limited by his/her inability to translation between modes of representation.

Our experience shows that it is also possible to utilize translation between representations for recognition of the level of understanding, also for searching

The research was supported by GACR, grant 406/02/0829 and GAUK, grant 306/2002/A-PP/PedF.

The preliminary version of this contribution was published in Beiträge zum Mathematikunterricht. Ed. M. Neubrand.: Franzbecker, Berlin 2000, (Tichá, M.: Textaufgaben mit Brüchen, die von Schülern konstruiert wurden, 671 – 674).

for misunderstandings and obstacles to understanding, and further as a tool for re-education. "To diagnose a student's learning difficulties, or to identify instructional opportunities, teachers can generate a variety of useful kinds of questions by presenting an idea in one representational mode and asking the student to illustrate, describe, or represent the same idea in another mode." ([6]).

This contribution deals with a student's formulations of word problems according to the given formula or the given visual (pictorial) representation. Our present investigations show that it is possible to get a lot of didactically interesting information from a student's formulations of word problems.

Two samples from the set of tasks that we assigned to students of the 7th and 8th grades (13 to 14-years-old) will be given. All these students had already dealt with fractions at school (i.e. with the notion of fraction, arithmetical operations with fractions, word problems with fractions). Students elaborated the task in a written form. The interview was conducted with some of them afterwards (especially with the authors of interesting or ambiguous solutions). Next, some problems formulated by students that contain the most frequent misunderstandings will be shown and an attempt will be made to outline suggestions how to re-educate these misunderstandings.

Task 1: Create a word problem that we solve by calculating $\frac{2}{3} + \frac{1}{4}$.

The samples of the students' work will be introduced by quoting part of the discussion among two girls and one boy – G1, G2, and B.

G1: *Jana and Hanka have ... possibly 20 sweets ... no, it's not possible ... 12 sweets, so that it is possible to divide it. They will share it so that Jana gets $\frac{2}{3}$ and Hanka $\frac{1}{4}$ ... but* (she looks unsure).

B: *And who will eat the rest?*

G1: *It is only approximately.*

B: *But there will be some rest. Who will have it?*

G1: *Yes ... then like this* (she draws a circle and marks $\frac{2}{3}$ and $\frac{1}{4}$). *They have a bag of sweets. Jana gets $\frac{2}{3}$ and Hanka $\frac{1}{4}$* (she shows it). *How many will they get together?*

G2: *But I don't know how many sweets they have. So I don't know how many they will eat. It is a piece of that circle.*

G1: *I see. So ... How much from the bag do they get together?*

This discussion outlines some phenomena that we often encounter. Students want to deal with concrete numbers (see also 1.3. below). Many students do not have any appropriate mental model (e.g. pictorial representation) that could help them to see the relations between the whole and its parts.

In this discussion there also appeared another very frequent phenomenon, which is characteristic for Czech students: In the formulation of questions, the students rather tend to use "How many ..." or "How much ..." and avoid the formulation "What part of ...". But that does not necessarily indicate misunderstanding, e.g.:

1.1. *Honza ate $\frac{2}{3}$ of a chocolate bar, daddy $\frac{1}{4}$. How much of the chocolate bar (or how much from the chocolate bar) remained for mammy?*

1.2. *Daddy mowed $\frac{2}{3}$ of the garden, Jirka $\frac{1}{4}$. How much was not mown?*

The students expressed the fact that they wanted to ask about the part of the whole by means of the language. The subsequent discussion with them complemented by the calculation and answer showed that students only formulated the question inaccurately. (Both these students wanted to create interesting problems and therefore they asked about the rest.)

$\frac{2}{3} + \frac{1}{4} = \frac{11}{12}$; $\frac{1}{12}$ *remained for mammy.*

$\frac{1}{12}$ *of the garden is not mown.*

Let's look at further types of problems that occurred in the students' works quite often.

1.3. *Petr ate $\frac{2}{3}$ of cake and later another $\frac{1}{4}$. How many bits of the cake did he eat?*

The students most often anticipate the answer *He ate 11 parts.* because they imagine (on the basis of calculation $\frac{2}{3} + \frac{1}{4} = \frac{11}{12}$) that the cake is divided into 12 parts.

The pictorial representation usually helps to remove this misunderstanding. We invite the students to draw a picture that corresponds to the given calculation (e.g. a circle divided into 24 segments or a circle as a whole).

