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The systematical construction of aims in mathematical education

Abstract. In education there are two elementary systems: the students and society, and one auxiliary, intermediary system: the teacher.
The construction of aims in mathematical instruction requires previous description of the most important educational proprieties and needs of both the society and the student, who is being gradually included in it.
It follows from that:

1. the basic subject of school education is the individual student and the essence of the educational process should be based on learning; whereas the teacher and the student's nearest environment (i.e. school) should be the organizer of learning,
2. school education should be a solid foundation for further education (including self-education), which equips the learner with relatively long-lasting and useful knowledge as well as with the skill of the widening and applying the knowledge.

Student's intellectual development is possible only in the process of active and directed work containing receipting, gaining, transforming and applying the information.

So it should be "education trough using mathematics, not only mathematics for itself" — the students should build their own cognitive schemas.

All statements above lead to three interrelated classifications:
type I — aims considering their contents at the level of generality,
type II — aims considering educational character,
type III — aims considering the level of realization (students achievements).

A set of all educational aims mentioned in classification I, II, III creates the didactic dimension of educational aims.

There are so many aspects of problems of aims of mathematical education. This domain has so many links with other domains; from the other side — in mathematical education researchers use various names, labels, expressions for the same phenomenon — even for the basic concepts and relations.

The attempt of systematic approach should rely on a distinction of the basic concepts, finding the basic relations between them and on exploration of the main properties of these relations.

In education two basic systems exist: student and society, and one intermediary system: the teacher.

The individual student lives and works in society. In a society different zones exists; some of them are closer to the pupil, some are further. In the educational process the pupil is the main subject for all teacher's activities. The whole society require from its members to increase their individual stocks and to cooperate with others. Each individual should know how utilize social stocks and how to develop his personality. Conclusion, which can be drawn from both these statements is as follows: **it is a necessity to postpone the "one faced" and uniformed system of teaching mathematics where *all* educational aims are realized in the same intensity.**

Construction of aims in mathematical education requires previous description of the most important educational properties and needs of both society and student, who is being gradually included in it.

Contemporary society still develops and one of the most important condition for such development is learning. Every day the world receives a huge doze of new knowledge, at the same time a huge amount of the "old" knowledge starts to be a non-actual and non-useful. Learning is a basic skill for the main part of society, and this skill is used also for self-learning, out of the school education.

Every student is an individual personality, who wants to develop itself, to know the world and other people. Obtaining a particular part of mathematical knowledge usually is not as the tool for realization of individual's needs. It follows from that:

1. the basic subject of school education is the individual student and the **essence of the educational process should be based on learning**; whereas the teacher and the student's nearest environment (i.e. school) should be the organizers of learning,
2. **school education should be a solid foundation for further education (including self-education)** which equips the learner with relatively long-lasting and useful knowledge as well as with the skill of widening and applying the knowledge.

Long lasting and various psychological researches show that this kind of intellectual development is possible only in the **active process, containing receiving, gaining, transforming, storing and applying information.** Those information are defined, on the one hand, by the society, on the other hand, by the students' necessities and possibilities. Information, external for the student, compared with his inside information brings a need for looking for a new knowledge and directs his further work. Motivation for this work depends on students' information about the world and about himself. Motivation like that is the most important factor, which eases the learning process, as the

more active a discovery of a new information the bigger student's intellectual satisfaction.

In a such context two basic questions appear:

?A — is mathematics, as entity, useful as a tool for rich those individual and social aims of education?

?B — if the answer to question A is "yes", how should we direct school mathematics to help to get the possible highest educational level?

To the first question we have a positive answer, based on the some theoretical analysis achieved by famous mathematicians, or gathered in the practice of teaching. According to that, we can define two statements. as the start point for identifying the aims of the mathematical education.

Z1. Mathematics is a very important part of the human culture.

Z2. In the process of mathematical education there is a correlation between stored mathematical knowledge and knowledge which is built.

