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Studia Mathematica IV

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Contents

Professor Andrzej Zajtz a scolar and educator

on the 70th birthday anniversary

Differential geometry was formed almost simultaneously with mathematical analysis. Its brisk development came with the success of the General Relativity Theory, which made extensive use of tensor calculus. The first Pole to do research on differential geometry at the beginning of the 20th century was Kazimierz Żorawski from the Jagiellonian University. Antoni Hoborski, another Cracovian mathematician, took it as his main sphere of interest and dreamed of the "Polish School" of differential geometry. Unfortunately, the Second World War and his sudden death in a concentration camp put an end to realization of his dreams. Effective steps aiming at fulfilling Hoborski's dream were undertaken by a student of his – Stanisław Gołąb. His interests revolved around the theory of geometric objects, which soon spread even beyond the Cracovian university centre. Professor Andrzej Zajtz proved himself a worthy successor of Professor's Gołąb's work in Kraków.

Andrzej Zajtz was born on December 9, 1934, in Radom. In 1955 he graduated from the Jagiellonian University, where he had studied mathematics. Even during his studies he worked as an assistant in the Institute of Mathematics of the AGH University of Science and Technology. In 1957 he took the position of an assistant in the Department of Geometry of the Jagiellonian University, whose head at time was Professor Gołąb. In 1961 he obtained a doctor's degree in the fields of mathematics and physics and in 1966 a habilitation in mathematics. He worked as an associate professor in the Institute of Mathematics of the Jagiellonian University until 1980 when he was honoured a professor's title. He was the head of the Geometry Department in the Institute of Mathematics of the Jagiellonian University for twenty years until 1990 when he moved to the Pedagogical University of Kraków. In 1994 Professor Zajtz became the head of the Department of Geometry and Differential Equations in the Institute of Mathematics of the Pedagogical University.

He frequently lectured at universities abroad. Between 1972 and 1975 he lectured at Ahmadu Bello University in Zaria, Nigeria, and from 1982 to 1983 at the Universidad Central de Venezuela in Caracas. He spent two years (1987-1989) as a professor at the Université de Tlemcen in Algeria and later (1990-1992) at the University of Zimbabwe in Harare. During his stay there, Professor Zajtz took an active part in the scientific and didactic work of these universities. Among his other duties, he acted as a consultant of various lecture programmes, especially in the field of geometry. He inspired and consulted scientific work of younger colleagues. Among other results of his activity one may mention four PhD theses supervised by him. This cooperation is being continued till now.

His scientific activity concentrated at first on differential geometry under weak differentiability conditions, and then on the theory of geometric objects, which was the main stream of geometrical research in the Cracovian mathematical centre. Geometric objects integrate the definitions of such significant concepts as tensors and connections. Their classification and study of their features had a great impact not only on the differential geometry but also on theoretical and practical applications in physics. Professor Zajtz classified important families of geometric objects and examined their properties in specific cases (see for instance [2]-[5], [8], [9]).

The theory of geometric objects is closely related to the theory of functional equations. It is also in this field that Professor Zajtz made significant discoveries [1] – [4]. His interests embraced differential geometry at its highest contemporary level. At the end of the 1970's the theory of geometric objects gained a new global description in the form of natural bundles and the natural prolongation functors. The academic textbook "Foundations of differential geometry of natural bundles" (1984), in cooperation with M. Paluszny, compiled A. Zajtz's lectures in Caracas. It included the latest outcomes in the natural bundles theory as well as generalizations of several important theorems.

A. Zajtz determined the sharp estimation of the order of natural bundles [6]. This fundamental result completed the long history of the order problem for differential geometric objects (in a new formalism – of natural functors), considered, in particular, by Penzov and Gołąb (1950's), up to the results of Palais-Terng and Epstein-Thurston (late 1970's). His estimation formula has found a direct application in the gauge-theory in theoretical physics. Further, he determined the sharp estimation of the order of natural functors on some basic local categories of manifolds (for instance manifolds with locally integrable volume form, symplectic manifolds and contact manifolds).

The natural differential operators theory has been developing together with the theory of natural functors. A. Zajtz investigated the problem of the row completion of natural operators and obtained some new general results [8]. Starting from a more general context, he contributed to the representation theory of certain groups of diffeomorphisms [7]-[9], [11]-[13]. He applied his own effective methods to the study of the equivalence and the order of certain types of representations. He also obtained positive results concerning the possibility of embedding diffeomorphisms in a smooth flow [14]. In parallel to these activities, but still within the general framework of his research, he proved some nonlinear Peetre-like theorems on local operators (such operators are very common in differential geometry) [10] – in his results, as opposed to previously formulated ones, references to the Whitney extension theorem are avoided.

The research outcomes of Professor Zajtz were presented in over 60 articles, both in Poland and abroad. He actively participated in many scientific conferences, where he delivered talks and was in charge of sessions or a member of scientific committees. In a short period of 1997-2001 his new original results were presented at ten conferences he attended.

Professor Zajtz was a supervisor of numerous MSc and thirteen PhD theses. Hitherto two of his former PhD students qualified as titular professors.

Scientific development of his numerous students participanting in his seminars has always been Professor Zajtz's particular concern. He encourages them to study advanced problems and to search for and draw their own conclusions. He is willing to discuss new trends and latest discoveries in differential geometry. There was only one person at the Department of Geometry holding doctor's degree at the time when Professor Zajtz became the head of that Department but in just twenty-year's time (1990) this number increased by seven – all but one worked under his supervision.

During the 50 years of his work he has taught the majority of main mathematical subjects. Moreover, he lectured on mathematics at other university institutes. It is worth mentioning that his lectures on analysis were keenly appreciated by the students of physics. Between 1976-1979 he was the Deputy Director of the Institute of Mathematics of the Jagiellonian University. At that time he was also the head of the Cracovian Branch of the Polish Mathematical Society. Twice he became a member of the Senate Board for the cooperation with the secondary educational system. From 1983 to 1990 he tutored secondary school "university classes" (with an originally designed advanced course of mathematics). In spite of his diverse duties he has never rejected requests to take part in the Mathematical Work Sessions for the Primary and Secondary Schools as a juror and chairman.

Prof. Zajtz was awarded the three most important medals: The Knight's Cross of Poland's Restoration (Krzyż Kawalerski Orderu Odrodzenia Polski), The Gold Medal of Merit (Złoty Krzyż Zasługi) and the National Education Committee Medal (Medal Komisji Edukacji Narodowej) for his didactic, scientific and organizational work.

It is said that the proof of the man is in his actions. Professor Zajtz has proved to be an excellent mathematician with broad interests. He is known as a man of mark, demanding and just. What he appreciates among his students is independent thinking and resolution to pursue the scientific truth. He is able to encourage young people to investigate puzzling questions of mathematics and stimulate their interest in new spheres of geometry.

It is only fair to add that Professor Andrzej Zajtz's profound influence on the Cracovian geometry research centre enabled it to establish itself as one of the most significant among such centres in Poland.

Zdzisław Pogoda

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Ludwik Byszewski

Strong maximum principles for implicit parabolic functional-differential problems together with initial inequalities

Dedicated to Professor Andrzej Zajtz
on the occasion of his 70th birthday

Abstract. The aim of the paper is to give strong maximum principles for implicit non-linear parabolic functional-differential problems together with initial inequalities in relatively arbitrary $(n+1)$-dimensional time - space sets more general than cylindrical domain.

1. Introduction

In this paper we consider implicit diagonal systems of non-linear parabolic functional-differential inequalities of the form

$$\begin{aligned} &F^i(t,x,u(t,x),u^i_t(t,x),u^i_x(t,x),u^i_{xx}(t,x),u) \\ &\qquad \geq F^i(t,x,v(t,x),v^i_t(t,x),v^i_x(t,x),v^i_{xx}(t,x),v) \\ &\qquad\qquad (i=1,\dots,m) \end{aligned} \tag{1.1}$$

for $(t,x)=(t,x_1,\dots,x_n)\in D$, where $D\subset(t_0,t_0+T]\times\mathbb{R}^n$ is one of three relatively arbitrary sets more general than the cylindrical domain $(t_0,t_0+T]\times D_0\subset\mathbb{R}^{n+1}$. The symbol w $(=u$ or $v)$ denotes the mapping

$$w\colon \tilde{D}\ni(t,x)\longrightarrow w(t,x)=(w^1(t,x),\dots,w^m(t,x))\in\mathbb{R}^m,$$

where $\tilde{D}$ is an arbitrary set contained in $(-\infty,t_0+T]\times\mathbb{R}^n$ such that $\bar{D}\subset\tilde{D}$; F^i $(i=1,\dots,m)$ are functionals of w; $w^i_x(t,x)=\mathrm{grad}_x w^i(t,x)$ $(i=1,\dots,m)$ and $w^i_{xx}(t,x)$ $(i=1,\dots,m)$ denote the matrices of second order derivatives with respect to x of $w^i(t,x)$ $(i=1,\dots,m)$. We give a lemma and a theorem on strong maximum principles for problems together with inequalities of types (1.1) and with initial inequalities.

AMS (2000) Subject Classification: 35R45, 35K20, 35K60, 35B50.

The results obtained are a generalization of some results given by R. Redheffer and W. Walter [4], by J. Szarski [5] and [6], by P. Besala [1], by W. Walter [8], by N. Yoshida [9], by the author [2] and [3], and base on those results. To prove the results of this paper we use the theorem on a strong maximum principle from [2].

2. Preliminaries

The notation and definitions given in this section are valid throughout this paper. Some of them are similar to those applied by J. Szarski [7], [6], by R. Redheffer and W. Walter [4], by P. Besala [1], by N. Yoshida [9] and by the author [3].

We use the following notation:

$$\mathbb{R} = (-\infty, \infty), \qquad \mathbb{N} = \{1, 2, \ldots\}, \qquad x = (x_1, \ldots, x_n)\ (n \in \mathbb{N}).$$

For any vectors $z = (z^1, \ldots, z^m) \in \mathbb{R}^m$, $\tilde{z} = (\tilde{z}^1, \ldots, \tilde{z}^m) \in \mathbb{R}^m$ we write

$$z \le \tilde{z} \qquad \text{if } z^i \le \tilde{z}^i\ (i = 1, \ldots, m).$$

Let t_0 be a real finite number and let $0 < T < \infty$. A set

$$D \subset \{(t, x) :\ t > t_0,\ x \in \mathbb{R}^n\}$$

(bounded or unbounded) is called a *set of type* (P) if:

(a) The projection of the interior of D on the t-axis is the interval $(t_0, t_0 + T)$.

(b) For every $(\tilde{t}, \tilde{x}) \in D$ there is a positive r such that

$$\left\{(t, x) :\ (t - \tilde{t})^2 + \sum_{i=1}^{n} (x_i - \tilde{x}_i)^2 < r,\ t < \tilde{t}\right\} \subset D.$$

We define the following sets:

$$S_{t_0} = \operatorname{int}\{x \in \mathbb{R}^n :\ (t_0, x) \in \bar{D}\} \qquad \text{and} \qquad \sigma_{t_0} = \operatorname{int}[\bar{D} \cap (\{t_0\} \times \mathbb{R}^n)].$$

Let $\tilde{D}$ be a set contained in $(-\infty, t_0 + T] \times \mathbb{R}^n$ such that $\bar{D} \subset \tilde{D}$. We introduce the following sets:

$$\partial_p D := \tilde{D} \setminus D \qquad \text{and} \qquad \Gamma := \partial_p D \setminus \sigma_{t_0}.$$

For an arbitrary fixed point $(\tilde{t}, \tilde{x}) \in D$ we denote by $S^-(\tilde{t}, \tilde{x})$ the set of points $(t, x) \in D$ that can be joined to $(\tilde{t}, \tilde{x})$ by a polygonal line contained in D along which the t-coordinate is weakly increasing from (t, x) to $(\tilde{t}, \tilde{x})$.

Let $Z_m(\tilde{D})$ denote the space of mappings

$$w\colon \tilde{D} \ni (t,x) \longrightarrow w(t,x) = (w^1(t,x),\dots,w^m(t,x)) \in \mathbb{R}^m$$

continuous in $\bar{D}$.

In the set of mappings bounded from above in $\tilde{D}$ and belonging to $Z_m(\tilde{D})$ we define the functional

$$[w]_t = \max_{i=1,\dots,m} \sup\{0, w^i(\tilde{t},x) : \ (\tilde{t},x) \in \tilde{D},\ \tilde{t} \le t\}, \qquad \text{where } t \le t_0 + T.$$

By $M_{n\times n}(\mathbb{R})$ we denote the space of real square symmetric matrices $r = [r_{jk}]_{n\times n}$.

A mapping $w \in Z_m(\tilde{D})$ is called *regular* in D if

$$w^i_t, \quad w^i_x = \operatorname{grad}_x w^i, \quad w^i_{xx} = [w^i_{x_j x_k}]_{n\times n} \qquad (i=1,\dots,m)$$

are continuous in D.

Let the mappings

$$F^i : D \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times M_{n\times n} \times Z_m(\tilde{D}) \ni (t,x,z,p,q,r,w) \longrightarrow F^i(t,x,z,p,q,r,w) \in \mathbb{R} \qquad (i=1,\dots,m)$$

be given and let for an arbitrary regular in D function $w \in Z_m(\tilde{D})$

$$F^i[t,x,w] := F^i(t,x,w(t,x),w^i_t(t,x),w^i_x(t,x),w^i_{xx}(t,x),w), \qquad (t,x) \in D \quad (i=1,\dots,m).$$

Each two regular in D mappings $u, v \in Z_m(\tilde{D})$ are said to be *solutions* of the system

$$F^i[t,x,u] \ge F^i[t,x,v] \qquad (i=1,\dots,m) \tag{2.1}$$

in D, if they satisfy (2.1) for all $(t,x) \in D$.

For a given regular mapping w in D and for an arbitrary fixed $i \in\{1,\dots,m\}$, the mapping F^i is called *uniformly parabolic* with respect to w in a subset $E \subset D$ if there is a constant $\kappa > 0$ (depending on E) such that for any two matrices $\tilde{r} = [\tilde{r}_{jk}], \hat{r} = [\hat{r}_{jk}] \in M_{n\times n}(\mathbb{R})$ and for all $(t,x) \in E$ we have

$$\begin{aligned}\tilde{r} \le \hat{r} \Longrightarrow F^i(t,x,w(t,x),w^i_t(t,x),w^i_x(t,x),\hat{r},w)& \\ - F^i(t,x,w(t,x),w^i_t(t,x),w^i_x(t,x),\tilde{r},w)& \\ \ge \kappa \sum_{j=1}^n (\hat{r}_{jj} - \tilde{r}_{jj}),&\end{aligned} \tag{2.2}$$

where $\tilde{r} \le \hat{r}$ means that $\sum_{j,k=1}^n (\tilde{r}_{jk} - \hat{r}_{jk})\lambda_j\lambda_k \le 0$ for every $(\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n$.

If (2.2) is satisfied for $\tilde{r} = w^i_{xx}(t,x)$, $\hat{r} = w^i_{xx}(t,x) + r$, $r \geq 0$ and $\kappa = 0$, then F^i is called *parabolic* with respect to w in E.

An unbounded set D of type (P) is called a *set of type* (P_Γ) if

$$\Gamma \cap \overline{\sigma}_{t_0} \neq \emptyset. \tag{2.3}$$

A bounded set D of type (P) is called a *set of type* (P_B).

It is easy to see that each set D of type (P_B) satisfies condition (2.3). Moreover, it is obvious that if D_0 is a bounded subset [D_0 is an unbounded proper subset] of $\mathbb{R}^n$, then $D = (t_0, t_0 + T] \times D_0$ is a set of type (P_B) [(P_Γ), respectively].

