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Generalized Erlang problem for queueing systems with accumulation

Abstract. We consider a non-classical $M/G/n/0$ type queueing system with random volume demands. Each demand requires m servers for its simultaneous service with probability q_m, $\sum_{m=1}^{n} g_m = 1$. Service time of the demand depends on the demand volume and m. The total volume of demands presenting in the system is restricted by memory volume V. For such system the stationary distribution of demands number and loss probability are determined.

Consider $M/G/n/0$ type queueing system. Let a be a parameter of demands entrance flow. Suppose that m servers are required for arbitrary demand service with probability q_m ($\sum_{m=1}^{n} q_m = 1$) independently of other demands. In this case all m servers function simultaneously with the same service time ξ_m, being a non-negative random variable. The demand required m servers for it's service we shall name m-demand or the demand of m-type. Assume, that each m-demand ($m = 1,...,n$) is characterized by some volume ζ_m, which is a non-negative random variable not dependent on volumes of other demands. Suppose, that service time, type of the demand and its volume are dependent. The distribution of m-demand volume and its service time may be define by the next joint distribution functions: $F_m(x,t) = \mathbf{P}\{\zeta_m < x, \xi_m < t\}$, $m = 1,...,n$. Then $L_m(x) = F_m(x,\infty)$ and $B_m(t) = F_m(\infty,t)$ be the distribution function of m-demand volume and service time accordingly. If there is no required number of free servers at the moment of demand entrance, the demand will be lost without any influence to future system behaviour.

Denote as $\eta(t)$ the number of demands presenting in the system at time moment t. Let $\sigma(t)$ be the total sum of such demands volumes. The $\sigma(t)$ process is named total volume. Suppose, that $\sigma(t)$ is restricted by $V > 0$ constant value, which is named memory volume. So, if there is sufficient number of servers for the demand service at arriving moment τ, this demand at least will be lost if its volume x is such that $\sigma(\tau - 0) + x > V$. If contrary $\sigma(\tau - 0) + x \leqslant V$, the demand's service begins and we have $\eta(\tau) = \eta(\tau - 0) + 1$, $\sigma(\tau) = \sigma(\tau - 0) + x$.

For this queueing system we define the stationary distribution of demands number and the loss probability.

Suppose that all demands presenting in the system under consideration are numbered in random order [1]. Denote as $\nu_j(t)$ the number of servers which are serving the j-th demand at the time moment t. Denote as $\xi_j^*(t)$ the length of time interval from the time moment t to finishing moment of j-th demand service. The system under consideration may be represented by the next (not markovian) random process

$$\left(\eta(t), \sigma(t), \nu_1(t), \xi_1^*(t), ..., \nu_{\eta(t)}, \xi_{\eta(t)}^*(t)\right), \tag{1}$$

which may be characterized by functions having the next probability sense:

$$\begin{aligned} &G_k(t, x, r_1, y_1, ..., r_k, y_k)\,\mathrm{d}x\,\mathrm{d}y_1 ... \mathrm{d}y_k \\ &\quad = \mathbf{P}\left\{\eta(t) = k,\ \sigma(t) \in \mathrm{d}x,\ \nu_i(t) = r_i,\ \xi_i^*(t) \in \mathrm{d}y_i,\ i = 1, ..., k\right\}, \\ &\qquad k = 1, ..., n;\ r_1 + ... + r_k \leqslant n; \end{aligned} \tag{2}$$

$$\begin{aligned} P_k(t, r_1, y_1, ..., r_k, y_k) = \int_0^V G_k(t, x, r_1, y_1, ..., r_k, y_k)\,\mathrm{d}x, \\ k = 1, ..., n;\ r_1 + ... + r_k \leqslant n; \end{aligned} \tag{3}$$

$$P_k(t, r_1, ..., r_k) = \int_0^\infty ... \int_0^\infty P_k(t, r_1, y_1, ..., r_k, y_k)\,\mathrm{d}y_1 ... \mathrm{d}y_k; \tag{4}$$

$$P_0(t) = \mathbf{P}\left\{\eta(t) = 0\right\}; \tag{5}$$

$$P_k(t) = \mathbf{P}\left\{\eta(t) = k\right\} = \sum_{r_1 + ... + r_k \leqslant n} P_k(t, r_1, ..., r_k),\ k = 1, ..., n. \tag{6}$$