Z3. The main ideas, principles, knowledge, methods and contexts useful for school mathematics are already described.

Intellectual personality level is determined first of all by two basic factors: range of learned information and skills to use them to reach their goals; in school teaching those goals are various and are described in the curriculum as the educational aims. The first factor is the **man's erudition**, the second — **his intellectual development**. Erudition consist of concrete information learned during the teaching process, it is a "potential energy" of the human mind. The development of thinking creates a possibility to transform erudition into a "kinetic energy", necessary to work.

Every human wants to know his position in society, wants to develop in a best way his abilities, to know the social significance of his activity. So, in the mathematical teaching (and not only) the pressure should be given on the intellectual development. It should be **"education through using mathematics, not only mathematics for itself"**: developing the basic cognitive skills — the general ones and the mathematical ones; abilities to ask questions and to search for answers, to be creative, to be self-dependent. Mathematics, like a few other school subject, can be a means to create an abstractive way of thinking. Although mathematics itself does not bring student's mind into the high level of thinking — it depends also on the general orientation of teaching. The orientation should use the operative character of mathematical concepts, so to have features of the teaching of mathematics based on the theory of interior construction of mathematical structure. Putting forward the postulates: 1. students' mathematical activity, 2.operative character of mathematics 3. structural orientations of its teaching we can get a necessary conclusion: **the students should construct their own cognitive schemas.**

Z. Krygowska proposed the following set of so called **mathematical activities for all**, which is a basement for construction of cognitive scheme: 1) to see and use analogies 2) to schematise 3) to define and interpret definition and to use definition in a rational way 4) to deduce and to reduce 5) to code and to construct symbolic language in the rational way 6) to algorithmize and to use algorithms in a rational way 7) to analyze, to classify, to put in order. This set should now be completed with activity: 8) to informaticize.

The basic systems which appear in education, that is: student and society (in which teacher and teaching program is used) have their own aims and, constantly changing in time but clear tendency and preparations to its realization. A lot of them show some pedagogical theories. The meaning of these aims can be as follows:

— to achieve some kind of wanted (planned) state of behavior (action),

– to achieve a change of the state or component of some kind of surrounding you people to their state of behavior.

These aims can be differentiated by the level of generality, it means: by the scope of notions and activities suitable for the development of personal beliefs. In each of those cases **the aim of teaching is the main and wanted student's properties**, concerning their knowledge, beliefs and general activity culture in problematic (open) situations.

All statements above lead to three interrelated classifications:
type I aims considering their contents at the level of generality,
type II aims considering educational character,
type III — aims considering the level of realization (students achievements).

Those classifications have their own hierarchy: higher category aims consist of lower category aims. That hierarchy gives us the chance to check the realization of the higher category aims using the lower category aims. Furthermore it eases reaching the equality between the lower and higher category aims, because very often in school practice the over-evaluation of lower category aims appears.

In type I classification we can differentiate two following aims:

N — **priorities** or **ideal** (main, global, essential), which are linked with a special school type: those aims *are a description of a wanted state of mind and personality of the student* after graduating. For the teaching of mathematics we can determine these aims: there are long-lasting and universal information, skills and beliefs which create a general mathematical culture, essential for everybody in the future life after graduating preliminary school. In other words, it is mathematical preparation for living in a society.

In the new secondary school (gymnasium) the aims are called "educational" and there are as follows:

1. Student's preparation for:

- *using mathematical knowledge to solve problems from various domains of schools education and from everyday situations,*
- *constructing mathematical models of real situations,*

2. *learning and using the mathematical language, to discover, formulate, solve and discuss mathematical problems,*
3. *developing spatial imagination.*

K — directive (specific, subjective, strategic) connected with mathematics as a teaching subject in a given school; goals are the *description of a wanted mathematical mind structure of a student* after graduating in that school.