3. Lemma

As a consequence of Theorem 3.1 from [2] we obtain the following:

Lemma 3.1
Assume that:

1° *D is a set of type (P).*

2° *The mappings F^i $(i = 1,\dots,m)$ are weakly increasing with respect to $z^1,\dots,z^{i-1},z^{i+1},\dots,z^m$ $(i = 1,\dots,m)$. Moreover, there is a positive constant $L > 0$ such that*

$$\begin{aligned} &F^i(t,x,z,p,q,r,w) - F^i(t,x,\tilde{z},p,\tilde{q},\tilde{r},\tilde{w}) \\ &\quad \leq L\Bigg(\max_{k=1,\dots,m} \mid z^k - \tilde{z}^k \mid + \mid x \mid \sum_{j=1}^{n} \mid q^j - \tilde{q}^j \mid \\ &\quad\quad + \mid x \mid^2 \sum_{j,k=1}^{n} \mid r_{jk} - \tilde{r}_{jk}] + [w - \tilde{w}]_t \Bigg) \end{aligned}$$

for all $(t,x) \in D$, $z, \tilde{z} \in \mathbb{R}^m$, $p \in \mathbb{R}$, $q, \tilde{q} \in \mathbb{R}^n$, $r, \tilde{r} \in M_{n\times n}(\mathbb{R})$, $w, \tilde{w} \in Z_m(\tilde{D})$, where $\sup_{(t,x)\in\tilde{D}}(w(t,x) - \tilde{w}(t,x)) < \infty$ $(i = 1,\dots,m)$.

3° *There are constants $C_i > 0$ $(i = 1, 2)$ such that*

$$F^i(t,x,z,p,q,r,w) - F^i(t,x,z,\tilde{p},q,r,w) < C_1(\tilde{p} - p) \qquad (i = 1,\dots,m)$$

for all $(t,x) \in D$, $z \in \mathbb{R}^m$, $p > \tilde{p}$, $q \in \mathbb{R}^n$, $r \in M_{n\times n}(\mathbb{R})$, $w \in Z_m(\tilde{D})$ and

$$F^i(t,x,z,p,q,r,w) - F^i(t,x,z,\tilde{p},q,r,w) < C_2(\tilde{p} - p) \qquad (i = 1,\dots,m)$$

for all $(t,x) \in D$, $z \in \mathbb{R}^m$, $p < \tilde{p}$, $q \in \mathbb{R}^n$, $r \in M_{n\times n}(\mathbb{R})$, $w \in Z_m(\tilde{D})$.

4° *The mapping* $u \in Z_m(\tilde{D})$ *is regular in* D*, and* $\sup_{(t,x)\in D} u(t,x) < \infty$.

5° $u(t,x) \leq K$ *for* $(t,x) \in \partial_p D$*, where* $K = (K^1, \dots, K^m)$ *is a constant mapping.*

6° *The mappings* u *and* K *are solutions of the system*

$$F^i[t,x,u] \geq F^i[t,x,K] \qquad (i = 1, \dots, m)$$

in D.

7° *The mappings* F^i $(i = 1, \dots, m)$ *are parabolic with respect to* u *in* D *and uniformly parabolic with respect to* K *in any compact subset of* D.

Then

$$u(t,x) \leq K \qquad \text{for } (t,x) \in \tilde{D}.$$

Moreover, if there is a point $(\tilde{t}, \tilde{x}) \in D$ *such that* $u(\tilde{t}, \tilde{x}) = K$ *then*

$$u(t,x) = K \qquad \text{for } (t,x) \in S^-(\tilde{t}, \tilde{x}).$$

4. Strong maximum principles together with initial inequalities in sets of types (P_Γ) and (P_B)

Now, we shall give the following theorem on strong maximum principles together with initial inequalities in sets of types (P_Γ) and (P_B):

Theorem 4.1
Assume that:

(i) D *is a set of type* (P_Γ) *or* (P_B) *and assumptions* 2^0 *and* 3^0 *of Lemma 3.1 are satisfied.*

(ii) *The mapping* $u \in Z_m(\tilde{D})$ *is regular in* D *and the maximum of* u *on* Γ *is attained. Moreover,*

$$K := \max_{(t,x)\in\Gamma} u(t,x). \tag{4.1}$$

(iii) *The inequality*

$$u(t_0, x) \leq K \qquad \text{for } x \in S_{t_0} \tag{4.2}$$

is satisfied.

(iv) *The maximum of* u *in* $\tilde{D}$ *is attained. Moreover,*

$$M := \max_{(t,x)\in\tilde{D}} u(t,x). \tag{4.3}$$

(v) *The mappings* u *and* M *are solutions of the system*

$$F^i[t,x,u] \geq F^i[t,x,M] \qquad (i=1,\dots,m)$$

in D.

(vi) *The mappings* F^i $(i=1,\dots,m)$ *are parabolic with respect to* u *in* D *and uniformly parabolic with respect to* M *in any compact subset of* D.

Then

$$\max_{(t,x)\in\tilde{D}} u(t,x) = \max_{(t,x)\in\Gamma} u(t,x). \tag{4.4}$$

Moreover, if there is a point $(\tilde{t},\tilde{x}) \in D$ *such that* $u(\tilde{t},\tilde{x}) = \max_{(t,x)\in\tilde{D}} u(t,x)$ *then*

$$u(t,x) = \max_{(t,x)\in\Gamma} u(t,x) \qquad \text{for } (t,x)\in S^-(\tilde{t},\tilde{x}).$$

Proof. We shall prove Theorem 4.1 for a set of type (P_Γ) only since the proof for a set of type (P_B) is analogous.

We shall argue by contradiction. Suppose

$$M \neq K. \tag{4.5}$$

From (4.1) and (4.3), we have

$$K \leq M. \tag{4.6}$$

Consequently

$$K < M. \tag{4.7}$$

Observe, from assumption (iv), that

$$\text{there is } (t^*,x^*) \in \tilde{D} \text{ such that } u(t^*,x^*) = M := \max_{(t,x)\in\tilde{D}} u(t,x). \tag{4.8}$$

By (4.8), by assumption (ii) and by (4.7), we have

$$(t^*,x^*) \in \tilde{D}\setminus\Gamma = D \cup \sigma_{t_0}. \tag{4.9}$$

Suppose that

$$(t^*,x^*) \in D. \tag{4.10}$$

From assumptions (ii) and (v), and from (4.8), we get

$$\begin{cases} u \in Z_m(\tilde{D}) \text{ and } u^i_t,\ u^i_x,\ u^i_{xx}\ (i=1,\dots,m) \text{ are continuous in } D,\\ F^i[t,x,u] \geq F^i[t,x,M] \text{ for } (t,x)\in D\ (i=1,\dots,m),\\ u(t,x) \leq M \text{ for } (t,x)\in\tilde{D},\\ u(t^*,x^*) = M. \end{cases} \tag{4.11}$$

The assumption that D is a set of type (P), assumptions 2° and 3° (see assumption (i)), formulas (4.10) and (4.11), and assumption (vi) imply, by Lemma 3.1, the equation

$$u(t,x) = M \qquad \text{for } (t,x) \in S^-(t^*,x^*). \tag{4.12}$$

On the other hand, from the definition of a set of type (P_Γ), there is a polygonal line $\gamma \subset S^-(t^*,x^*)$ such that

$$\overline{\gamma} \cap \Gamma \neq \emptyset. \tag{4.13}$$

Since $u \in C(\bar{D},\mathbb{R}^m)$, we have a contradiction of formulas (4.12) and (4.13) with formulas (4.1) and (4.7). Therefore, $(t^*,x^*) \notin D$ and, consequently, from (4.9), $(t^*,x^*) \in \sigma_{t_0}$. But this leads, by (4.7), to a contradiction of (4.2) with (4.8). The proof of (4.4) is complete.

The second part of Theorem 4.1 is a consequence of equality (4.4) and of Lemma 3.1. Therefore, the proof of Theorem 4.1 is complete.

Remark 4.1

If D is a set of type (P_B) and if $\tilde{D} = \bar{D}$ then the first part of assumption (ii) of Theorem 4.1 relative to the maximum of u and the first part of assumption (iv) of this theorem are trivially satisfied since $u, v \in C(\bar{D},\mathbb{R}^m)$ and Γ is bounded and closed set in this case.

Remark 4.2

If the mappings F^i $(i = 1,\ldots,m)$ do not depend on the functional argument w then Lemma 3.1 and Theorem 4.1 reduce to the lemma and the theorem, respectively, on parabolic differential inequalities including terms

$$F^i(t,x,u(t,x),u^i_t(t,x),u^i_x(t,x),u^i_{xx}(t,x)) \qquad (i = 1,\ldots,m)$$

and in this case we can put $\tilde{D} = \bar{D}$.

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Functorial prolongations of some functional bundles

To Andrzej Zajtz, on the occasion of his 70th birthday

Abstract. We discuss two kinds of functorial prolongations of the functional bundle of all smooth maps between the fibers over the same base point of two fibered manifolds over the same base. We study the prolongation of vector fields in both cases and we prove that the bracket is preserved. Our proof is based on several new results concerning the finite dimensional Weil bundles.

Introduction

Let E_1 and E_2 be two classical fiber bundles over the same base M. The differential geometric investigation of the functional bundle $\mathcal{F}(E_1, E_2) \longrightarrow M$ of all smooth maps from a fiber of E_1 into the fiber of E_2 over the same base point was iniciated by the paper by A. Jadczyk and M. Modugno on the Schrödinger connection, [6], [7]. The simpliest cases of the tangent bundle $T\mathcal{F}(E_1, E_2) \longrightarrow TM$ and of the r-th jet prolongation $J^r\mathcal{F}(E_1, E_2) \longrightarrow M$ are discussed in [1]. In the present paper we first clarify that the essential assumption for these constructions is that T is a product preserving bundle functor on the classical category $\mathcal{M}f$ of all smooth manifolds and all smooth maps and J^r is a fiber product preserving bundle functor on the category $\mathcal{FM}_m$ of all fibered manifolds with m-dimensional bases and of all fibered manifold morphisms covering local diffeomorphisms. Every product preserving bundle functor F on $\mathcal{M}f$ is a Weil functor $F = T^A$, where A is a Weil algebra, [12]. The general construction of $T^A\mathcal{F}(E_1, E_2) \longrightarrow T^AM$ was presented by the third author in [9], [10], see also Section 2 of the present paper. We underline that this construction is based on the covariant approach to Weil bundles and their natural transformations, [8], [12]. On the other hand, in [13] it was deduced that every fiber product preserving bundle functor G on $\mathcal{FM}_m$ is of

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the form $G = (A, H, t)$, where A is a Weil algebra, H is a group homomorphism $H\colon G^r_m \longrightarrow \operatorname{Aut} A$ of the r-th jet group G^r_m in dimension m into the group of all algebra automorphisms of A and $t\colon \mathbb{D}^r_m \longrightarrow A$ is an equivariant algebra homomorphism, where $\mathbb{D}^r_m = J^r_0(\mathbb{R}^m, \mathbb{R})$ is the Weil algebra corresponding to the functor of (m, r)-velocities. In Section 6 of the present paper we construct $G\mathcal{F}(E_1, E_2) \longrightarrow M$ in a way that generalizes the case of $J^r\mathcal{F}(E_1, E_2) \longrightarrow M$.

Our main geometric problem is the prolongation of vector fields on $\mathcal{F}(E_1, E_2)$ with respect to F and G. Since we cannot use the flow in the functional case, we start from the fact that the classical flow prolongation with respect to T^A of a vector field $M \longrightarrow TM$ coincides with the composition of its T^A-prolongation $T^AM \longrightarrow T^ATM$ with the exchange map $\kappa^A_M\colon T^ATM \longrightarrow TT^AM$. We apply this idea to a vector field X on $\mathcal{F}(E_1, E_2)$ and we say the composition $\mathcal{T}^AX = \kappa^A_{\mathcal{F}(E_1,E_2)} \circ T^AX$ to be the field prolongation of X. The bracket of vector fields on $\mathcal{F}(E_1, E_2)$ is defined in terms of the strong difference, [1], [12]. Proposition 3.2 in Section 3 reads that $\mathcal{T}^A$ preserves the bracket of vector fields even in the functional case. To deduce it, we develop, in Sections 4 and 5, a purely algebraic proof of the fact that $\mathcal{T}^A$ preserves bracket in the manifold case. For this purpose we need certain new lemmas concerning the classical Weil bundles, which are collected in Sections 4 and 5. In particular, we present a complete description of the strong difference in terms of Weil algebras. In Section 7 we study the prolongation of vector fields to $G\mathcal{F}(E_1, E_2)$ and we prove that the bracket is preserved even in this case. Finally we remark that an interesting kind of exchange morphism, which was introduced recently for the manifold case in [11], can be extended to the functional bundles as well.

In Section 1 we present a simplified version of the theory of smooth spaces in the sense of A. Frölicher, [4], which we call F-smooth spaces, and of F-smooth bundles. Special attention is paid to the functorial character of the construction of $\mathcal{F}(E_1, E_2)$ and to the concept of finite order morphism.

If we deal with finite dimensional manifolds and maps between them, we always assume they are of class C^∞, i.e. smooth in the classical sense. Unless otherwise specified, we use the terminology and notation from the monograph [12].

1. F-smooth bundles

We shall use the following simplified version, [2], of the theory of smooth spaces by A. Frölicher, [4].

Definition 1.1
An F-smooth space is a set S along with a set C_S of maps $c\colon \mathbb{R} \longrightarrow S$, which are called F-smooth curves, satisfying the following two conditions:

(i) each constant curve $\mathbb{R} \longrightarrow S$ belongs to C_S ,

(ii) if $c \in C_S$ and $\gamma \in C^\infty(\mathbb{R}, \mathbb{R})$, then $c \circ \gamma \in C_S$.

If $(S', C_{S'})$ is another F-smooth space, a map $f\colon S \longrightarrow S'$ is said to be F-smooth, if $f \circ c$ is an F-smooth curve on S' for every F-smooth curve c on S.

So we obtain the category $\mathcal{S}$ of F-smooth spaces. Every subset $\bar{S} \subset S$ is also an F-smooth space, if we define $C_{\bar{S}} \subset C_S$ to be the subset of the curves with values in $\bar{S}$. In particular every smooth manifold M turns out to be an F-smooth space by assuming as F-smooth curves just the smooth curves. Moreover, a map between smooth manifolds is F-smooth, if and only if it is smooth.

We find it useful to define the concept of F-smooth bundle in a more general form than in [2].

Definition 1.2
An F-smooth bundle is a triple of an F-smooth space S, a smooth manifold M and a surjective F-smooth map $p\colon S \longrightarrow M$. If $p'\colon S' \longrightarrow M'$ is another F-smooth bundle, then a morphism of S into S' is a pair of an F-smooth map $f\colon S \longrightarrow S'$ and a smooth map $\underline{f}\colon M \longrightarrow M'$ satisfying $\underline{f} \circ p = p' \circ f$.

Thus we obtain the category $\mathcal{SB}$ of F-smooth bundles. Every subset $\bar{S} \subset S$ satisfying $p(\bar{S}) = M$ is also an F-smooth bundle.

An important class of F-smooth bundles are the bundles of smooth maps between the fibers over the same base point of two classical fibered manifolds $p_1\colon E_1 \longrightarrow M$ and $p_2\colon E_2 \longrightarrow M$. We write

$$\mathcal{F}(E_1, E_2) = \bigcup_{x \in M} C^\infty(E_{1x}, E_{2x})$$

and denote by $p\colon \mathcal{F}(E_1, E_2) \longrightarrow M$ the canonical projection. A curve $c\colon \mathbb{R} \longrightarrow \mathcal{F}(E_1, E_2)$ is called F-smooth, if $\underline{c} := p \circ c\colon \mathbb{R} \longrightarrow M$ is a smooth map and the induced map

$$\tilde{c}\colon \underline{c}^* E_1 \longrightarrow E_2, \qquad \tilde{c}(t, y) = c(t)(y), \qquad p_1(y) = \underline{c}(t)$$

is also smooth, [1].

Write $\mathcal{FM}^I \subset \mathcal{FM}$ for the subcategory of locally trivial fibered manifolds whose morphisms are diffeomorphisms on the fibers. Let $\mathcal{FM}^I \times_{\mathcal{B}} \mathcal{FM}$ denote the category whose objects are pairs (E_1, E_2) with $E_1 \longrightarrow M$ in $\mathcal{FM}^I$ and $E_2 \longrightarrow M$ in $\mathcal{FM}$ and morphisms are pairs (f_1, f_2) with $f_1\colon E_1 \longrightarrow E_3$ in $\mathcal{FM}^I$ and $f_2\colon E_2 \longrightarrow E_4$ in $\mathcal{FM}$ over the same base map $\underline{f}\colon M \longrightarrow N$, where N is the common base of E_3 and E_4. If we define $\mathcal{F}(f_1, f_2)\colon \mathcal{F}(E_1, E_2) \longrightarrow \mathcal{F}(E_3, E_4)$ by

$$\mathcal{F}(f_1, f_2)(h) = f_2(x) \circ h \circ f_1^{-1}(\underline{f}(x)), \qquad h \in C^\infty(E_{1x}, E_{2x}), \tag{1.1}$$

then $\mathcal{F}$ is a functor on $\mathcal{FM}^I \times_{\mathcal{B}} \mathcal{FM}$ with values in the category $\mathcal{SB}$.

