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The optimal strategy in one stochastic game

Abstract. The article deals with the stochastic game "Portion on three fractions". The optimal solution in this game is investigated.

From ancient times games belong among favourite human activities. A game of appropriate choice can be a very good educational means.

There are several games where our success or failure depends on chance, so called *stochastic games.* On the contrary we can also find games where maths can help a clever player to influence his chance to win, it means the result doesn't depend only on favourable circumstances, but also on our sagacity and preliminary consideration. The player who has found the main of the game (has found *rational strategy*), will have bigger chances to win comparing his opponent who has not got into the math main yet.

Now we are going to deal with example of one game and looking for its rational strategy, it means, process how to play under given circumstances as well as possible.

The game, *divide into three parts,* is determined for two players H_A and H_B. Each of them will divide number 9000 into three parts in the way that his opponent doesn't see what way he did it in. Then both players will compare the size of their parts (the biggest against the biggest, the middle size against the middle size and the smallest against the smallest). The player with bigger parts will get a point for each compared couple. The player who gets two points is the winner. How big part is the player H_A to create in case he knows that his opponent will divide 9000 by chance into three parts so that all permissible decompositions are of the same probability?

Let's indicate the size of particular parts as $x,\ y,\ 9000-x-y$. To make it simple we can suppose that $x \leqslant y \leqslant 9000-x-y$. All permissible decompositions are shown in fig. 1. (each of such decompositions corresponds with one point of a triangle OPQ).

Let's suppose that the player H_A divided 9000 into three parts $a, b, 9000-a-b$ and it is hold that: $a \leqslant b \leqslant 9000-a-b$, it means that player H_A in the triangle OPQ has chosen the points of coordinates $[a, b]$ (see Fig. 2.).

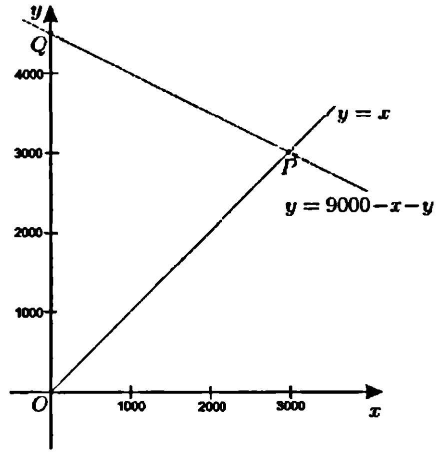

Figure 1. The permissible decompositions

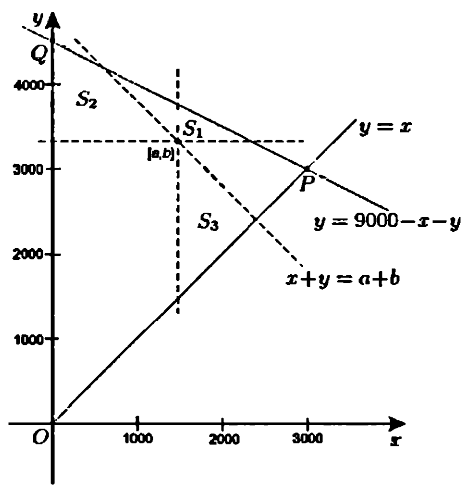

Figure 2. The favorable areas for player H_B

We specify which points of the triangle OPQ the player H_B can chose, provided he wants to win. Let's consider that the player H_B will make decomposition $m \leqslant n \leqslant 9000 - m - n$ and than he can win in case thet one of the following possibilities comes true.

1. $a < m$ and simultaneously $b < n$, (all the points which correspond with this situation make triangle S_1, see Fig. 2),
2. $b < n$ and simultaneously $9000-a-b < 9000-m-n$, (all the points which correspond with this situation make triangle tetragon S_2, see Fig. 2),

3. $a < m$ and simultaneously $9000 - a - b < 9000 - m - n$, (all the points which correspond with this situation make triangle triangle S_3, see Fig. 2).

Player H_A is looking for an optimal strategy in this game, then he must choose such point $[a, b]$ to odds of his opponent was minimal it means to get a minimal sum of the areas fig. S_1, S_2 a S_3. Regarding the fact that for different values $[a, b]$ there can be figures S_1, S_2 a S_3 triangles or tetragons the player H_A must differentiate 4 cases:

1. $a + b \geqslant 4500, \quad b < 3000,$
2. $a + b \geqslant 4500, \quad b \geqslant 3000,$
3. $a + b < 4500, \quad b < 3000,$
4. $a + b < 4500, \quad b \geqslant 3000.$

In each of these cases we will find a point for which the sum of areas is minimal. The point which defines the smallest sum of these 4 cases is the searched optimal point.

Let's show the whole process in the case 1 (see Fig. 2). First we count the area of the triangle S_1. The vertexes of triangle have coordinates

$$[a, b]; \quad [9000 - 2b, b]; \quad \left[a, \frac{9000 - a}{2}\right]$$

and its area is $\frac{1}{4}(9000 - a - 2b)^2$.

The coordinates of vertexes tetragon S_2 are

$$[0, b]; \quad [a, b]; \quad [2a + 2b - 9000, 9000 - a - b]; \quad [0, 4500].$$

The area of this figure is $(a + b - 4500)^2 + \frac{1}{2}(3a + 2b - 9000)(9000 - a - 2b)$.

The coordinates of triangle S_3 are

$$[a, a]; \quad \left[\frac{a + b}{2}, \frac{a + b}{2}\right]; \quad [a, b].$$

Its area is $\frac{1}{4}(b - a)^2$.

After simplification we get that for the searched sum S of areas it is hold that:

$$S = 4500a - \frac{3}{2}ab + \frac{1}{4}b^2.$$

The minimum of this function of two variables will be in points where the partial derivatives equal 0.

$$\frac{\partial}{\partial a}\left(4500a - \frac{3}{2}ab + \frac{1}{4}b^2\right) = 4500 - \frac{3}{2}b = 0,$$

$$\frac{\partial}{\partial b}\left(4500a - \frac{3}{2}ab + \frac{1}{4}b^2\right) = -\frac{3}{2}a + \frac{1}{2}b = 0.$$

The solutions of the system is $a = \frac{27000}{13} \approx 2076,92$ a $b = \frac{45000}{13} \approx 3461,53$. This point lies on the abscissa PQ, it means the area of figures S_1 equals 0. The sum of the areas of figures S_2 a S_3 equals in this case $\frac{20250000}{13} \approx 1557692,308$ and the area of triangle OPQ is 6750000. So, for the point chosen in this way the probability of the player's H_A win in the game is approximately 0,77.
It is possible to show that this point is simultaneously *optimal solution* of the whole problem it means, the player H_A has to divide 9000 into three parts in this way:
$a = \frac{27000}{13} \approx 2076,92$, $b = \frac{45000}{13} \approx 3461,54$, $9000 - a - b = \frac{45000}{13} \approx 3461,54$.

We can see that the process of finding optimal solutions of even such a simple game calls for managing of quite complicated mathematical methods. The whole procedure can be make easier by using a suitable mathematical software which enables the user to practice lots of calculations without any deeper getting into the mathematical main of the problem.

Completely different approach is based on finding an approximate solution which in this case is completely sufficient for a common player. In this case program Cabri seems to be the most suitable means which is able to count individual areas of figures S_1, S_2 a S_3 in dependance on the changing positions of the point $[a, b]$.

References

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