In the new secondary school (gymnasium) the following set of aims was taken, called school duties: *1. Forming the thinking and clearly answering abilities, 2. Developing abilities of understanding a text written in the mathematical language, 3. Developing the abilities of describing an easy situations in mathematical language, 4. Making it easier to describe problems and to explore them in some situations by using some typical mathematical strategies.*

E — staged (intermediary , tactical) connected with the curriculum of a particular class (Vth class) or some unit from the teaching program (polygons) or one of thematic line (measurement). *Those aims are description of the subsystem of the K aim system or describes the cognitive substructures existing in students' mind* after graduating in that school. Those aims are not always described directly, they are possible to undercover according to the previous types of aims and to proposed content of the teaching program.

In the new secondary school (gymnasium) a list of aims was defined. Aims were connected with achievements: *1. conducting a non-complicated mathematical reasoning, 2. using numbers, operation properties and figure properties in problem solving, 3. using a calculator to solve typical tasks, 4. perceiving, using and interpreting functions, interpreting relations made by formulas, graphs, schemas, diagrams, tables. 5. using the mathematical language to present solutions of mathematical problems.*

S — detailed (behaviors, executive, achievements, objectives). Because of their character they are called **operative** (according to the general system theory). Operative aims are the aims of the specific teacher's work related to one teaching unit (1, 2 school hours). Those aims *are the exact description of elements of the conceptual structure of the student.*

Elements are defined as an activity, which the student should understand and practice in typical situations after knowing what the topic of the lesson is. The operational aim is a description of student's state of mind, and of his behavior after finishing a particular lesson. The aim like that should consists of:

— description of operations and their final results,

— conditions (inner and external) of executing those operations,

— models (standards) of making operations like that and coding the final results.

Ending operations should be described by using so called operational verbs: to recognise, to count, to draw, to measure, to represent. Verbs like: to understand, to know, to learn should not be used here, because on this level they are two-faced. For lesson "measurement of angles" in IV class of primary school the operative aims can be as following: projecting and giving names to the measure unit, using a protractor for measuring angles, drawing the angle when measure is given, finding the relation between the type of angle and its measure.

Among external conditions of executing final operations are:

— tools and contexts of the operations, like: straight line and circles, and compass and ruler in drawing pictures of geometrical figures,

— mathematical restrictions in executing the operations and in using those tools, and context (using a ruler for measurement is not allowed),

– features of personality (taking into account the needs and abilities of the student; the sense of self-motivation, giving high expectations and motivating them to work, regular control and assessment).

Standards of executing the operations are described variously, even in the same subject range. The level of the aim's realization should be described by one example that we can consider a positive one. In this way we can get praxeological efficacy of learning and teaching criteria. In manuals and exercise books these criteria appear as the set of typical tasks and their solutions, presented as a model. Teachers can use those models during their preparation for lessons.

Aims N, K are in the capacity of qualitative, and E, S — quantitative ones.

In type II classification we can differentiate the following aims: b

P **cognitive**; all information and skills should be contained in the cognitive structure. The whole amount of those information and skills students learn during the educational process. On the class level aims are described by eligible *teaching material* and *requirement* (in the curriculum for gymnasium it is called as *attainments*). Teaching material, requirements and directive aims create a system called *educational content*. On the primary school level cognitive aims are called basic information and skills. For example, in the high school lyceum) curriculum in 1966 aims were described as follows:

a) mastering by the student of basic mathematical information (i.e. scientific, useful, basic) in their modern shape,

b) practicing skills of using mathematical methods for solving problems connected with mathematics curriculum topics, as well as with technical and natural science problems,

c) developing logical thinking and using mathematical symbols and notions.

O — personality aims (educational, instructive); There are general features of mind and intellectual beliefs, created by teachers during a long-term educational process (connected not only with mathematics, but also with other school subjects). At the same time on the school level those aims serve as detailed description of N-aims. There are qualitative aims, which describe general results of education "through using mathematics".