Definition 1.3
Every F-smooth subbundle $S \subset \mathcal{F}(E_1, E_2)$ will be called a functional F-smooth bundle.

If $S' \subset \mathcal{F}(E_3, E_4)$ is another functional F-smooth bundle and (f_1, f_2) has the property $\mathcal{F}(f_1, f_2)(S) \subset S'$, then the restricted and corestricted map will be interpreted as an $\mathcal{SB}$-morphism $S \longrightarrow S'$.

Consider a smooth map $q\colon E_3 \longrightarrow E_1$.

Definition 1.4
An $\mathcal{SB}$-morphism $D\colon \mathcal{F}(E_1, E_2) \longrightarrow \mathcal{F}(E_3, E_4)$ is said to be of the order r, if for every $\varphi, \psi\colon E_{1x} \longrightarrow E_{2x}$ and $v \in E_3$, $p_1(q(v)) = x$,

$$j^r_{q(v)}\varphi = j^r_{q(v)}\psi \quad \text{implies} \quad D(\varphi)(v) = D(\psi)(v). \tag{1.2}$$

Consider the fibered manifold

$$\mathcal{F}J^r(E_1, E_2) = \bigcup_{x \in M} J^r(E_{1x}, E_{2x}) \longrightarrow E_1\,. \tag{1.3}$$

By (1.2), D induces the so called associated map

$$\mathcal{D}\colon \mathcal{F}J^r(E_1, E_2) \times_{E_1} E_3 \longrightarrow E_4\,.$$

In the same way as in [1] one proves that $\mathcal{D}$ is a smooth map.

We express the coordinate form of $\mathcal{D}$ in the case $q\colon E_3 \longrightarrow E_1$ is an $\mathcal{FM}$-morphism that is a surjective submersion on each fiber of E_3. Let x^i or u^a be some local coordinates on M or N and y^p or z^s or (y^p, v^b) or w^c be some additional fiber coordinates on E_1 or E_2 or E_3 or E_4, respectively. Then z^s_α are the induced coordinates on $\mathcal{F}J^r(E_1, E_2)$, where $0 \le |\alpha| \le r$ is a multiindex, the range of which is the fiber dimension of E_1, and the coordinate expression of $\mathcal{D}$ is

$$u^a = f^a(x^i), \qquad w^c = f^c(x^i, y^p, z^s_\alpha, v^b), \tag{1.4}$$

where f^a and f^c are smooth functions.

The concept of r-th order morphism can be modified to a functional F-smooth bundle $S \subset \mathcal{F}(E_1, E_2)$ analogously to [12], Section 18.

2. The tangent-like case

Let A be a Weil algebra of the width k. Under the covariant approach, [8], [12], the elements of a Weil bundle $T^A M$ are the A-velocities $j^A g$ of smooth maps $g\colon \mathbb{R}^k \longrightarrow M$. For a smooth map $f\colon M \longrightarrow N$, we define $T^A f\colon T^A M \longrightarrow T^A N$ by

$$T^A f(j^A g) = j^A (f \circ g). \tag{2.1}$$

If B is another Weil algebra of the width l, then every algebra homomorphism $\mu\colon A \longrightarrow B$ can be generated by a B-velocity $j^B h$ of a map $h\colon \mathbb{R}^l \longrightarrow \mathbb{R}^k$. The natural transformation $\mu_M\colon T^A M \longrightarrow T^B M$ induced by μ has the form of a reparametrization

$$\mu_M(j^A g) = j^B (g \circ h). \tag{2.2}$$

Consider $\mathcal{F}(E_1, E_2)$. We have $T^A p_i\colon T^A E_i \longrightarrow T^A M$ and we write

$$T^A_X E_i := (T^A p_i)^{-1}(X), \qquad X \in T^A M,\ i = 1, 2.$$

Let $g_1, g_2\colon \mathbb{R}^k \longrightarrow \mathcal{F}(E_1, E_2)$ be two F-smooth maps satisfying $j^A(p \circ g_1) = j^A(p \circ g_2) \in T^A M$. Then we construct the associated maps $T^A_0 g_i\colon T^A_X E_1 \longrightarrow T^A_X E_2$,

$$T^A_0 g_i(j^A f(u)) = j^A g_i(u)(f(u)), \qquad u \in \mathbb{R}^k,$$

where $f\colon \mathbb{R}^k \longrightarrow E_1$ satisfies $p \circ g_i = p_1 \circ f$, $i = 1, 2$. If $T^A_0 g_1 = T^A_0 g_2$, we say that g_1 and g_2 determine the same A-velocity $j^A g_1 = j^A g_2$. The set $T^A \mathcal{F}(E_1, E_2)$ of all such A-velocities is a subspace in $\mathcal{F}(T^A E_1, T^A E_2) \longrightarrow T^A M$, so a functional F-smooth bundle. In the product case $E_i = M \times Q_i$, $i = 1, 2$, the third author deduced in [9]

$$T^A(M \times Q_1, M \times Q_2) = T^A M \times C^\infty(Q_1, T^A Q_2). \tag{2.3}$$

In [9] it was also clarified that the idea of reparametrization (2.2) can be applied to $j^A g \in T^A \mathcal{F}(E_1, E_2)$ as well. So every algebra homomorphism $\mu = j^B h\colon A \longrightarrow B$ induces an F-smooth map

$$\mu_{\mathcal{F}(E_1,E_2)}\colon T^A \mathcal{F}(E_1, E_2) \longrightarrow T^B \mathcal{F}(E_1, E_2), \qquad j^A g \longmapsto j^B (g \circ h). \tag{2.4}$$

Consider a functional F-smooth bundle $S \subset \mathcal{F}(E_1, E_2)$. Then $T^A S \subset T^A \mathcal{F}(E_1, E_2)$ means the subset of all $j^A g$, $g\colon \mathbb{R}^k \longrightarrow S$.

Definition 2.1
An $\mathcal{SB}$-morphism $D\colon S \longrightarrow \mathcal{F}(E_3, E_4)$ is called A-differentiable, if the rule

$$T^A D(j^A g) = j^A (D \circ g)$$

defines an F-smooth map $T^A S \longrightarrow T^A \mathcal{F}(E_3, E_4)$. We say D is strongly differentiable, if it is A-differentiable for every Weil algebra A.

If D is strongly differentiable, then $T^A D$ is also strongly differentiable. Indeed, analogously to the finite dimensional case one verifies easily $T^B(T^A D) = T^{B\otimes A} D$. In particular, every finite order morphism is strongly differentiable, for its associated map is smooth. Further, each morphism $\mathcal{F}(f_1, f_2)$ is strongly differentiable and we have

$$T^A\mathcal{F}(f_1, f_2)(j^A g(u)) = j^A\big(f_2(p(g(u))) \circ g(u) \circ f_1^{-1}(\underline{f}(p(g(u))))\big)\,.$$

Thus, $T^A\mathcal{F}$ is a functor on the category $\mathcal{FM}^I \times_{\mathcal{B}} \mathcal{FM}$ with values in $\mathcal{SB}$.

Analogously to the finite dimensional case, [3], we define an A-field on $\mathcal{F}(E_1, E_2)$ as a strongly differentiable section $\mathcal{F}(E_1, E_2) \longrightarrow T^A\mathcal{F}(E_1, E_2)$. In the case $A = \mathbb{D}$ of the algebra of dual numbers, we obtain a vector field $X\colon \mathcal{F}(E_1, E_2) \longrightarrow T\mathcal{F}(E_1, E_2)$.

3. Prolongation of vector fields

In the manifold case, the exchange algebra homomorphism $\kappa^A\colon A \otimes \mathbb{D} \longrightarrow \mathbb{D}\otimes A$ defines a natural transformation $\kappa^A_M\colon T^A TM \longrightarrow TT^A M$. For a classical vector field $X\colon M \longrightarrow TM$, its flow prolongation $\mathcal{T}^A X\colon T^A M \longrightarrow TT^A M$ coincides with $\kappa^A_M \circ T^A X$, [12]. For a vector field $X\colon \mathcal{F}(E_1, E_2) \longrightarrow T\mathcal{F}(E_1, E_2)$, we also can construct $T^A X\colon T^A\mathcal{F}(E_1, E_2) \longrightarrow T^A T\mathcal{F}(E_1, E_2)$ and apply $\kappa^A_{\mathcal{F}(E_1,E_2)}\colon T^A T\mathcal{F}(E_1, E_2) \longrightarrow TT^A\mathcal{F}(E_1, E_2)$. In this way we obtain a vector field on $T^A\mathcal{F}(E_1, E_2)$.

Definition 3.1
The vector field $\mathcal{T}^A X := \kappa^A_{\mathcal{F}(E_1,E_2)} \circ T^A X$ will be called the field prolongation of X.

We recall that the bracket of two vector fields X, Y on $\mathcal{F}(E_1, E_2)$ was defined by using the strong difference, [1],

$$[X, Y] = (TY \circ X) \div (TX \circ Y). \tag{3.1}$$

(For classical vector fields $X, Y\colon M \longrightarrow TM$, (3.1) coincides with the classical bracket, [1].) We are going to deduce

Proposition 3.2
For every vector fields X, Y on $\mathcal{F}(E_1, E_2)$,

$$\mathcal{T}^A([X, Y]) = [\mathcal{T}^A X, \mathcal{T}^A Y]. \tag{3.2}$$

The proof will be based on the algebraic results of the next two sections.

4. The algebraic form of the strong difference

Write $p_M^T \colon TM \longrightarrow M$ for the bundle projection. We recall that two elements $X, Y \in TT_xM$ satisfying

$$p_{TM}^T X = Tp_M^T Y, \qquad p_{TM}^T Y = Tp_M^T X \tag{4.1}$$

determine the strong difference

$$X \div Y \in T_xM, \tag{4.2}$$

[12]. Denote by SM the domain of definition of the strong difference, i.e., $SM \subset TTM \times_M TTM$ is the subset of all pairs (X, Y) satisfying (4.1), and by $\sigma_M \colon SM \longrightarrow TM$ the map (4.2). For every smooth map $f \colon M \longrightarrow N$, one verifies easily that (TTf, TTf) transforms SM into SN. So we obtain a map

$$Sf \colon SM \longrightarrow SN$$

and S is a bundle functor on $\mathcal{M}f$. Moreover, the strong difference map is a natural transformation

$$\sigma_M \colon SM \longrightarrow TM. \tag{4.3}$$

The fact $S\mathbb{R}^m = \overset{5}{\times}\mathbb{R}^m$ implies that S preserves products. Write $\mathbb{S}$ for the corresponding Weil algebra. In general, the sum of two Weil algebras $A = \mathbb{R} \times N_A$ and $B = \mathbb{R} \times N_B$ is defined by

$$A + B = \mathbb{R} \times N_A \times N_B$$

with the induced multiplication that satisfies $ab = 0$ for all $a \in N_A$, $b \in N_B$. Clearly, we have

$$T^AM \times_M T^BM = T^{A+B}M.$$

Write $\mathbb{D} = \{a_0 + a_1 e\}$, $e^2 = 0$. Then TT corresponds to $\mathbb{D} \otimes \mathbb{D}$, which is linearly generated by 1, e_1, e_2, e_1e_2. Let $\{1, E_1, E_2, E_1E_2\}$ be the linear generators of another copy of $\mathbb{D} \otimes \mathbb{D}$. So $\mathbb{S}$ is a subalgebra of $\mathbb{D} \otimes \mathbb{D} + \mathbb{D} \otimes \mathbb{D}$ and (4.1) implies directly that the elements of $\mathbb{S}$ are of the form

$$X = a_0 + a_1(e_1 + E_2) + a_2(e_2 + E_1) + a_3e_1e_2 + a_4E_1E_2,$$

$a_0, \ldots, a_4 \in \mathbb{R}$. By the definition of the strong difference, [12], the algebra homomorphism $\sigma \colon \mathbb{S} \longrightarrow \mathbb{D}$ corresponding to (4.2) is

$$\sigma(X) = a_0 + (a_3 - a_4)e. \tag{4.4}$$

Write $p_M^A \colon T^AM \longrightarrow M$ for the bundle projection. Since $SM \subset TTM \times_M TTM$ is defined by (4.1), $T^ASM \subset T^ATTM \times_{T^AM} T^ATM$ is the set of all pairs (X, Y) satisfying

$$T^A p^T_{TM} X = T^A T p^T_M Y, \qquad T^A p^T_{TM} Y = T^A T p^T_M X. \tag{4.5}$$

On the other hand, $ST^A M \subset TTT^A M \times_{T^A M} TTT^A M$ is characterized by

$$p^T_{TT^A M} X = T p^T_{T^A M} Y, \qquad p^T_{TT^A M} Y = T p^T_{T^A M} X. \tag{4.6}$$

We have $T^A \sigma_M \colon T^A SM \longrightarrow T^A TM$, $\kappa^A_{TM} \colon T^A TTM \longrightarrow TT^A TM$ and $T\kappa^A_M \colon TT^A TM \longrightarrow TTT^A M$. For technical reasons, we postpone the proof of the following assertion to Section 5.

Proposition 4.1
The map $T\kappa^A_M \circ \kappa^A_{TM} \colon T^A TTM \longrightarrow TTT^A M$ induces a diffeomorphism $K^A_M \colon T^A SM \longrightarrow ST^A M$ and the following diagram commutes

$$\begin{array}{ccc} T^A SM & \xrightarrow{K^A_M} & ST^A M \\ {\scriptstyle T^A\sigma_M}\downarrow & & \downarrow{\scriptstyle \sigma_{T^A M}} \\ T^A TM & \xrightarrow{\kappa^A_M} & TT^A M \end{array} \tag{4.7}$$

Now we first show how (4.7) implies that the flow prolongation $\mathcal{T}^A$ of classical vector fields $X, Y \colon M \longrightarrow TM$ preserves the bracket. We have $(TY \circ X, TX \circ Y) \colon M \longrightarrow SM$ and

$$[X, Y] = \sigma_M \circ (TY \circ X, TX \circ Y). \tag{4.8}$$

Then $T^A(TY \circ X, TX \circ Y) \colon T^A M \longrightarrow T^A SM$. Adding K^A_M we obtain

$$\begin{aligned} T\kappa^A_M \circ \kappa^A_{TM} \circ T^A TY \circ T^A X &= T\kappa^A_M \circ TT^A Y \circ \kappa^A_M \circ T^A X \\ &= T\mathcal{T}^A Y \circ \mathcal{T}^A X \end{aligned}$$

and the same for $TX \circ Y$. So in (4.7) we clockwise obtain $[\mathcal{T}^A X, \mathcal{T}^A Y]$. Counterclockwise, we first get $T^A[X, Y]$ and then $\mathcal{T}^A[X, Y]$.

Consider now the case of $\mathcal{F}(E_1, E_2)$. According to the general fact that the homomorphisms of Weil algebras extend to the functional case, (4.7) yields a commutative diagram

$$\begin{array}{ccc} T^A S\mathcal{F}(E_1, E_2) & \xrightarrow{K^A_{\mathcal{F}(E_1,E_2)}} & ST^A \mathcal{F}(E_1, E_2) \\ {\scriptstyle \kappa^A_{\mathcal{F}(E_1,E_2)}}\downarrow & & \downarrow{\scriptstyle \sigma_{T^A \mathcal{F}(E_1,E_2)}} \\ T^A T\mathcal{F}(E_1, E_2) & \xrightarrow{\kappa^A_{\mathcal{F}(E_1,E_2)}} & TT^A \mathcal{F}(E_1, E_2) \end{array} \tag{4.9}$$

For two vector fields X, Y on $\mathcal{F}(E_1, E_2)$, we first construct

$$(TY \circ X, TX \circ Y) \colon \mathcal{F}(E_1, E_2) \longrightarrow S\mathcal{F}(E_1, E_2).$$

Then we deduce (3.2) in the same way as in the manifold case. This proves Proposition 3.2.