Some students formulated the problem in a similar manner. But they immediately realized that they were not able to answer the question "How many parts ..." and therefore they supplemented the problem setting by additional data, e.g.:

1.4. *Vojta had 60 Crowns. He spent $\frac{2}{3}$ for a toy car for his brother and $\frac{1}{4}$ for a dolly for the sister. How many Crowns remained to him?*

The problems formulated in this way also point out that the students better accept and also preferably formulate problems in discrete environments and also demonstrate that many students predominantly understand fractions as operators.

1.5. *Hanka ate $\frac{2}{3}$ of the cake and later $\frac{1}{4}$ of the rest. What part of the cake did she eat?*

When formulating problems of type 1.5. students were evidently strongly influenced by problems that they often solved in mathematics lessons. As was confirmed during the subsequent discussion, they did not realize that such problems did not fit the setting. It is possible that the students grasped the notation $\frac{2}{3} + \frac{1}{4}$ as a process ([4]).

The mistake concerns therefore the whole which the second fraction refers to. Here as a rule, the pictorial representation helps again to eliminate this mistake. We invite the student to draw consecutive pictures which represent how the action went (this time it is the pictorial representation of the suggested word problem). Some students discover their mistake if we ask them to repeat their considerations for the calculation $\frac{1}{4} + \frac{2}{3}$.

1.6. *In the class there are $\frac{2}{3}$ boys and $\frac{1}{4}$ girls* (meant: $\frac{2}{3}$ and $\frac{1}{4}$ from the whole number of students). *How many students are there in the class?*

This type of problems appeared quite often, too, in different contexts (men – women, firs – spruces, geese – hens, ...). It indicates that the student does not understand that he/she does not have a proper image of the situation, fractions and classification. He/she composed the problem by mechanically associating the given data and writing the usual question. It is necessary in the course of discussion to invite the student who composed this problem to visualize the problem and to solve it. The challenge to check the correctness of their own answer (e.g. *There are 24 students in the class.*) helps to some of them to realize their own mistake.

The following problem and its solution demonstrate a very serious type of misunderstanding. Students do not realize that they work with different wholes. Unfortunately, this misunderstanding appears often, many students in our research created problems of this type.

1.7. *Classes 6.A a 6.B elected "the teacher of the year". The greatest number of votes was for Mrs. Daniela Novakova. From class 6.A, $\frac{1}{4}$ of students voted for her, from class 6.B, $\frac{2}{3}$. What part of both classes voted for Mrs. Novakova? (How many students from both classes voted for Mrs. Novakova?*

Solution: $\frac{1}{4} + \frac{2}{3} = \frac{3}{12} + \frac{8}{12} = \frac{11}{12}$.

Answer: $\frac{11}{12}$ from the total number of students of both classes voted for Mrs. Novakova.

In this case, it is suitable to model the situation on the concrete sample and not only to count but also to visualize the situation, e.g. using segments. It is also helpful to modify the problem created by the student, e.g. to add class 6.C in which $\frac{3}{5}$ voted for Mrs. Novakova and to discuss the result.

Task 2: Petr recorded the setting of a word problem with fractions using the following picture. Write the word problem that he could have recorded like this.

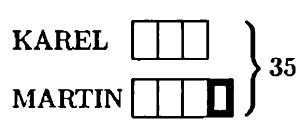

Figure 1.

I will again present some didactically interesting examples of problems.

2.1. *Karel and Martin have together 35 marbles. Karel has $\frac{3}{7}$. How many marbles does each of them have?*

This is one of the most frequent types of problems created in our research. The student supposed that the whole was divided into 7 equal parts and therefore he calculated $\frac{3}{7}$ *from 35.*

The student who composed the following type of problem grasped task 2 in a different way.

2.2. *Karel and Martin have together 35 marbles. Karel has $\frac{3}{7}$. Martin has by $\frac{1}{7}$ more than Karel. How many marbles does each of them have?*

In the interview, it transpired that the author of this problem grasped the fraction "one seventh" as a new object, the unit of a new kind for which he used the notation $\frac{1}{7}$. In the interview, he explained that "one seventh" plays in his problem the same role as "basket" in the following problem: *Karel had 3 baskets of apples, Martin had by 1 basket more than Karel. Together they had 35 kg of apples. How many kg were there in each basket assuming that there was the same amount of apples in each basket?*

Another student formulated the problem at first glance similarly:

2.3. *Karel and Martin have together 35 tin soldiers. Martin has by $\frac{1}{7}$ of soldiers more than Karel. How many soldiers does each of them have?*

But he did not regard the conjunction "than" as an indicator of the whole. For him the whole indicates the statement "they have together 35 ...".