Among other, the list of such aims includes:

- aims concerning general intellectual activity: developing imagination, intuition, creativity; dveloping skills in analyzing and synthesizing; developing of perseverance in intellectual effort to put tasks and to find solutions,
- aims concerning general skills in reasoning: educate general technique of solving problems; educate in logical thinking, to put in order information, recognizing causes and results, putting hypothesis; differentiate between proof and example, differentiate between truth and false,
- developing the aesthetic sense: sense of mathematical beauty in its inner construction and practical applications; clearness and elegance in pronouncements, clean view of notes.

Personality qualifications mentioned above are acknowledged as necessary elements of general education. They are necessary for everybody, whatever his/her profession and domain of activity. Prof. Z. Krygowska wrote: *The process of learning mathematics should be steered in such a way, that gathering knowledge and practicing skills take place at the some time of the process of development of beliefs and intellectual techniques typical for mathematics. Those techniques and beliefs should be not only the base for more general beliefs and intellectual techniques, but also the base for opinion about the role of science and theoretical thinking in our technological world.*

Type III classification concerns the level of learning the subject matter included in the curriculum. In Poland the most popular is a register of aims called ABCD. Aims A,B concern learning information, aims C and D concern learning skills.

In this list there are the following aims:

A — **remembrance of information**; it is an aptitude to remember particular terms, statements, (axioms, definitions, theorems, proofs), rules of action (algorithms, methods). Remembrance should be complete and ready tu use. Information can not be incorrect or deformed.

B — **understanding of information**; we expect here that the student is able to present some information in adifferent (but equivalent) way, to get some typical examples of concepts, to explore the possibilities for extending some situations, to see some basic relations, to make some conclusions.

C — **applying information in typical situations**: this aim means getting right result in solving a task, which is not much different from task

solved earlier. Applying knowledge rely not only on calling out from long-term memory, but also on making comparisons, analyzing data, recognizing rules and isomorphisms. Students can choose themselves algorithms or actions performed in the right order.

D — applying information in problem situations: the student can solve a problem new for him — he has to formulate the problem, to do analysis and synthesis of information and skills from different domains, to create a plan of action, to construct a new hypothesis and models, and to asses the final results.

All the personality aims and the cognitive high level aims (the D-level) are only orientation points for the teaching process. The teacher can only approach those aims using various ways and methods. But the main and necessary factor, from which the educational success depends — it is the authentic students' mathematical activitiy. Students' work has to be directed by well-prepared curriculum — not too rich and not too poor as well.

The set of all educational aims , indicated according to the classifications I, II, III — this means the Cartesian product I = {S, E, K, N}, II = {P, O}, III = {A, B, C, D}, together with the relation of inclusion in those sets — create **the didactic space of educational aims**. From the considerations above we can conclude that that space is not full; it means there are the empty cells in it.

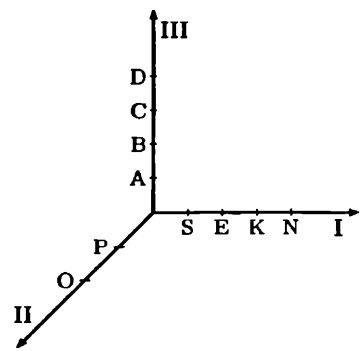

Figure 1. Dimension I, II, III of didactical space of educational aims

All aims having dimension (K, C, P) create **standards of educational quality.**

Passing to the more general aim — in the framework of the same classification — requires the realization of at least one of the aims from the lover level. This means, the structure of didactical space of educational aims has a cumulative character. This cumulativity is not rigid. Influence of the lover-level aims on the higher-level aims depends on many factors; this means, in the school reality, that this influence is changeable. The system of educational aims should involve a correction system, concerning not only the system of aims, but also the teaching material and the method of its realization.

If we put on the didactical space the educational program we can separate some **lines of realization of the teaching content**. Those lines can be as the necessary — but not sufficient — disposition of the teaching material, which can make possible to reach some aims from the I, II classifications on one of the A, B, C, D levels.

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