5. Some Weilian lemmas

The elements of $A = T^A\mathbb{R}$ are of the form $j^A g$, $g\colon \mathbb{R}^k \longrightarrow \mathbb{R}$. For a vector space V, the map $V \times A \longrightarrow T^A V$, $(v, j^A g) \longmapsto j^A(gv)$ is bilinear and defines an identification $T^A V = V \otimes A$. If W is another vector space and $f\colon V \longrightarrow W$ is a linear map, then $T^A f\colon T^A V \longrightarrow T^A W$ is of the form

$$T^A f = f \otimes \mathrm{id}_A \colon V \otimes A \longrightarrow W \otimes A, \tag{5.1}$$

[12]. Further, let $\mu\colon A \longrightarrow B$ be an algebra homomorphism. Then the induced natural transformation $\mu_V\colon T^A V \longrightarrow T^B V$ is of the form

$$\mu_V = \mathrm{id}_V \otimes \mu\colon V \otimes A \longrightarrow V \otimes B. \tag{5.2}$$

This follows from the fact that V is isomorphic to $\mathbb{R}^n$ and we have a product preserving functor.

In particular, if C is another Weil algebra, then (5.1) implies that the natural transformation $T^C\mu_M\colon T^C T^A M \longrightarrow T^C T^B M$ corresponds to the algebra homomorphism

$$\mathrm{id}_C \otimes \mu\colon C \otimes A \longrightarrow C \otimes B. \tag{5.3}$$

Further, the maps $\mu_{T^C M}\colon T^A T^C M \longrightarrow T^B T^C M$ form a natural transformation $T^A T^C \longrightarrow T^B T^C$ that corresponds to the algebra homomorphism

$$\mu \otimes \mathrm{id}_C\colon A \otimes C \longrightarrow B \otimes C. \tag{5.4}$$

The trivial bundle functor on $\mathcal{M}f$ transforming every manifold M into $\mathrm{id}_M\colon M \longrightarrow M$ and every smooth map f into (f, f) corresponds to the trivial Weil algebra $\mathbb{R}$. The natural transformation $p^A_M\colon T^A M \longrightarrow M$ is determined by the canonical "real part projection" $\rho_A\colon A = \mathbb{R} \times N_A \longrightarrow \mathbb{R}$. So $T^B p^A_M\colon T^B T^A M \longrightarrow T^B M$ corresponds to the canonical map

$$\mathrm{id}_B \otimes \rho_A\colon B \otimes A \longrightarrow B \otimes \mathbb{R} = B. \tag{5.5}$$

Write $\kappa^{A,B}\colon A \otimes B \longrightarrow B \otimes A$ for the exchange map. This defines the exchange natural transformation $\kappa^{A,B}_M\colon T^A T^B M \longrightarrow T^B T^A M$. By (5.4), $\kappa^{A,B}_{T^C M}\colon T^A T^B T^C M \longrightarrow T^B T^A T^C M$ corresponds to the exchange $A \otimes B \otimes C \longrightarrow B \otimes A \otimes C$. By (5.3), $T^B\kappa^{A,C}_M\colon T^B T^A T^C M \longrightarrow T^B T^C T^A M$ corresponds to the exchange $B \otimes A \otimes C \longrightarrow B \otimes C \otimes A$.

Lemma 5.1
The following diagram commutes

$$\begin{array}{ccc}
T^A T^B T^C M & \xrightarrow{\;T^B\kappa^{A,C}_M \circ \kappa^{A,B}_{T^C M}\;} & T^B T^C T^A M \\
{\scriptstyle T^A p^B_{T^C M}}\downarrow & & \downarrow{\scriptstyle p^B_{T^C T^A M}} \\
T^A T^C M & \xrightarrow{\;\kappa^{A,C}_M\;} & T^C T^A M
\end{array} \tag{5.6}$$

Proof. At the algebra level, we have a commutative diagram

$$\begin{array}{ccccc} A\otimes B\otimes C & \longrightarrow & B\otimes A\otimes C & \longrightarrow & B\otimes C\otimes A \\ \downarrow & & & & \downarrow \\ A\otimes C & & \longrightarrow & & C\otimes A \end{array}$$

Now we are in position to prove Proposition 4.1. Comparing our general case with the situation in Section 4, we see $\kappa^{A,\mathbb{D}} = \kappa^A$ and $p^{\mathbb{D}}_M = p^T_M$. So if we put $B = \mathbb{D} = C$ into (5.6), we obtain

$$p^T_{TT^AM} \circ T\kappa^A_M \circ \kappa^A_{TM} = \kappa^A_M \circ T^A p^T_{TM}\,. \tag{5.7}$$

Every $X, Y \in T^A SM$ satisfy (4.5). The naturality of κ^A on $p^T_M \colon TM \longrightarrow M$ yields

$$\kappa^A_M \circ T^A T p^T_M = TT^A p^T_M \circ \kappa^A_{TM} \tag{5.8}$$

and the standard relation $p^T_{T^AM} \circ \kappa^A_M = T^A p^T_M$ implies

$$Tp^T_{T^AM} \circ T\kappa^A_M = TT^A p^T_M\,. \tag{5.9}$$

Hence we have

$$\begin{aligned}(p^T_{TT^AM} \circ T\kappa^A_M \circ \kappa^A_{TM})(X) &= \kappa^A_M(T^A p^T_{TM}(X)) = \kappa^A_M(T^A T p^T_M(Y)) \\ &= (TT^A p^T_M \circ \kappa^A_{TM})(Y) \\ &= (Tp^T_{T^AM} \circ T\kappa^A_M \circ \kappa^A_{TM})(Y).\end{aligned}$$

Thus, $(T\kappa^A_M \circ \kappa^A_{TM})(X)$ and $(T\kappa^A_M \circ \kappa^A_{TM})(Y)$ satisfy (4.6), so that K^A_M maps $T^A SM$ into $ST^A M$. In the case $M = \mathbb{R}^m$, we have $S\mathbb{R}^m = \overset{5}{\times}\mathbb{R}^m$ and $T^A\mathbb{R}^m = A^m$, so that $T^A S\mathbb{R}^m = \overset{5}{\times}A^m$ and $ST^A\mathbb{R}^m = \overset{5}{\times}A^m$. In this situation, $K^A_{\mathbb{R}^m}$ is the identity of $\overset{5}{\times}A^m$. Moreover, by (4.4) $\sigma_{\mathbb{R}^m}$ is determined by the difference of the fourth and fifth components. Taking into account that the vector addition in A is the T^A-prolongation of the addition of reals, we deduce that the diagram (4.7) commutes.

6. The jet-like case

Every fiber product preserving bundle functor G on $\mathcal{FM}_m$ is of the form $G = (A, H, t)$ where A is a Weil algebra, $H \colon G^r_m \longrightarrow \operatorname{Aut} A$ is a group homomorphism and $t \colon \mathbb{D}^r_m \longrightarrow A$ is an equivariant algebra homomorphism, [13]. For every manifold N, the natural transformations corresponding to $\operatorname{Aut} A$ determine an action H_N of G^r_m on $T^A N$. So we can construct the associated bundle

$P^rM[T^AN, H_N]$, where $P^rM \subset T^r_mM$ is the r-th order frame bundle of M. For a fibered manifold $\pi\colon E \longrightarrow M$, we define GE as a subset of $P^rM[T^AE, H_E]$ characterized by

$$GE = \{\{u, Z\},\ t_M u = T^A\pi(Z)\}, \qquad u \in P^rM,\ Z \in T^AE. \tag{6.1}$$

For an $\mathcal{FM}_m$-morphism $f\colon E \longrightarrow \bar{E}$ over a local diffeomorphism $\underline{f}\colon M \longrightarrow \bar{M}$, we have the induced principal bundle morphism $P^r\underline{f}\colon P^rM \longrightarrow P^r\bar{M}$ and an G^r_m-equivariant map $T^Af\colon T^AE \longrightarrow T^A\bar{E}$. So we can construct $P^r\underline{f}[T^Af]\colon P^rM[T^AE] \longrightarrow P^r\bar{M}[T^A\bar{E}]$ and we define

$$Gf = P^r\underline{f}[T^Af]|GE. \tag{6.2}$$

In the product case $E = \mathbb{R}^m \times Q$, we have $GE = \mathbb{R}^m \times T^AQ$, [13].

This construction extends directly to $\mathcal{F}(E_1, E_2)$. By (2.4), each element of $\operatorname{Aut} A$ determines an F-smooth isomorphism $T^A\mathcal{F}(E_1, E_2) \longrightarrow T^A\mathcal{F}(E_1, E_2)$. So we have an action $H_{\mathcal{F}(E_1,E_2)}$ of G^r_m on $T^A\mathcal{F}(E_1, E_2)$ and we can construct the F-smooth associated bundle

$$P^rM[T^A\mathcal{F}(E_1, E_2), H_{\mathcal{F}(E_1,E_2)}]. \tag{6.3}$$

Then we define $G\mathcal{F}(E_1, E_2)$ as the subset of (6.3) characterized by

$$\begin{aligned} G\mathcal{F}(E_1, E_2) = \{\{u, Z\},\ t_M u = T^Ap(Z)\},& \\ u \in P^rM,\ Z \in T^A\mathcal{F}(E_1, E_2).& \end{aligned} \tag{6.4}$$

Write $\mathcal{FM}^I_m = \mathcal{FM}^I \cap \mathcal{FM}_m$. For $(f_1, f_2) \in \mathcal{FM}^I_m \times_{\mathcal{B}} \mathcal{FM}_m$ with the common base map $\underline{f}$, we define

$$G\mathcal{F}(f_1, f_2) = P^r\underline{f}[T^A\mathcal{F}(f_1, f_2)]|G\mathcal{F}(E_1, E_2). \tag{6.5}$$

Hence $G\mathcal{F}$ is a functor on $\mathcal{FM}^I_m \times_{\mathcal{B}} \mathcal{FM}_m$ with values in $\mathcal{SB}$.

In the product case $E_1 = \mathbb{R}^m \times Q_1$, $E_2 = \mathbb{R}^m \times Q_2$, we have

$$G\mathcal{F}(E_1, E_2) = \mathbb{R}^m \times C^\infty(Q_1, T^AQ_2). \tag{6.6}$$

This shows that for $J^r = (\mathbb{D}^r_m, \mathrm{id}_{G^r_m}, \mathrm{id}_{\mathbb{D}^r_m})$ we obtain $J^r\mathcal{F}(E_1, E_2)$ constructed by means of the fiber r-jets in [1].

7. Vector fields in the jet-like case

In the manifold case, [11], if we have a principal bundle $P(M, C)$ with structure group C and a left C-space S, a right-invariant vector field φ on P and a left-invariant vector field ψ on S, the product vector field (φ, ψ) on

$P \times S$ is projectable to a vector field $\{\varphi, \psi\}$ on the associated bundle $P[S]$. In particular, if η is a projectable vector field on $E \longrightarrow M$ over a vector field ξ on M, then the flow prolongation $\mathcal{P}^r\xi$ is right-invariant on P^rM and $\mathcal{T}^A\eta$ is left-invariant on T^AE. In [11] we deduced that the flow prolongation $\mathcal{G}\eta$ of η coincides with the restriction of $\{\mathcal{P}^r\xi, \mathcal{T}^A\eta\}$ to $GE \subset P^rM[T^AE]$.

In the functional case, consider a vector field $X\colon \mathcal{F}(E_1, E_2) \longrightarrow T\mathcal{F}(E_1, E_2)$ over $\xi\colon M \longrightarrow TM$. Then (2.4) implies that the field prolongation $\mathcal{T}^AX$ is $H_{\mathcal{F}(E_1,E_2)}$-invariant. Hence we have the vector field $\{\mathcal{P}^r\xi, \mathcal{T}^AX\}$ on $P^rM[T^A\mathcal{F}(E_1, E_2)]$ and we define the field prolongation $\mathcal{G}X$ of X by

$$\mathcal{G}X = \{\mathcal{P}^r\xi, \mathcal{T}^AX\}|G\mathcal{F}(E_1, E_2). \tag{7.1}$$

This is a vector field $G\mathcal{F}(E_1, E_2) \longrightarrow TG\mathcal{F}(E_1, E_2)$ over ξ. For two vector fields X_i on $\mathcal{F}(E_1, E_2)$ over ξ_i , $i = 1, 2$, we have by the basic properties of the strong difference

$$[\mathcal{G}X_1, \mathcal{G}X_2] = \{[\mathcal{P}^r\xi_1, \mathcal{P}^r\xi_2], [\mathcal{T}^AX_1, \mathcal{T}^AX_2]\}.$$

Hence Proposition 3.2 yields

Proposition 7.1
We have

$$[\mathcal{G}X_1, \mathcal{G}X_2] = \mathcal{G}[X_1, X_2]\,.$$

At the end we remark that the third author, [11], constructed a map

$$\mu^G_E\colon J^rTM \times_{GTM} GTE \longrightarrow TGE$$

with the property that for every projectable vector field η on E over ξ on M

$$\mathcal{G}\eta = \mu^G_E \circ (j^r\xi \times_M G\eta)\,,$$

where $j^r\xi\colon M \longrightarrow J^rTM$ is the r-th jet prolongation of the section $\xi\colon M \longrightarrow TM$ and $G\eta : GE \longrightarrow GTE$ is the induced morphism. Analyzing this construction, one realizes that each step can be extended to our functional case. In other words, one can introduce in the same way an F-smooth morphism

$$\mu^G_{\mathcal{F}(E_1,E_2)}\colon J^rTM \times_{GTM} G\mathcal{F}(E_1, E_2) \longrightarrow TG\mathcal{F}(E_1, E_2)$$

with the property

$$\mathcal{G}X = \mu^G_{\mathcal{F}(E_1,E_2)} \circ (j^r\xi \times_M GX)$$

for every vector field X on $\mathcal{F}(E_1, E_2)$ with underlying vector field ξ on M.

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On some functional equations related to Steffensen's inequality

Dedicated to Professor Andrzej Zajtz on his seventieth birthday

Abstract. We consider the problem, proposed by the second author (cf. [1]) of solving functional equations stemming from the Steffensen integral inequality (S), which is applicable in actuarial problems, cf. [4]. Imposing some regularity conditions we find solutions of two equations in two variables, one with two and another with three unknown functions.

1. Introduction

J.F. Steffensen proved in his paper [4] from 1918 entitled *On certain inequalities between mean values and their applications to actuarial problems* the following

Proposition

If $f\colon [a,b] \to \mathbb{R}$ *is a decreasing function and* $g\colon [a,b] \to [0,1]$ *is an integrable function, then*

$$\int_{b-c}^{b} f(t)\,dt \le \int_{a}^{b} f(t)g(t)\,dt \le \int_{a}^{a+c} f(t)\,dt, \qquad c := \int_{a}^{b} g(t)\,dt. \tag{S}$$

[Cf. also J. Dieudonné [2], p. 50, and, for this and for several related inequalities, D.S. Mitrinović [3], Section 2.16, pp. 105-116.]

The second author proposed in [1] to look for f and g such that the medial term in inequalities (S) is the arithmetic mean of the two others. Let x and y vary in $[a,b]$ and let us write the relevant functional equation with the unknown functions f and g:

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$$\int_x^{x+\gamma(x,y)} f(t)\,dt + \int_{y-\gamma(x,y)}^{y} f(t)\,dt = 2\int_x^y f(t)g(t)\,dt, \qquad \gamma(x,y) := \int_x^y g(t)\,dt, \tag{E}$$

where $(x, y) \in [a, b]^2$.

In this paper we deal with equation (E) for differentiable f and continuous g. We also consider a functional equation related to (E), with three, sufficiently regular, unknown functions: f, g and h, the latter replacing those limits of integration in (E) which contain $\gamma(x, y)$, cf. [1].

2. Equation with two unknown functions

Let us first note that if f in (E) is a constant function then the equation is satisfied by an arbitrary (integrable) function g. In the theorem that follows we determine functions f, corresponding to a wide variety of functions g, such that (f, g) be the solution to (E). It turns out that in most cases f is a constant function.

THEOREM 1
Assume that $g\colon [a, b] \to [0, 1]$ *is a continuous function and either:*

(i) $g(x) = K$ *for* $x \in [a, b]$, *and* $K \notin \{0, 1, \frac{1}{2}\}$

or

(ii) $0 < g(x) < 1$, $x \in (a, b)$ *and either* $g(a) = 0$, $g(b) = 1$ *or* $g(a) = 1$, $g(b) = 0$.