To make students aware of the misunderstanding, it is helpful to give them the following problem: *Karel and Martin have together 45 tin soldiers. Martin has by $\frac{1}{4}$ of soldiers more than Karel. How many soldiers does each of them have?*

Problem 2.3. documents a serious shortcoming – students are not able to distinguish what the whole is and what the part is. This fact is even more strongly demonstrated by the following answer to the given task:

2.4. *There are two possibilities:*

Karel and Martin had together 35 Crowns.

Karel had by $\frac{1}{3}$ less than Martin or Martin had by $\frac{1}{3}$ more than Karel.

How many Crowns did each of them have?

With the students who created problems of types 2.3. and 2.4., we should go back to the creation of the stock of different models to improve their imagination of fractions and especially focus on the relation between the part and the whole.

Remarks

In each group of students that formulated problems we always find new didactically interesting types of problems (which we have not met until now). Sometimes a feature that we have already discovered formerly appears in a more expressive form.

In the Czech Republic various modes of representation in mathematics lessons at the elementary level (from the 1st to 5th grade) are introduced. But in the higher grades teachers mainly use verbal and symbolic modes of representations and enactive and iconic modes are neglected. The translation among representations is hardly ever fostered.

The teachers in whose classes we carried out our research were often surprised by the great amount and diversity of the information about the students' imagination, level of understanding etc. they gained during our research. They came to see that the posing of problems according to the given calculations and pictorial representations is a method that enables us to identify or at least outline problems and difficulties of each student, and that it is possible to convince the students who made mistakes in the posing of problems by asking him/her or his/her classmates to solve the created problem. This procedure is also a tool of re-education and disposal of errors, misunderstandings and elimination of obstacles.

Our research shows that especially younger students and students of the school for kindergarten teachers created didactically very interesting problems. The considerable (often negative) influence of textbooks appeared in works of students of the 7th and 8th grades.

Our investigations also included other calculations (e.g.: $\frac{2}{3}-\frac{1}{4}$, $\frac{2}{3}\times\frac{1}{4}$, $\frac{4}{5}\times 20$) and different pictures. There were very interesting and surprising problems, e.g. problems that students posed to the calculation $\frac{4}{5}\times 20$ and especially the difficulties that it brought to them were important.

It emerged again that the students' difficulties were due to problems not only of mathematics but also of the Czech language, of the ability to formulate, express, communicate. The competence to think about what we have just presented belongs, in our opinion, to the general literacy and therefore it is very important to devote attention to this question, too, in mathematics education.

References

[1] Ch. Bills, E. Gray, Pupils' images of teachers' representations, in: O. Zaslavsky (Ed.), *23rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, Haifa-Israel 1999, 113-120.

[2] D. Bönig, *Problems with transfer between different modes of representation in operating with multiplication and division in primary school classes*, in: L.Bazzini, H.G.Steiner (Eds.), 2nd Italian-German Symposium on Didactics of Mathematics. IDM, Bielefeld 1994.

[3] G. S. Halford, *Children's Understanding: The Development of Mental Models*, Lawrence Erlbaum Associates, Hillsdale 1993.

[4] M. Hejný, *Představa celku a jeho části* (Conception of the whole and its part, in Czech), in: Jak učit matematice žáky ve věku 10-15 let, Frýdek-Místek 1999, 5-21.

[5] C. Janvier, *Representation and understanding: the notion of function as an example*, in: C. Janvier (Ed.), Problems of representation in the teaching and learning of mathematics, Lawrence Erlbaum Associates, Hillsdalle 1987, 67-71.

[6] R. Lesh, T. Post, M. Behr, *Representations and translations among representations in mathematics learning and problem solving*, in: C. Janvier (Ed.), Problems of representation in the teaching and learning of mathematics, Lawrence Erlbaum Associates, Hillsdalle 1987, 33-40.

Academy of Sciences of the Czech Republic
Mathematical Institute
Department of Didactics of Mathematics
Žitná 25
115 67 Praha 1
Czech Republic
E-mail: ticha@math.cas.cz