In the case of $\rho = a(q_1\beta_{11} + ... + q_n\beta_{n1}) < \infty$ the next limits exist:

$$\begin{aligned} g_k(x, r_1, y_1, ..., r_k, y_k) = \lim_{t\to\infty} G_k(t, x, r_1, y_1, ..., r_k, y_k), \\ k = 1, ..., n;\ r_1 + ... + r_k \leqslant n; \end{aligned} \tag{7}$$

$$\begin{aligned} p_k(r_1, y_1, ..., r_k, y_k) &= \lim_{t\to\infty} P_k(t, r_1, y_1, ..., r_k, y_k) \\ &= \int_0^V g_k(x, r_1, y_1, ..., r_k, y_k)\,\mathrm{d}x; \end{aligned} \tag{8}$$

$$\begin{aligned} p_k(r_1, ..., r_k) &= \lim_{t\to\infty} P_k(t, r_1, ..., r_k) \\ &= \int_0^V ... \int_0^V p_k(r_1, y_1, ..., r_k, y_k)\,\mathrm{d}y_1 ... \mathrm{d}y_k, \\ &\qquad k = 1, ..., n;\ r_1 + ... + r_k \leqslant n; \end{aligned} \tag{9}$$

$$p_0 = \lim_{t\to\infty} P_0(t); \tag{10}$$

$$p_k = \lim_{t\to\infty} P_k(t) = \sum_{r_1+\ldots+r_k\leqslant n} p_k(r_1,\ldots,r_k),\ k=1,\ldots,n. \tag{11}$$

For simplicity we assume that the density $f_m(x,t)$ of (ζ_m,ξ_m) random vector exists, $m = 1,\ldots,n$. All results of the paper are relevant without this assumption. It may be easy shown by the additional variables method, that the stationary functions (7) – (9) satisfy the next equations.

$$0 = -ap_0 \sum_{m=1}^{n} q_m L_m(V) + \sum_{m=1}^{n} p_1(m,0); \tag{12}$$

$$\begin{aligned}-\frac{\partial p_1(m,y)}{\partial y} &= aq_m p_0 \int_0^V f_m(x,y)\,\mathrm{d}x \\ &- a\sum_{j=1}^{n-m} q_j g_1(x,m,y)L_j(V-x)\,\mathrm{d}x \\ &+ \sum_{j=1}^{n-m}[p_2(j,m,0,y)+p_2(m,y,j,0)],\ m=1,\ldots,n-1;\end{aligned} \tag{13}$$

$$-\frac{\partial p_1(n,y)}{\partial y} = aq_n p_0 \int_0^V f_n(x,y)\,\mathrm{d}x; \tag{14}$$

$$\begin{aligned}&-\sum_{j=1}^{k}\frac{\partial p_k(r_1,y_1,\ldots,y_k)}{\partial y_j} \\ &= \frac{a}{k}\sum_{j=1}^{k} q_{r_j}\int_{x=0}^{V} g_{k-1}(x,r_1,y_1,\ldots,r_{j-1},y_{j-1},r_{j+1},y_{j+1},\ldots,r_k,y_k) \\ &\quad\times\int_{u=0}^{V-x} f_{r_j}(u,y_j)\,\mathrm{d}u\,\mathrm{d}x \\ &\quad -a\sum_{m=1}^{n-r_1-\ldots-r_k} q_m\int_0^V g_k(x,r_1,y_1,\ldots,r_k,y_k)L_m(V-x)\,\mathrm{d}x \\ &\quad +\sum_{j=1}^{k+1}\sum_{m=1}^{n-r_1-\ldots-r_k} p_{k+1}(r_1,y_1,\ldots,r_{j-1},y_{j-1},m,0,r_{j+1},y_{j+1},\ldots,r_k,y_k), \\ &\qquad\qquad k=2,\ldots,n-1;\ r_1+\ldots+r_k<n;\end{aligned} \tag{15}$$

$$-\sum_{j=1}^{k}\frac{\partial p_k(r_1,y_1,...,r_k,y_k)}{\partial y_j}$$
$$=\frac{a}{k}\sum_{j=1}^{k}q_{r_j}\int_{x=0}^{V}g_{k-1}(x,r_1,y_1,...,r_{j-1},y_{j-1},r_{j+1},y_{j+1},...,r_k,y_k) \tag{16}$$
$$\times\int_{u=0}^{V-x}f_{r_j}(u,y_j)\,\mathrm{d}u\,\mathrm{d}x,\ k=2,...,n-1;\ r_1+...+r_k=n;$$

$$-\sum_{j=1}^{n}\frac{\partial p_n(1,y_1,...,1,y_n)}{\partial y_j}$$
$$=\frac{a}{n}\sum_{j=1}^{n}q_1\int_{x=0}^{V}g_{n-1}(x,1,y_1,...,1,y_{j-1},1,y_{j+1},...,1,y_n) \tag{17}$$
$$\times\int_{u=0}^{V-x}f_1(u,y_j)\,\mathrm{d}u\,\mathrm{d}x.$$