Then the function $f\colon [a, b] \to \mathbb{R}$, *differentiable in* $[a, b]$, *satisfies equation* (E) *if and only if it is of the form:*

in case (i)

$$f(x) = \alpha x + \beta, \qquad x \in [a, b],$$

in case (ii)

$$f(x) = A, \qquad x \in [a, b],$$

where α, β, A *are arbitrary real numbers.*

Proof. Assume (i). Since now $\gamma(x, y) = K(y - x)$, equation (E) becomes:

$$\int_x^{x+K(y-x)} f(t)\,dt + \int_{y-K(y-x)}^{y} f(t)\,dt = 2K\int_x^y f(t)\,dt, \qquad (x, y) \in [a, b]^2. \tag{1}$$

We take the derivatives, with respect to x of both sides of (1):

$$f(x+K(y-x))(1-K)-f(x)-f(y-K(y-x))K=-2Kf(x).$$

as f is continuous in $[a,b]$. We differentiate again, but with respect to y, and divide the formula obtained by $K(1-K)$:

$$f'(x+K(y-x))-f'(y-K(y-x))=0, \qquad (x,y)\in[a,b]^2.$$

The transformation

$$T\colon \mathbb{R}^2\to\mathbb{R}^2; \quad (x,y)\mapsto(X,Y):=((1-K)x+Ky,Kx+(1-K)y)$$

maps bijectively $[a,b]^2$ onto itself. Indeed, $K\neq\frac{1}{2}$ implies injectivity of T and $K\in(0,1)$ says that both X and Y are convex combinations of x and y. Therefore we have the relation

$$f'(t)=f'(s), \qquad (s,t)\in[a,b]^2.$$

Consequently, the function f' is constant and f is affine, on $[a,b]$, as claimed.

It is the matter of a straightforward calculation to verify that the functions given by $f(x)=\alpha x+\beta$, $g(x)=K$, $x\in[a,b]$, satisfy equation (1) for every real α and β.

Assume (ii). We differentiate equation (E) twice, first with respect to x, then with respect to y. Since $\gamma(x,y):=\int_x^y g(t)\,dt$ and g is continuous, $\frac{\partial\gamma}{\partial x}=-g(x)$, $\frac{\partial\gamma}{\partial y}=g(y)$. We get, consecutively,

$$f(x+\gamma(x,y))(1-g(x))-f(x)-f(y-\gamma(x,y))(-g(x))=-2f(x)g(x),$$

$$f'(x+\gamma(x,y))(1-g(x))g(y)-f'(y-\gamma(x,y))g(x)(1-g(y))=0. \qquad (\mathrm{E}')$$

Both equalities hold for $(x,y)\in[a,b]^2$.

Let now $g(a)=0$ and $g(b)=1$. We put $x=a$ in (E′):

$$f'(a+\gamma(a,y))g(y)=0, \qquad y\in(a,b].$$

Since the function $u\colon(a,b]\to\mathbb{R}$, $u(y)=a+\gamma(a,y)$, is strictly increasing in (a,b) ($u'(y)=g(y)>0$), it maps $(a,b]$ onto $(u(a),u(b)]=(a,a+c]$ (where $c=\int_a^b g(t)\,dt$, cf. (S)). Thus

$$f'(t)=0 \qquad (2)$$

for $t\in(a,a+c]$. Now we put $y=b$ in (E′):

$$f'(x+\gamma(x,b))(1-g(x))=0, \qquad x\in[a,b).$$

The function $v\colon[a,b)\to\mathbb{R}$, $v(x)=x+\gamma(x,b)$, is a strictly increasing bijection ($v'(x)=1-g(x)>0$) of $[a,b)$ onto $[a+c,b)$, whence (2) holds for $t\in(a+c,b]$. Finally, (2) is valid in (a,b), f is a constant function, $f(x)=A$ in (a,b). By the continuity, f is constant in $[a,b]$, as claimed.

Similarly, $g(a)=1$ implies (2) in $(a,b-c]$, whereas $g(b)=0$ yields (2) in $[b-c,b)$. It follows that $f'(t)$ vanishes in (a,b) also in the other case listed in (ii), whence f is a constant function on $[a,b]$.

To complete the proof let us remind that if f in (E) is a constant function then the equation is satisfied for every integrable function g.

Remark 1
If in case (i) we have $K \in \{0, 1, \frac{1}{2}\}$, then equation (E) is an identity and f may be an arbitrary function.

Remark 2
If the inequalities of (ii) hold but $g(a) = g(b) = 0$, or $g(a) = g(b) = 1$. then $f'(t) = 0$ in $(a, a+c] \cup [b-c, b)$. In the case where both intervals are disjoint, we get only the information that the restrictions of f to each of the intervals is a constant function.

Remark 3
For $[a, b]$ replaced by $\mathbb{R}$ and $g(x) = K$, $x \in \mathbb{R}$, $|K| > 1$, the formula $f(x) = \alpha x + \beta$, $x \in \mathbb{R}$ also presents all differentiable in $\mathbb{R}$ solution of (E). Indeed, we may repeat the proof of (i) of Theorem 1, because the transformation T exploited there maps bijectively the plane onto itself.

3. Equation with three unknown functions

We pass to examining the equation related to (E) in which the limits of integration $x + \gamma(x, y)$, resp. $y - \gamma(x, y)$, are replaced by $h(xy + x + y)$, resp. $h(xy - x - y)$, where h is also an unknown function. Moreover, a "correcting term" has been added. The equation to be solved, with the unknown functions f, g, h, reads

$$\begin{aligned}\int_x^{h(xy+x+y)} f(t)\,dt + \int_{h(xy-x-y)}^{y} f(t)\,dt + \int_{h(y^2+2y)}^{h(y^2-2y)} f(t)\,dt \\ = 2\int_x^y f(t)g(t)\,dt,\end{aligned} \tag{H}$$

First of all, assuming the necessary regularity of the functions involved, on differentiating both sides of equation (H) with respect to x we obtain the equation

$$\begin{aligned}&(y+1)h'(xy+x+y)f(h(xy+x+y)) \\ &\qquad - (y-1)h'(xy-x-y)f(h(xy-x-y)) \\ &\quad = f(x)[1-2g(x)].\end{aligned} \tag{H$'$}$$

The subsequent lemma establishes the equivalence of equations (H) and (H$'$).

Lemma 1
The functions: $h, f, g\colon \mathbb{R} \to \mathbb{R}$; h differentiable on $\mathbb{R}$; f and g continuous on $\mathbb{R}$, satisfy equation (H) *on $\mathbb{R}^2$ if and only if they satisfy equation* (H$'$) *on $\mathbb{R}^2$.*

Proof. Clearly $(H) \Rightarrow (H')$. To get the converse implication let us rewrite equation (H′) as follows (cf. (4)):

$$\begin{aligned} &-(y+1)h'(sy+s+y)f(h(sy+s+y) \\ &\qquad +(y-1)h'(sy-s-y)f(h(sy-s-y))+f(s) \\ &= 2f(s)g(s). \end{aligned} \tag{3}$$

and integrate with respect to s their sides, LHS and RHS, over the interval $[x,y]$. After executing the substitutions: $t=h(sy+s+y)$ in the first integral of the LHS of (3) and $t=h(sy-s-y)$ in the second one we obtain

$$\int_{h(y^2+2y)}^{h(xy+x+y)} f(t)\,dt + \int_{h(y^2-2y)}^{h(xy-x-y)} f(t)\,dt + \int_x^y f(t)\,dt.$$

After adding and subtracting integrals with suitable limits of integration we find that this sum of integrals equals to the LHS of (H). The integral over $[x,y]$ of the RHS of (3) and RHS of (H) are the same expressions. Hence $(H') \Rightarrow (H)$.

In the sequel J will stand for an interval contained either in $(-\infty,-1)$ or in $(-1,1)$, or in $(1,+\infty)$. We are in position to prove the following

THEOREM 2
If the function $h\colon \mathbb{R} \to J$ is three times differentiable on $\mathbb{R}$; the function $f\colon J \to \mathbb{R}$ is twice differentiable on J; and $g\colon J \to \mathbb{R}$ is continuous in J, then equation (H) *when postulated for $(x,y) \in J^2$, is equivalent to the system of the equalities (both valid for $x \in J$)*

$$\begin{cases} (x+1)h'(x)f(h(x)) = \alpha x + \beta; \\ (x^2-1)f(x)(1-2g(x)) = 2(\alpha x^2 - \beta), \end{cases} \tag{C}$$

where α and β are arbitrary real numbers, but $\alpha^2+\beta^2>0$.

Proof. According to the Lemma it is enough to solve equation (H′). We denote, for short, by A and B the factors of the first product of functions occurring on the LHS of (H′):

$$A(x,y) := (y+1)h'(xy+x+y); \qquad B(x,y) := f(h(xy+x+y)).$$

Then the factors of the other product in LHS of (3) are equal

$$(y-1)h'(xy-x-y) = -\,A(-x,-y); \qquad f(h(xy-x-y)) = B(-x,-y).$$

Applying to A and B Maclaurin's formula (with the Peano reminder) in a neighbourhood of $y=0$ yields:

$$\begin{aligned}A(x,y) = h'(x) &+ y[h'(x) + (x+1)h''(x)] \\ &+ \frac{1}{2}y^2[2(x+1)h''(x) + (x+1)^2 h'''(x)] \\ &+ o(y^2),\end{aligned}$$

$$\begin{aligned}B(x,y) = f(h(x)) &+ y[(x+1)h'(x)f'(h(x))] \\ &+ \frac{1}{2}y^2[(x+1)^2 h''(x)f'(h(x)) + (x+1)^2(h'(x))^2 f''(h(x))] \\ &+ o(y^2),\end{aligned}$$

where the Landau symbol o refers to $y \to 0$.

Now we calculate the LHS(H$'$) $= A(x,y)B(x,y) + A(-x,-y)B(-x,-y)$, insert the formula obtained to (H$'$) and compare the free terms and the coefficients of y and y^2 of the resulting equation. This yields the following equalities:

$$h'(x)f(h(x)) + h'(-x)f(h(-x)) = f(x)[1 - 2g(x)], \tag{4}$$

$$F(x) = F(-x), \tag{5}$$

where

$$F(x) := [(x+1)h'(x)f(h(x))]' \tag{6}$$

and

$$(x+1)F'(x) = (x-1)F'(-x). \tag{7}$$

From (5) we have $F'(x) = -F'(-x)$. Eliminating $F'(-x)$ from this equation and from (7) we get $xF'(x) = 0$, $x \in J$, whence $F(x) = \alpha$ for $x \in J$. Integrating (6) we get the first equation of system (C). Inserting the resulting formula to (4) we arrive at the other equation of (C).

On the other hand, given some functions f, g, h satisfying (C) and regular as required in the theorem one checks by a direct calculation that they satisfy equation (H$'$) and, by Lemma, also equation (H). This completes the proof of the theorem.

Remark 4

The three unknown functions f, g, h, are linked by two conditions (C) only. Thus given arbitrarily one of the functions one may determine the others. For instance, if one of the functions h or f is the identity, we get the following triplets of solutions to (H):

$$\begin{cases} f(x) = \dfrac{\alpha x + \beta}{x+1}, \\ g(x) = \dfrac{(\beta - \alpha)x}{2(x-1)(\alpha x + \beta)}, \\ h(x) = x, \qquad\qquad x \in J, \alpha = 0 \text{ or } -\frac{\beta}{\alpha} \notin J; \end{cases}$$

$$\begin{cases} f(x) = x, \\ g(x) = \dfrac{1}{2} + \dfrac{\alpha x^2 - \beta}{x(1-x^2)}, \\ [h(x)]^2 = 2\alpha x + 2(\beta - \alpha)\log|x+1| + C, \qquad |x| > 1, \end{cases}$$

provided that the constants are so chosen that $h(x) \in J$. Among solutions of (H) when g is the identity there are those corresponding to $\alpha = \beta \neq 0$ in (C):

$$\begin{cases} f(x) = \dfrac{2\alpha}{1-2x}, \\ g(x) = x, \\ h(x) = C\ e^{-x} + \dfrac{1}{2}, \qquad |x| > 1, \end{cases}$$

since in this case h is a solution of the equation $2h'(x) + 2h(x) = 1$.

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Curvature properties of some submanifolds in space forms

Dedicated to Professor Dr. Andrzej Zajtz on his seventieth birthday

Abstract. Curvature properties of pseudosymmetry type of some submanifolds of codimension greater then 1 immersed isometrically in semi-Riemannian spaces of constant curvature are given.

1. Introduction

Theorem 3.1 of [20] states that if at every point x of a hypersurface M immersed isometrically in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 3$, its second fundamental tensor H has the form

$$H = \alpha v \otimes v + \beta w \otimes w, \qquad v, w \in T_x^* M, \ \alpha, \beta \in \mathbb{R}, \tag{1}$$

then on M we have

$$R \cdot R = \frac{\tilde{\kappa}}{n(n+1)} Q(g, R), \tag{2}$$

which means that M is a pseudosymmetric hypersurface. In particular, if the ambient space is a non-flat manifold then M is non-semisymmetric. Evidently, if the ambient space is a semi-Euclidean space $\mathbb{E}_s^{n+1}$ then (1) reduces to

$$R \cdot R = 0, \tag{3}$$

which means that M is a semisymmetric hypersurface. In this paper we prove, that under some additional assumptions, the mentioned above results remain also true when the codimension of a submanifold M in a semi-Riemannian space of constant curvature is greater than 1.

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2. Conditions of pseudosymmetry type

In this section we give a review on manifolds of pseudosymmetry type. We refer to [2], [17] and [33] for a survey of results related to this subject.

Let (M, g), $n = \dim M \geq 3$, be a connected semi-Riemannian manifold of class C^{∞}. We define on M the endomorphisms $\mathcal{R}(X, Y)$ and $X \wedge_A Y$ by

$$\mathcal{R}(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$
$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y,$$

respectively, where A is a symmetric $(0,2)$-tensor, ∇ is the Levi-Civita connection of (M, g) and $X, Y, Z \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M. Furthermore, we define the Riemann-Christoffel curvature tensor R and the $(0,4)$-tensor G of (M, g) by

$$R(X_1, X_2, X_3, X_4) = g(\mathcal{R}(X_1, X_2)X_3, X_4),$$
$$G(X_1, X_2, X_3, X_4) = g((X_1 \wedge_g X_2)X_3, X_4),$$

respectively. We denote by S and κ the Ricci tensor and the scalar curvature of (M, g), respectively. For a $(0, k)$-tensor field T on M, $k \geq 1$ and a symmetric $(0,2)$-tensor A we define the $(0, k+2)$-tensors $R \cdot T$ and $Q(A, T)$ by

$$\begin{aligned}
(R \cdot T)(X_1, \ldots, X_k; X, Y) &= (\mathcal{R}(X,Y) \cdot T)(X_1, \ldots, X_k)\\
&= -T(\mathcal{R}(X,Y)X_1, X_2, \ldots, X_k)\\
&\quad - \ldots - T(X_1, \ldots, X_{k-1}, \mathcal{R}(X,Y)X_k),\\
Q(g,T)(X_1, \ldots, X_k; X, Y) &= ((X \wedge_A Y) \cdot T)(X_1, \ldots, X_k)\\
&= -T((X \wedge_A Y)X_1, X_2, \ldots, X_k)\\
&\quad - \ldots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k).
\end{aligned}$$

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be a *pseudosymmetric manifold* ([17, Section 3.1], [33]) if at every point of M the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent. Thus the manifold (M, g) is pseudosymmetric if and only if

$$R \cdot R = L_R\, Q(g, R) \tag{4}$$

on $U_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)} G \neq 0 \text{ at } x\}$, where L_R is some function on U_R. It is clear that every *semisymmetric manifold* ($R \cdot R = 0$, [32]) is pseudosymmetric. The condition (4) arose during the study on totally umbilical submanifolds of semisymmetric manifolds as well as when considering geodesic mappings of semisymmetric manifolds ([17, Sections 10 and 13], [33]). There exist pseudosymmetric manifolds which are non-semisymmetric. For instance, in [18] (see Example 3.1 and Theorem 4.1) it was shown that the warped product $S^p \times_F S^{n-p}$, $p \geq 2$, $n - p \geq 1$, of the standard spheres S^p and S^{n-p}, with some function F, is pseudosymmetric.

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be a *Ricci-pseudosymmetric manifold* ([17, Section 3.4]) if at every point of M the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent. Thus the manifold (M, g) is Ricci-pseudosymmetric if and only if

$$R \cdot S = L_S\, Q(g, S) \tag{5}$$

on $U_S = \{x \in M \mid\ S - \frac{\kappa}{n}\, g \neq 0 \text{ at } x\}$, where L_S is some function on U_S. It is clear that if (4) is satisfied at a point x of a manifold (M, g) then also (5) holds at x. The converse statement is not true. E.g. every warped product $M_1 \times_F M_2$, $\dim M_1 = 1$, $\dim M_2 = n - 1 \geq 3$, of a manifold $(M_1, \overline{g})$ and a non-pseudosymmetric Einstein manifold $(M_2, \tilde{g})$ is a non-pseudosymmetric, Ricci-pseudosymmetric manifold. It is also known that the Cartan hypersurfaces of dimensions 6, 12 or 24 are non-pseudosymmetric Ricci-pseudosymmetric manifolds ([24]).

For any $X, Y \in \Xi(M)$ we define the endomorphism $\mathcal{C}(X, Y)$ by

$$\mathcal{C}(X, Y) \ = \ \mathcal{R}(X, Y) - \frac{1}{n-2} \left(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y \right).$$

The Ricci operator $\mathcal{S}$ and the Weyl conformal curvature tensor C of (M, g) are defined by

$$g(\mathcal{S}X, Y) = S(X, Y),$$
$$C(X_1, X_2, X_3, X_4) = g(\mathcal{C}(X_1, X_2)X_3, X_4),$$

respectively. Now we define the $(0, 6)$-tensor $C \cdot C$ by

$$\begin{aligned}(C \cdot C)(X_1, X_2, X_3, X_4; X, Y) &= (\mathcal{C}(X, Y) \cdot C)(X_1, X_2, X_3, X_4)\\ &= -C(\mathcal{C}(X, Y)X_1, X_2, X_3, X_4)\\ &\qquad - \ldots - C(X_1, X_2, X_3, \mathcal{C}(X, Y)X_4).\end{aligned}$$

A semi-Riemannian manifold (M, g), $n \geq 4$, is said to be a *manifold with pseudosymmetric Weyl tensor* ([17, Section 12.6]) if at every point of M the tensors $C \cdot C$ and $Q(g, C)$ are linearly dependent. Thus the manifold (M, g) is a manifold with pseudosymmetric Weyl tensor if and only if

$$C \cdot C = L_C\, Q(g, C) \tag{6}$$

on $U_C = \{x \in M \mid\ C \neq 0 \text{ at } x\}$, where L_C is some function on U_C. It is known that (6) is fulfilled at every point of the warped product $M_1 \times_F M_2$,

$\dim M_1 = \dim M_2 = 2$ ([15], Theorem 2). An example of a 4-dimensional Riemannian manifold satisfying (6), which is not a warped product, was found in [25].

A semi-Riemannian manifold (M, g), $n \geq 4$, is said to be a *Weyl-pseudosymmetric manifold* if then at every point of M the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent. Thus the manifold (M, g) is a Weyl-pseudosymmetric manifold if and only if

$$R \cdot C = L\, Q(g, C) \tag{7}$$

on U_C, where L is some function on U_C. Every pseudosymmetric manifold is Weyl-pseudosymmetric. The converse statement is not true ([13]). Evidently, any *Weyl-semisymmetric manifold* ($R \cdot C = 0$) is Weyl-pseudosymmetric. We refer to [1] for a review of results on Weyl-pseudosymmetric manifolds.

It is easy to see that at every point of a pseudosymmetric Einstein manifold the tensors $R \cdot R - Q(S, R)$ and $Q(g, C)$ are linearly dependent. We also mention that any 3-dimensional semi-Riemannian manifold fulfils ([14], Theorem 3.1)

$$R \cdot R = Q(S, R). \tag{8}$$

Moreover, every hypersurface M immersed isometrically in an $(n+1)$-dimensional semi-Euclidean space $\mathbb{E}^{n+1}_s$ with signature $(n+1-s, s)$, $n \geq 3$, satisfies (8) ([21], Corollary 3.1). A review of results on manifolds satisfying (8) is given in Section 5 of [17].

Semi-Riemannian manifolds fulfilling the above presented conditions or other conditions of this kind are called *manifolds of pseudosymmetry type* ([17], [33]). Recently, a review of results on pseudosymmetry type manifolds was given in [2].

Further, for a symmetric $(0, 2)$-tensor fileds A and B on M we define their Kulkarni-Nomizu product $A \wedge B$ by

$$\begin{aligned}(A \wedge B)(X_1, X_2, X_3, X_4) = {} & A(X_1, X_4)B(X_2, X_3) + A(X_2, X_3)B(X_1, X_4) \\ & - A(X_1, X_3)B(X_2, X_4) - A(X_2, X_4)B(X_1, X_3).\end{aligned}$$

Further, for a symmetric $(0, 2)$-tensor field A on M we define the endomorphism $\mathcal{A}$ of $\Xi(M)$ and the $(0, 2)$-tensors A^2 and A^3 by

$$\left.\begin{aligned} g(\mathcal{A}X, Y) &= A(X, Y), \\ A^2(X, Y) &= A(\mathcal{A}X, Y), \\ A^3(X, Y) &= A^2(\mathcal{A}X, Y), \end{aligned}\right\} \tag{9}$$

respectively. We end this section with the following statement.

Lemma 2.1 ([20])
Let at a point x of a semi-Riemannian manifold (M,g), $n \geq 3$, be given a $(0,2)$-tensor A having the form

$$A = \alpha v \otimes v + \beta w \otimes w, \qquad v, w \in T_x^*M,\ \alpha, \beta \in \mathbb{R}. \tag{10}$$

Then the following relations are fulfilled at x

$$Q(A^2, A \wedge A) = 0, \tag{11}$$

$$A^3 = tr(A)A^2 + \lambda A, \quad \lambda = \alpha\beta((g(V,W))^2 - g(V,V)g(W,W)), \tag{12}$$

where the vectors $V, W \in T_xM$ are related to the covectors v, w by $v(X) = g(V,X)$ and $w(X) = g(W,X)$, respectively, and $X \in T_xM$.

3. Quasi-umbilical hypersurfaces

Let M be a connected submanifold immersed isometrically in a semi-Riemannian manifold $(N, \tilde{g})$, $3 \leq n = \dim M < n + k = \dim N$, $k \geq 1$. We denote by g the metric tensor induced on M from the metric tensor $\tilde{g}$. We denote by $\tilde{\nabla}$ and ∇ the Levi-Civita connections correspondig to the metric tensors $\tilde{g}$ and g, respectively. The *Gauss formula* of M in N is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{13}$$

where h is the second fundamental form of M in N and X, Y are vector fields tangent to M. Further, for any vector field ξ normal to M and for any vector field X tangent to M we have the *Weingarten formula* of M in N

$$\tilde{\nabla}_X \xi = -\mathcal{A}_\xi X + D_X \xi\,, \tag{14}$$

where D denotes the normal connection induced in the normal bundle $N(M)$ of M in N and $\mathcal{A}$, defined by $\mathcal{A}(\xi, X) = \mathcal{A}_\xi X$, is the *Weingarten map* (the *shape operator*) of M in N. We have

$$g(\mathcal{A}_\xi X, Y) \ = \ \tilde{g}(h(X,Y), \xi). \tag{15}$$

A submanifold M in a semi-Riemannian manifold $(N, \tilde{g})$ is said to be *quasi-umbilical* with respect to the normal direction ξ at a point $x \in M$ (cf. [21], [22]) if at x its second fundamental tensor H_ξ satisfies the equality

$$H_\xi = \alpha_\xi\, g + \beta_\xi\, v_\xi \otimes v_\xi\,, \qquad v_\xi \in T_x^*M,\ \alpha_\xi, \beta_\xi \in \mathbb{R}. \tag{16}$$

If $\alpha_\xi = 0$ (resp., $\beta_\xi = 0$ or $\alpha_\xi = \beta_\xi = 0$) holds at x then it is called *cylindrical* (resp., *umbilical* or *geodesic*) w.r.t. ξ at p. If (16) is fulfilled at every point of M then M is called a *quasi-umbilical hypersurface* w.r.t. ξ. Let now M be a submanifold immersed isometrically in a Riemannian manifold $(N,\tilde{g})$ and let ξ be a local unit normal vector field on M in N. In this case we can prove that the above notion of quasi-umbilicity equivalent to the following definitions ([5], [8], [9]): The submanifold M immersed isometrically in a Riemannian manifold $(N,\tilde{g})$ is said to be *quasi-umbilical* w.r.t. ξ at a point $x \in M$ when it has a principal curvature with multiplicity $\geq n-1$, i.e. when the principal curvatures of M at x w.r.t. ξ are given by μ_ξ, $\lambda_\xi, \ldots, \lambda_\xi$, where λ_ξ occurs $(n-1)$-times. In particular, when $\mu_\xi = \lambda_\xi$ (resp., $\mu_\xi = \lambda_\xi = 0$), then M is *umbilical* (resp., *geodesic*) at x w.r.t. ξ. If we have μ_ξ, $0, \ldots, 0$, where 0 occurs $(n-1)$-times then M is *cylindrical* at x w.r.t. ξ.

The following statement gives a curvature characterization of quasi-umbilical hypersurfaces in Euclidean spaces.

Theorem 3.1 ([3])
A hypersurface M immersed isometrically in a Euclidean space $\mathbb{E}^{n+1}$, $n \geq 4$, is quasi-umbilical if and only if it is conformally flat.

By the invariance of the multiplicity of principal curvatures of submanifolds under conformal changes of the metric of the ambient space we obtain the following result.

Theorem 3.2 ([31])
A hypersurface M, $n \geq 4$, immersed isometrically in a Riemannian conformally flat manifold is quasi-umbilical if and only if it is conformally flat.

The assertion of Theorem 3.2 is not true when $n = 3$. Namely, there exist conformally flat hypersurfaces in $\mathbb{E}^4$, which are not quasi-umbilical, i.e. hypersurfaces with three distinct principal curvatures ([26]). A generalization of Theorem 3.2, for the case when the ambient space is a semi-Riemannian manifold, was given in [21].

Theorem 3.3 ([21], Theorem 4.1)
A hypersurface M, $n \geq 4$, immersed isometrically in a semi-Riemannian conformally flat manifold is quasi-umbilical if and only if it is conformally flat.

A submanifold M immersed isometrically in a semi-Riemannian manifold $(N,\tilde{g})$ is said to be *2-quasi-umbilical* w.r.t. ξ at a point $x \in M$ (cf. [22], [23]) if at x the second fundamental tensor H_ξ of M satisfies the equality

$$H_\xi = \alpha_\xi g + \beta_\xi v_\xi \otimes v_\xi + \gamma_\xi w_\xi \otimes w_\xi\,, \quad v_\xi, w_\xi \in T^*_x M,\ \alpha_\xi, \beta_\xi, \gamma_\xi \in \mathbb{R}, \tag{17}$$

where $U_\xi, V_\xi \in T_x M$, $g(U_\xi, V_\xi) = 0$, $u_\xi(X) = g(U_\xi, X)$, $v_\xi(X) = g(V_\xi, X)$ for any vector $X \in T_x M$. If (17) is fulfilled at every point of M then it is called a *2-quasi-umbilical submanifold* w.r.t. ξ. It is clear that if the ambient space $(N, \tilde{g})$ is a Riemannian manifold then the above definition of a 2-quasi-umbilical submanifold M w.r.t. ξ at a point x is equivalent to the following definition (cf. [6]): The submanifold M, $n \geq 4$, immersed isometrically in a Riemannian manifold $(N, \tilde{g})$ is said to be *2-quasi-umbilical* w.r.t. ξ at a point $x \in M$ when it has a principal curvature w.r.t. ξ with multiplicity $\geq n-2$, i.e. when the principal curvatures of M at x w.r.t. ξ are given by μ_ξ, ν_ξ, λ_ξ, …, λ_ξ, where λ_ξ occurs $(n-2)$-times. Hypersurfaces with pseudosymmetric Weyl tensor immersed isometrically in Euclidean spaces were considered in [12]. The main result of [12] is the following

THEOREM 3.4 ([12], Theorem 1)
A hypersurface M immersed isometrically in a Euclidean space $\mathbb{E}^{n+1}$, $n \geq 4$, is a manifold with pseudosymmetric Weyl tensor if and only if at every point of the set $U_C \subset M$, M has at most three distinct principal curvatures. Moreover, if x is a point of U_C, at which M has exactly three distinct principal curvatures, then their multiplicities are the following: $1, 1, n-2$*, i.e., M is 2-quasi-umbilical at x.*

Examples of hypersurfaces in $\mathbb{E}^{n+1}$, $n \geq 4$, with pseudosymmetric Weyl tensor are also given in [12]. A review of results on hypersurfaces satisfying (6) is given in [23].

4. Quasiumbilical submanifolds of codimension two

Let M be a submanifold immersed isometrically in a semi-Riemannian manifold N, $n = \dim M \geq k = \operatorname{codim} M$. Let $\xi_1, \ldots, \xi_k$ be mutually orthogonal units normal local vector fields on M and let $\tilde{g}(\xi_y, \xi_y) = e_y$, $e_y = \pm 1$, $x, y, z = 1, \ldots, k$. From (15) we get

$$h(X,Y) = \sum_y H_y(X,Y)\, \xi_y\,. \tag{18}$$

The scalar valued form H_y is called the second fundamental tensor with respect to the normal section ξ_y. We denote by R and $\tilde{R}$ the Riemann-Christoffel curvature tensors of M and N, respectively. The *Gauss equation* of M in N has the following form

$$\begin{aligned} R(X_1, \ldots, X_4) = \tilde{g}(h(&X_1, X_4), h(X_2, X_3)) \\ &- \tilde{g}(h(X_1, X_3), h(X_2, X_4)) \\ &+ \tilde{R}(X_1, \ldots, X_4)\,, \end{aligned} \tag{19}$$

where $X_1, \ldots, X_4$ are vector fields tangent to M.

The submanifold M in a semi-Riemannian manifold N, $n = \dim M \geq k = \operatorname{codim} M$, is said to be *quasi-umbilical* if at every point $x \in M$ there exist mutually orthogonal units normal vector fields $\xi_1, \ldots, \xi_k$, defined on a neighbourhood $\mathcal{U}$ of x such that on $\mathcal{U}$ we have

$$H_y = \alpha_y g + \beta_y \bar{u}_y \otimes \bar{u}_y \, , \tag{20}$$

where α_y and β_y are some functions and u_y is some 1-forms on $\mathcal{U}$, respectively, $x = 1, \ldots, k$, and the vector fields U_y related with $\bar{u}_y$ by $\bar{u}_y(X) = g(U_y, X)$, $X \in T_x\mathcal{U}$, satisfy

$$g(U_y, U_z) = 0, \quad y \neq z, \qquad \text{and} \qquad g(U_y, U_y) = \bar{e}_y, \quad \bar{e}_y = \pm 1. \tag{21}$$

Quasi-umbilical submanifolds were studied among others in: [6]-[9], [27]-[30] and [34].

From now we will assume that the ambient space $(N, \tilde{g})$ is a semi-Riemannian space of constant curvature $N_s^{n+k}(c)$ with signature $(n+k-s, s)$, $n \geq 4$. The Gauss equation (22) of M in $N_s^{n+k}(c)$ reads

$$\begin{aligned} R(X_1, \ldots, X_4) &= \tilde{g}(h(X_1, X_4), h(X_2, X_3)) - \tilde{g}(h(X_1, X_3), h(X_2, X_4)) \\ &\quad + \frac{\tilde{\kappa}}{(n+k-1)(n+k)} G(X_1, \ldots, X_4), \end{aligned} \tag{22}$$

where $\tilde{\kappa}$ denotes the scalar curvature of the ambient space. Further, if M is quasi-umbilical with respect to $\xi_1, \ldots, \xi_k$, then (22) turns into

$$R(X_1, \ldots, X_4) = (g \wedge u)(X_1, \ldots, X_4) + \eta \, G(X_1, \ldots, X_4), \tag{23}$$

where

$$\left.\begin{aligned} \eta &= \frac{\tilde{\kappa}}{(n+k-1)(n+k)} + \sum_{y=1}^{k} \bar{e}_y \alpha_y^2 \\ \text{and} \quad & \\ u(Y, Z) &= \sum_{y=1}^{k} \bar{e}_y \alpha_y \beta u_y(Y) u_y(Z). \end{aligned}\right\} \tag{24}$$

Using (23) we can present the Ricci tensor S of (M, g) in the form

$$\begin{aligned} S(X_1, X_4) &= \rho \, g(X_1, X_4) + (n-2) \, u(X_1, X_4), \\ \rho &= (n-1)\eta + tr_g u. \end{aligned} \tag{25}$$

We note that form (23), by an application of (25), it follows that the Weyl curvature tensor C of (M, g) vanishes identically on M (cf. [5]). From (25) we get easily

$$S(U_y \, , Z) = (\rho + (n-2) e_y \bar{e}_y \alpha_y \beta_y) g(U_y \, , Y), \qquad y = 1, \ldots, k. \tag{26}$$

We denote by $\mathcal{S}$ the Ricci operator of S. Now (26) turns into

$$\mathcal{S}U_y = \tilde{\tau}_y U_y\,, \qquad \tilde{\tau}_y = \rho + (n-2)\tau_y\,,\ \tau_y = e_y \bar{e}_y \alpha_y \beta_y\,,\ y = 1, \ldots, k. \quad (27)$$

We have the following generalizations of Theorem 3.1 for the case when codimension of a submanifold is ≥ 1.