For the system under consideration the boundary and normalization conditions may be represented in the next form:

$$p_1(m,0)=aq_mp_0L_m(V),\ m=1,...,n; \tag{18}$$

$$kp_k(r_1,y_1,...,r_{j-1},y_{j-1},m,0,r_{j+1},y_{j+1},...,r_k,y_k)$$
$$=aq_m\int_0^V g_{k-1}(x,r_1,y_1,...,r_{j-1},y_{j-1},r_{j+1},y_{j+1},...,r_k,y_k)L_m(V-x)\,\mathrm{d}x, \tag{19}$$
$$k=2,...,n;\ j=1,...,k;\ r_1+...+r_k\leqslant n;$$

$$p_0+\sum_{k=1}^{n}\sum_{r_1+...+r_k\leqslant n}\int_0^\infty...\int_0^\infty p_k(r_1,y_1,...,r_k,y_k)\,\mathrm{d}y_1...\mathrm{d}y_k=1. \tag{20}$$

If we introduce the notation $H_{my}(x)=\mathbf{P}\{\zeta_m<x,\xi_m\geqslant y\}$, then $\mathrm{d}H_{my}(x)=\int_{u=y}^{\infty}\mathrm{d}F_m(x,u)$ and by means of direct substitution, taking into account (18) – (20) expressions, we can easy show, that the solution of (12) – (17) equations is

$$g_k(x,r_1,y_1,...,r_k,y_k)dx=p_0\frac{a^k}{k!}\prod_{j=1}^{k}q_{r_j}\,\mathrm{d}\,(H_{r_1y_1}*...*H_{r_ky_k}(x)), \tag{21}$$
$$k=1,...,n;\ r_1+...+r_k\leqslant n,$$

where $H_{r_1 y_1} * ... * H_{r_k y_k}(x)$ is the Stieltjes convolution of $H_{r_1 y_1}(x)$,...,$H_{r_k y_k}(x)$ functions. From (8) relations we obtain

$$p_k(r_1, y_1, ..., r_k, y_k) = p_0 \frac{a^k}{k!} \prod_{j=1}^{k} q_{r_j} H_{r_1 y_1} * ... * H_{r_k y_k}(V), \qquad (22)$$
$$k = 1, ..., n; \ r_1 + ... + r_k \leqslant n.$$

Denote as

$$R_m(x) = \int_{u=0}^{x} \int_{y=0}^{\infty} y \, \mathrm{d}F_m(u, y), m = 1, ..., n.$$

Then it is follows from (9), that

$$p_k(r_1, ..., r_k) = p_0 \frac{a^k}{k!} R_{r_1} * ... * R_{r_k}(V) \prod_{j=1}^{k} q_{r_j}, \qquad (23)$$
$$k = 1, ..., n; \ r_1 + ... + r_k \leqslant n.$$

And from (11) formula we have

$$p_k = p_0 \frac{a^k}{k!} \sum_{r_1 + ... + r_k \leqslant n} R_{r_1} * ... * R_{r_k}(V) \prod_{j=1}^{k} q_{r_j}, \ k = 1, ..., n.$$

From the last formula we finally obtain

$$p_k = p_0 \frac{a^k}{k!} \sum_{r_1 + ... + r_k \leqslant n} R_{r_1} * ... * R_{r_k}(V) \prod_{j=1}^{k} q_{r_j}, \ k = 1, ..., n, \qquad (24)$$

where

$$p_0 = \left[1 + \sum_{k=1}^{n} \frac{a^k}{k!} \sum_{r_1 + ... + r_k \leqslant n} R_{r_1} * ... * R_{r_k}(V) \prod_{j=1}^{k} q_{r_j} \right]^{-1} \qquad (25)$$

The stationary loss probability p_{ms} of m-demand, $m = 1, ..., n$, is determined taking into account the symmetry of $p_k(r_1, ..., r_k)$ functions about rearrangements of arguments pairs (r_j, y_j), $j = 1, ..., k$, from the next equilibrium conditions:

$$aq_m(1 - p_{ms})$$
$$= \sum_{j=1}^{n-m+1} j \sum_{r_1 + ... + r_{j-1} \leqslant n-m} \int_0^{\infty} ... \int_0^{\infty} p_j(r_1, y_1, ..., r_{j-1}, y_{j-1}, m, 0) \times \mathrm{d}y_1 ... \mathrm{d}y_{j-1},$$
$$m = 1, ..., n.$$

From the last relation taking into account (22) and (25) formulas we have

$$p_{ms} = 1 - p_0 \sum_{j=0}^{n-m} \frac{a^j}{j!} \sum_{r_1+\ldots+r_j \leqslant n-m} L_m * R_{r_1} * \ldots * R_{r_j}(V) \prod_{l=1}^{j} q_{r_l}.$$

The total loss probability evidently is equal

$$\begin{aligned} p_s &= \sum_{m=1}^{n} q_m p_{ms} \\ &= 1 - p_0 \sum_{m=1}^{n} q_m \sum_{j=0}^{n-m} \frac{a^j}{j!} \sum_{r_1+\ldots+r_j \leqslant n+m} L_m * R_{r_1} * \ldots * R_{r_j}(V) \prod_{l=1}^{j} q_{r_l}. \end{aligned}$$

References

[1] O. M. Tikhonenko, *Queueing models in information systems*, Universitetskoe, Minsk 1990 (in Russian).

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