Remark 4.1

(i) Let M be a n-dimensional submanifold in $\mathbb{E}^{n+k}$, $n \geq 4$.

  (a) ([7]) The submanifold M, with a flat normal connection and such that $1 \leq k \leq n-3$, is quasi-umbilical if and only if it is conformally flat.

  (b) ([28]) The submanifold M, such that $1 \leq k \leq \inf(4, n-3)$, is quasi-umbilical if and only if it is conformally flat.

(ii) An example of a non quasi-umbilical conformally flat submanifold of codimension 2 in a Euclidean space $\mathbb{E}^6$ is given in [34] (Chapter 5, p. 100).

Let M, $n = \dim M \geq 3$, be quasi-umbilical submanifold, with respect to the normal sections $\xi_1, \ldots, \xi_k$, in a Riemannian space of constant curvature $N^{n+k}(c)$, $k \geq 2$. From (27) we have $\mathcal{S}U_y = (\rho + (n-2)\alpha_y\beta_y)U_y$, $y = 1, \ldots, k$. Further, we note that if V is a vector such that $g(U_y, V) = 0$, then from (27) we have $\mathcal{S}V = \rho V$. Thus we see that ρ, $\rho + (n-2)\,\alpha_1\,\beta_1, \ldots, \rho + (n-2)\,\alpha_k\,\beta_k$, are eigenvalues of the Ricci operator $\mathcal{S}$ of M. In [11] (Theorem 1.3) it was shown that a conformally flat Riemannian manifold (M, g) is pseudosymmetric if and only if at every point of M its Ricci operator $\mathcal{S}$ has at most two distinct eigenvalues. Thus we have

Theorem 4.1

Let M, $n \geq 3$, be quasi-umbilical submanifold, with respect to the normal sections $\xi_1, \ldots, \xi_p$, in a Riemannian space of constant curvature $N^{n+k}(c)$, $k \geq 2$. Then M is pseudosymmetric if and only if at every point of M the Ricci operator $\mathcal{S}$ of M has at most two distinct eigenvalues ρ_1, ρ_2, i.e., the set $\{\rho, \rho + (n-2)\alpha_1\beta_1, \ldots, \rho + (n-2)\alpha_k\beta_k\}$ has at most two distinct numbers.

From the last theorem it follows

Corollary 4.1

Let M, $n \geq 3$, be a quasi-umbilical submanifold, with respect to the normal sections ξ_1, ξ_2, in a Riemannian space of constant curvature $N^{n+2}(c)$. Then M is pseudosymmetric if and only if at every point $x \in M$ we have: M is umbilical or cylindrical with respect to ξ_1 or ξ_2 at x or M is non-umbilical and non-cylindrical quasi-umbilical with respect to ξ_1 or ξ_2 at x and $\alpha_1\beta_2 = \alpha_2\beta_2$.

Let M be a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 3$, and let ξ_1 be the normal sections of M in $N_s^{n+1}(c)$. Now the Gauss equation (22) reads

$$R - \frac{\tilde{\kappa}}{n(n+1)}\, G = \frac{\varepsilon}{2}\, H \wedge H, \tag{28}$$

where H is the second fundamental tensor of M in $N_s^{n+1}(c)$. From (28) we get immediately

$$S - \frac{(n-1)\tilde{\kappa}}{n(n+1)}\, g = \varepsilon\, (tr\,(H)H - H^2). \tag{29}$$

Further, applying Lemma 2.1 of [21] into (28) we obtain

$$\begin{aligned} &\left(R - \frac{\tilde{\kappa}}{n(n+1)}\, G\right) \cdot \left(R - \frac{\tilde{\kappa}}{n(n+1)}\, G\right) \\ &\quad = Q\left(S - \frac{(n-1)\kappa}{n(n+1)}\, g, R - \frac{\tilde{\kappa}}{n(n+1)}\, G\right), \end{aligned} \tag{30}$$

whence by making use of (28) and (29) we obtain

$$\left(R - \frac{\tilde{\kappa}}{n(n+1)} G\right) \cdot R = -\frac{1}{2}\, Q(H^2, H \wedge H). \tag{31}$$

Using Lemma 2.1 and (31) we can prove

THEOREM 4.2 ([20], Theorem 3.1)
Let M be a hypersurface immersed isometrically in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 3$. If at every point $x \in M$ its second fundamental tensor H has the form

$$H = \alpha v \otimes v + \beta w \otimes w, \qquad v, w \in T_x^*M,\ \alpha, \beta \in \mathbb{R}, \tag{32}$$

then the following relation is satisfied on M

$$R \cdot R = \frac{\tilde{\kappa}}{n(n+1)}\, Q(g, R). \tag{33}$$

In particular, from this it follows that if at every point of a hypersurface M in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, M has three distinct principal curvatures $\lambda, \mu, 0, \ldots, 0$, where 0 occurs $(n-2)$-times, then M is a pseudosymmetric manifold.

We give now an extension of the last theorem on the case of the codimension greater then 1. Let M, $n \geq 3$, be a submanifold in a Riemannian space of constant curvature $N^{n+k}(c)$, $k > 1$, and let $\xi_1, \ldots, \xi_k$ be the normal sections of M in $N_s^{n+k}(c)$. The Gauss equation (22) of M reads

$$R_{sijk} = \sum_x \varepsilon_x (H_{x\,sk} H_{x\,ij} - H_{x\,sj} H_{x\,ik}) + \frac{\tilde{\kappa}}{(n+k-1)(n+k)}\, G_{hijk}\,. \tag{34}$$

Transvecting this with $R^s{}_{hef}$ and using (34) we get

$$\begin{aligned} R_{sijk} R^s{}_{hef} = & \sum_x \big(- H_{x\,ik}(H_{x\,he} H^2_{x\,jf} - H_{x\,hf} H^2_{x\,je}) \\ & \qquad + H_{x\,ij}(H_{x\,he} H^2_{x\,kf} - H_{x\,hf} H^2_{x\,ke})\big) \\ & + \sum_{x\neq y} \big(- \varepsilon_x \varepsilon_y\, H_{x\,ik}(H_{y\,he} H_{xy\,jf} - H_{y\,hf} H_{xy\,je}) \\ & \qquad + \varepsilon_x \varepsilon_y\, H_{x\,ij}(H_{y\,he} H_{xy\,kf} - H_{y\,hf} H_{xy\,ke})\big) \\ & + \sum_{x\neq y} \big(- \varepsilon_x \varepsilon_y\, H_{y\,ik}(H_{x\,he} H_{yx\,jf} - H_{x\,hf} H_{yx\,je}) \\ & \qquad + \varepsilon_x \varepsilon_y\, H_{y\,ij}(H_{y\,he} H_{yx\,kf} - H_{y\,hf} H_{yx\,ke})\big) \\ & + \frac{\tilde{\kappa}}{(n+k-1)(n+k)} (g_{ij} R_{khef} - g_{ik} R_{jhef} \\ & + g_{he} R_{fijk} - g_{hf} R_{eijk}), \end{aligned} \tag{35}$$

where

$$H_{xy\,ij} = H_{x\,is} g^{sr} H_{y\,jr}\,. \tag{36}$$

Applying now (35) to the idendity

$$\begin{aligned} & (R\cdot R)_{hijklm} \\ & \quad = R_{sijk} R^s{}_{hef} - R_{shjk} R^s{}_{ief} + R_{skhi} R^s{}_{jef} - R_{sjhi} R^s{}_{kef}\,, \end{aligned} \tag{37}$$

and using the definition of the tensor $Q(A,T)$, where A is a symmetric $(0,2)$-tensor and T a generalized curvature tensor, we find

$$\begin{aligned} R\cdot R = & \frac{\tilde{\kappa}}{(n+k-1)(n+k)} Q(g,R) - \frac{1}{2} \sum_x Q(H_x^2, H_x \wedge H_x) \\ & - \sum_{x\neq y} \varepsilon_x \varepsilon_y\, Q(H_{xy} + H_{yx}, H_x \wedge H_y). \end{aligned} \tag{38}$$

Theorem 4.3
Let M, $n = \dim M \geq 3$, be a submanifold in a Riemannian space of constant curvature $N^{n+k}(c)$, $k \geq 2$, and let ξ_z, $z = 1, \ldots, k$, be the normal sections of M in $N^{n+k}(c)$. If at every point $x \in M$ the second fundamental tensors H_z of M satisfy the following relations:

$$H_z = \alpha_z u_z \otimes u_z + \beta_z w_z \otimes w_z\,, \qquad u_z, w_z \in T_x^* M,\ \alpha_z, \beta_z, \in \mathbb{R}, \tag{39}$$

and the subspaces $\mathrm{lin}\,\{U_x, W_x\}$ *and* $\mathrm{lin}\,\{U_y, W_y\}$, *for* $x \neq y$, *are mutually orthogonal then*

$$R \cdot R = \frac{\tilde{\kappa}}{(n+k-1)(n+k)}\, Q(g, R), \tag{40}$$

where the vectors $U_x, W_x \in T_xM$ *are related to the covectors* u_x, w_x *by* $u_z(X) = g(U_z, X)$, $w_z(X) = g(W_z, X)$ *and* $X \in T_xM$.

Proof. From Lemma 2.1 it follows immediately that $Q(H_z^2, H_z \wedge H_z) = 0$. Further, we have

$$\begin{aligned}(H_{xy} + H_{yx})_{ij} = {} & \alpha_x\alpha_y g(U_x, U_y)\,(u_{x\,i}u_{y\,j} + u_{x\,i}u_{y\,j}) \\ & + \beta_x\alpha_y g(W_x, U_y)\,(w_{x\,i}u_{y\,j} + w_{x\,j}u_{y\,i}) \\ & + \alpha_x\beta_y g(U_x, W_y)\,(w_{x\,i}u_{y\,j} + w_{x\,j}u_{y\,i}) \\ & + \beta_x\beta_y g(W_x, W_y)\,(w_{x\,i}w_{y\,j} + w_{x\,i}w_{y\,j}),\end{aligned} \tag{41}$$

where $u_{z\,j}$, $w_{z\,j}$ are the local components of the covectors u_z and w_z . By our assumptions, (41) reduces to $H_{xy} + H_{yx} = 0$, whence $Q(H_{xy} + H_{yx}, H_x \wedge H_y) = 0$. Now (38) turns into (40), completing the proof.

From the above theorem it follows immediately the following

Theorem 4.4
Let M, $n = \dim M \geq 3$, *be a submanifold in a semi-Euclidean space* $\mathbb{E}_s^{n+k}$ *and let* ξ_z, $z = 1, \ldots, k$, *be the normal sections of* M *in* $\mathbb{E}_s^{n+k}$. *If at every point* $x \in M$ *the second fundamental tensors* H_z *of* M *satisfy* (39) *and for any* $x \neq y$ *the subspaces* $\mathrm{lin}\,\{U_x, W_x\}$ *and* $\mathrm{lin}\,\{U_y, W_y\}$ *are mutually orthogonal then* M *is a semisymmetric manifold.*

Example 4.1 (cf. [34], Chapter VII, Theorem 1)
First of all we note that the product manifold of k, $k \geq 2$, semisymmetric manifolds is also a semisymmetric manifold. Let now M_a, $\dim M_a = n_a$, be a hypersurface of rank 2 immersed isometrically in a Euclidean space $\mathbb{E}^{n_a}$, $a = 1, \ldots, k$. Such hypersurface is a semisymmetric manifold (cf. Theorem 4.2). By an standard construction, the Cartesian product manifold $M_1 \times \ldots \times M_k$ of the manifolds $M_1, \ldots, M_k$ is a semisymmetric submanifold in a Euclidean space $\mathbb{E}^{n_1+n_2+k}$ such that (39) is satisfied and for any $x \neq y$ $\mathrm{lin}\,\{U_x, W_x\}$ and $\mathrm{lin}\,\{U_y, W_y\}$ are orthogonal.

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Underlying functors on fibered manifolds

To Andrzej Zajtz, on the occasion of his 70th birthday

Abstract. For a product preserving bundle functor on the category of fibered manifolds we describe subordinated functors and we introduce the concept of the underlying functor. We also show that there is an affine bundle structure on product preserving functors on fibered manifolds.

Introduction

Let $\mathcal{M}f$ be the category of all manifolds and all smooth maps, $\mathcal{FM}$ be the category of fibered manifolds and all fiber preserving maps and $\mathcal{FM}_m$ be the category of fibered manifolds over m-dimensional bases and fibered manifold morphisms with local diffeomorphisms as base maps. It is well known that the product preserving bundle functors on $\mathcal{M}f$ coincide with Weil functors and their natural transformations are in bijection with the algebra homomorphisms, [6]. In particular, for every product preserving bundle functor F on $\mathcal{M}f$ there exists a Weil algebra A such that F is a Weil functor of the form $F = T^A$. Further, W.M. Mikulski [9] has clarified that all product preserving bundle functors on $\mathcal{FM}$ are of the form T^μ, where $\mu\colon A \to B$ is a homomorphism of Weil algebras. Finally, I. Kolář and W.M. Mikulski have characterized all fiber product preserving functors on $\mathcal{FM}_m$ in terms of Weil algebras, [7].

Recently it has been also pointed out that one can introduce an affine bundle structure on product preserving bundles. The first general result from this field is the paper [4] by I. Kolář, who described the affine structure on product preserving bundles on $\mathcal{M}f$. In particular, he introduced the underlying k-th order Weil functor T^{A_k} for every r-th order Weil functor T^A and proved that $T^A M \to T^{A_{r-1}} M$ is an affine bundle. Further, in [2] we introduced the general concept of a subordinated functor and we showed that there is an affine structure on the fiber product preserving functors on $\mathcal{FM}_m$.

The aim of this paper is to define underlying functors for every product preserving bundle functor on $\mathcal{FM}$ and to describe affine bundle structure on

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such functors. In Section 1 we recall some properties of product preserving functors on $\mathcal{FM}$ and we describe natural transformations between such functors. In Section 2 we study subordinated functors on $\mathcal{FM}$ and in Section 3 we introduce the concept of an underlying functor on $\mathcal{FM}$. Finally, in Section 4 we show that there is an affine bundle structure on product preserving functors on $\mathcal{FM}$.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from the book [6].

1. Product preserving bundle functors on fibered manifolds

First we recall the result by W.M. Mikulski, who has characterized all product preserving bundle functors defined on the category $\mathcal{FM}$ of fibered manifolds in terms of Weil algebras, [9]. A product preserving bundle functor F on $\mathcal{FM}$ determines a homomorphism of Weil algebras $\mu\colon A \to B$ in the following way. Denote by $i, j\colon \mathcal{M}f \to \mathcal{FM}$ two canonical bundle functors defined by $i(M) = \mathrm{id}_M\colon M \to M$, $i(f) = (f, f)$, $j(M) = \mathrm{pt}_M\colon M \to \mathrm{pt}$, $j(f) = (f, \mathrm{id}_{\mathrm{pt}})$, where pt denote one element manifold. Then $t_M = (\mathrm{id}_M, \mathrm{pt}_M)\colon i(M) \to j(M)$ is the identity natural transformation. Applying functor F, we obtain two product preserving functors $F \circ i$, $F \circ j$ on $\mathcal{M}f$ and a natural transformation $F \circ t\colon F \circ i \to F \circ j$. By the theory of product preserving bundle functors [6], there exist two Weil algebras A and B such that $F \circ i$ and $F \circ j$ are Weil functors of the form $F \circ i = T^A$, $F \circ j = T^B$ and we have a Weil algebra homomorphism $\mu\colon A \to B$ such that $F \circ t = \mu$.

On the other hand, consider an arbitrary homomorphism of Weil algebras $\mu\colon A \to B$. Then μ induces a bundle functor T^μ on $\mathcal{FM}$ in the following way. First, μ determines two bundle functors T^A and T^B on $\mathcal{M}f$ and a natural transformation (denoted by the same symbol) $\mu\colon T^A \to T^B$. If $p\colon Y \to M$ is a fibered manifold, then $T^B p\colon T^B Y \to T^B M$ and we can construct the induced bundle $T^\mu Y$ as the pull back $T^\mu Y = \mu_M^* T^B Y$ with respect to $\mu_M\colon T^A M \to T^B M$. In other words,

$$\begin{aligned} T^\mu Y &= T^A M \times_{T^B M} T^B Y \\ &= \{(U, V) \in T^A M \times T^B(Y);\ \mu_M(U) = T^B p(V)\}. \end{aligned} \tag{1}$$

Given a fibered manifold morphism $f\colon Y \to \overline{Y}$ over a base map $\underline{f}\colon M \to \overline{M}$, we have $T^B f\colon T^B Y \to T^B \overline{Y}$ and we can construct the induced map

$$T^\mu f = T^A \underline{f} \times_{T^B \underline{f}} T^B f\colon T^\mu Y \to T^\mu \overline{Y}.$$

This defines a product preserving bundle functor T^μ on $\mathcal{FM}$. W.M. Mikulski has deduced that every product preserving bundle functor F on $\mathcal{FM}$ is naturally equivalent to T^μ for some Weil algebra homomorphism $\mu\colon A \to B$, [9] (see also [3] for a simplified proof).

Consider two algebra homomorphisms $\mu\colon A \to B$, $\nu\colon C \to D$ and two bundle functors T^μ and T^ν on $\mathcal{FM}$. W.M. Mikulski has also proved that natural transformations $T^\mu \to T^\nu$ are in bijection with couples of algebra homomorphisms $f_1\colon A \to C$, $f_2\colon B \to D$ such that

$$\nu \circ f_1 = f_2 \circ \mu. \tag{2}$$

The algebra homomorphisms f_1 and f_2 induce natural transformations (denoted by the same symbols) $f_1\colon T^A \to T^C$ and $f_2\colon T^B \to T^D$. Then we can construct a map

$$(f_{1M}, f_{2Y})\colon T^A M \times T^B Y \to T^C M \times T^D Y.$$

We have

PROPOSITION 1
Natural transformations $T^\mu \to T^\nu$ are of the form (f_{1M}, f_{2Y}).

Proof. Consider an element $(U, V) \in T^\mu Y \subset T^A M \times T^B Y$. Then

$$(f_{1M}(U), f_{2Y}(V)) \in T^C M \times T^D Y \supset T^\nu Y.$$

By (2), $\nu_M(f_{1M}(U)) = f_{2M}(\mu_M(U))$. Further, the following diagram commutes

$$\begin{array}{ccc} T^B Y & \xrightarrow{\ f_{2Y}\ } & T^D Y \\ {\scriptstyle T^B p}\big\downarrow & & \big\downarrow{\scriptstyle T^D p} \\ T^B M & \xrightarrow{\ f_{2M}\ } & T^D M \end{array}$$

Thus, we have $\nu_M(f_{1M}(U)) = f_{2M}(\mu_M(U)) = f_{2M}(T^B p(V)) = T^D p(f_{2Y}(V))$, which yields $(f_{1M}(U), f_{2Y}(V)) \in T^\nu(Y)$.

By [6], the order of a bundle functor on $\mathcal{FM}$ is determined by three numbers (q, s, r) and is based on the concept of (q, s, r)-jet, $s \geq q \leq r$. Consider two fibered manifold morphisms $f, g\colon Y \to \overline{Y}$ with base maps $\underline{f}, \underline{g}\colon M \to \overline{M}$. We say that f and g determine the same (q, s, r)-jet at $y \in Y$, $j_y^{q,s,r} f = j_y^{q,s,r} g$, if

$$j_y^q f = j_y^q g, \quad j_y^s(f|Y_x) = j_y^s(g|Y_x) \quad \text{and} \quad j_x^r \underline{f} = j_x^r \underline{g}, \quad x = p(y),$$

where $p\colon Y \to M$ is a fibered manifold projection. A bundle functor F on $\mathcal{FM}$ is said to be of the order (q, s, r), if $j_y^{q,s,r} f = j_y^{q,s,r} g$ implies $Ff|F_y Y = Fg|F_y Y$. In such a case the integer r is called the base order of F, s is called the fiber order of F and q is called the total order of F, see [2].

Consider a bundle functor $F = T^\mu$ determined by $\mu\colon A \to B$ and denote by N_A and N_B the ideals of all nilpotent elements of A and B, respectively. The nilpotency implies $\mu(N_A) \subset N_B$. For $t \geq 1$ we have $N_B^t \subset N_B$, which yields $\mu(N_A) N_B^t \subset N_B$. So there exists the smallest integer t such that $\mu(N_A) N_B^t = 0$.

By A. Cabras and I. Kolář [1], the smallest integer t satisfying $\mu(N_A)N_B^t = 0$ is called the order of μ and is denoted by $\mathrm{ord}(\mu)$. A. Cabras and I. Kolář have also deduced that the order (q, s, r) of a bundle functor T^μ on $\mathcal{FM}$ is of the form

$$q = \mathrm{ord}(\mu), \quad s = \mathrm{ord}(B), \quad r = \max(\mathrm{ord}(A), \mathrm{ord}(\mu)),$$

see [1]. Obviously, we have $\mu(N_A)N_B^s \subset N_B^{s+1} = 0$, which implies the condition $s \geq q \leq r$.

2. Subordinated functors on fibered manifolds

Let $A = \mathbb{R} \times N_A$ be a Weil algebra of order r, where N_A is the ideal of all nilpotent elements of A. I. Kolář has recently introduced the underlying algebra of order k as the factor algebra $A_k = A/N_A^{k+1}$. The corresponding Weil functor T^{A_k} is said to be the underlying k-th order functor of T^A, [4]. In [2] we have introduced the more general concept of a subordinated functor. In general, G is called a subordinated functor of a functor F, if there exists a surjective natural transformation

$$t\colon F \to G.$$

In such a case we also say that G is dominated by F. A Weil algebra $\widetilde{A}$ is said to be dominated by A, if the Weil functor $T^{\widetilde{A}}$ is dominated by T^A. By [2], $\widetilde{A}$ is dominated by A if and only if we have an algebra epimorphism $A \to \widetilde{A}$. This yields

$$\widetilde{A} = A/I$$

for some ideal $I \subset A$. Clearly, for $I = N_A^{k+1}$ we obtain the particular concept of the underlying algebra A_k from [4]. In [2] we have also proved

Lemma 1
Every k-th order Weil algebra $\widetilde{A}$, which is dominated by A, is also dominated by A_k. So there is an epimorphism $\varphi\colon A_k \to \widetilde{A}$.

Proposition 2
Let $F = T^\mu$ and $G = T^\nu$ be two bundle functors on $\mathcal{FM}$ determined by algebra homomorphisms $\mu\colon A \to B$ and $\nu\colon C \to D$. The functor T^ν is dominated by T^μ if and only if the following conditions are satisfied:

(i) *$C = A/I$ is dominated by A,*

(ii) *$D = B/J$ is dominated by J,*

(iii) *the ideals $I \subset A$ and $J \subset B$ satisfy $\mu(I) \subset J$.*

Proof. Suppose first that G is dominated by F. Then there are algebra epimorphisms $f_1\colon A \to C$ and $f_2\colon B \to D$ such that

$$\nu \circ f_1 = f_2 \circ \mu. \tag{3}$$

Hence the Weil algebra C is dominated by A and D is dominated by B, so that we may write $C = A/I$ and $D = B/J$ for some ideals $I \subset A$ and $J \subset B$. Further, we have $f_2(\mu(I)) = \nu(f_1(I)) = \nu(0) = 0$ which reads $\mu(I) \subset \ker(f_2) = J$.

On the other hand, consider a bundle functor $F = T^\mu$, $\mu\colon A \to B$ and two ideals $I \subset A$, $J \subset B$ satisfying $\mu(I) \subset J$. Clearly, we can define an algebra homomorphism

$$\nu\colon A/I \to B/J \quad \text{by } \nu(a+I) = \mu(a) + J \tag{4}$$

such that (3) is true. So the functor T^ν is dominated by T^μ.

Let T^A and $T^{\widetilde{A}}$ be two Weil functors such that $T^{\widetilde{A}}$ is dominated by T^A. By [2], if the order of T^A is r, then the order of $T^{\widetilde{A}}$ is at most r. A similar result is true also for bundle functors defined on $\mathcal{FM}$.

Proposition 3

Let T^μ and T^ν be two bundle functors on $\mathcal{FM}$, $\mu\colon A \to B$ and $\nu\colon C \to D$. Denote by (q,s,r) the order of T^μ and by $(\overline{q},\overline{s},\overline{r})$ the order of T^ν. If T^ν is dominated by T^μ, then $\overline{q} \le q$, $\overline{s} \le s$ and $\overline{r} \le r$.

Proof. By [1], the order (q,s,r) is given by $q = \operatorname{ord}(\mu)$, $s = \operatorname{ord}(B)$, $r = \max(\operatorname{ord}(A), \operatorname{ord}(\mu))$. The condition $\overline{s} \le s$ follows from the fact that the Weil algebra D is dominated by B. Denote by N_A, N_B, N_C and N_D the ideals of nilpotent elements. From the epimorphisms $f_1\colon A \to C$ and $f_2\colon B \to D$ it follows $N_C = f_1(N_A)$ and $N_D = f_2(N_B)$. So we have

$$\begin{aligned}\nu(N_C)N_D^q &= \nu(f_1(N_A))f_2(N_B^q) = f_2(\mu(N_A))f_2(N_B^q) = f_2(\mu(N_A)N_B^q)\\ &= 0,\end{aligned}$$

which yields $\overline{q} \le q$. Finally, the algebra C is dominated by A, so that $\operatorname{ord}(C) \le \operatorname{ord}(A)$, which implies $\overline{r} \le r$.

Example 1

Write $J^{q,s,r}(Y,\overline{Y})$ for the space of all (q,s,r)-jets of the $\mathcal{FM}$-morphisms of Y into $\overline{Y}$. Denoting by $\mathbb{R}^{k,\ell} = \mathbb{R}^k \times \mathbb{R}^\ell \to \mathbb{R}^k$ the product fibered manifold, we can define a bundle functor $T_{k,\ell}^{q,s,r}$ of fibered velocities of dimension (k,ℓ) and order (q,s,r) by

$$T_{k,\ell}^{q,s,r} = J_{(0,0)}^{q,s,r}(\mathbb{R}^{k,\ell}, Y).$$

Clearly, the functor $T_{k,\ell}^{q,s,r}$ has a subordinated functor $T_{k,\ell}^{\overline{q},\overline{s},\overline{r}}$ for every $\overline{q} \le q$, $\overline{s} \le s$ and $\overline{r} \le r$. Moreover, A. Cabras and I. Kolář have deduced, [1], that for every product preserving bundle functor T^μ on $\mathcal{FM}$ of the order (q,s,r) there exists a velocities functor $T^{q,s,r}_{k,\ell}$ and a surjective natural transformation $T^{q,s,r}_{k,\ell} \to T^\mu$. Hence every bundle functor T^μ of the order (q,s,r) is dominated by a velocities functor $T^{q,s,r}_{k,\ell}$.

Example 2
If $A = B$ and $\mu = \mathrm{id}_A$, then $T^\mu Y = T^A M \times_{T^A M} T^A Y = T^A Y \to Y$. Clearly, the order of T^μ is (r,r,r), where $r = \mathrm{ord}(A)$. Further, if $C = D$ and $\nu = \mathrm{id}_C$, then $T^\nu Y = T^C Y \to Y$. If the Weil algebra C is dominated by A, then the functor T^ν is dominated by T^μ.

Example 3
Consider a product preserving functor T^μ, $\mu\colon A \to B$ and write $A_m = A/N_A^{m+1}$ for the underlying algebra of order m. Further, for $\ell \geq k$ we can define a factor algebra $B^\mu_{k,\ell} = B/\left\langle \mu(N_A)N_B^k, N_B^{\ell+1}\right\rangle$. If $m \geq k$, then we have $\mu(N_A^{m+1}) \subset \mu(N_A)N_B^k$, so that there is an induced algebra homomorphism (4)

$$\mu_{k,\ell,m}\colon A_m \to B^\mu_{k,\ell}, \quad \mu_{k,\ell,m}(a + N_A^{m+1}) = \mu(a) + \left\langle \mu(N_A)N_B^k, N_B^{\ell+1}\right\rangle .$$

By [8], the functor $T^{\mu_{k,\ell,m}}$ is dominated by T^μ.

Example 4
Consider a product preserving functor T^μ, $\mu\colon A \to B$ and write $I = N_A^{m+1}$, $J = N_B^{\ell+1}$, $A_m = A/I$, $B_\ell = B/J$. If $m \geq \ell$, then we have $\mu(I) \subset J$. By Proposition 2, for $m \geq \ell$ there exists an induced algebra homomorphism (4)

$$\mu_{\ell,m}\colon A_m \to B_\ell, \quad \mu_{\ell,m}(a + N_A^{m+1}) = \mu(a) + N_B^{\ell+1}$$

such that the functor $T^{\mu_{\ell,m}}$ is dominated by T^μ. One evaluates directly that if the order of T^μ is (q,s,r), then the order of $T^{\mu_{\ell,m}}$ is (q,ℓ,m).

Example 5
Consider a product preserving functor T^μ, $\mu\colon A \to B$. Write $I = 0$ and let $J \subset B$ be an arbitrary ideal. Then $A = A/I$ and the Weil algebra $D = B/J$ is dominated by B. Clearly, the condition $\mu(I) \subset J$ from Proposition 2 is satisfied for an arbitrary ideal $J \subset B$. So we can define an induced algebra homomorphism (4)

$$\nu\colon A \to D = B/J \quad \text{by } \nu(a) = \mu(a) + J.$$

By Proposition 2, the functor T^ν is dominated by T^μ.

Example 6
Write $J = N_B$ in Example 5. Then we have $D = B/N_B = \mathbb{R}$ and the induced Weil algebra homomorphism (4) $\nu\colon A \to D = \mathbb{R}$ is of the form $\nu(r,n) = r$. So the subordinated functor T^ν is of the form $T^\nu Y = T^A M \times_M Y$. Clearly, the

order of T^ν is $(0, 0, \mathrm{ord}(A))$ and the surjective natural transformation $T^\mu \to T^\nu$ is of the form

$$(T^A M \times_{T^B M} T^B Y) \to (T^A M \times_M Y), \quad (U, V) \mapsto (U, q^B_Y(V)),$$

where $q^B_Y \colon T^B Y \to Y$ is the bundle projection. Moreover, one can verify directly that the fiber order of an arbitrary functor T^α on $\mathcal{FM}$ is zero if and only if $T^\alpha Y = T^A M \times_M Y$.

Example 7
Write $J = N_B^{\ell+1}$ in Example 5. Then we have $D = B_\ell = B/N_B^{\ell+1}$. By Proposition 2, there is an induced algebra homomorphism (4)

$$\mu_\ell \colon A \to B_\ell, \quad \mu_\ell(a) = \mu(a) + N_B^{\ell+1}$$

such that the functor T^{μ_ℓ} is dominated by T^μ. Denote by $f \colon B \to B_\ell$ the algebra epimorphism given by $f(b) = b + N_B^{\ell+1}$ and suppose that the order of T^μ is (q, s, r). We can write

$$\begin{aligned} \mu_\ell(N_A)N_D^t &= f(\mu(N_A))(N_B + N_B^{\ell+1})^t = (\mu(N_A) + N_B^{\ell+1})(N_B + N_B^{\ell+1})^t \\ &= \mu(N_A)N_B^t. \end{aligned}$$

This yields $\mathrm{ord}(\mu_\ell) = \mathrm{ord}(\mu) = q$. Thus, we have proved: If the order of T^μ is (q, s, r), then the order of T^{μ_ℓ} is (q, ℓ, r).

In what follows we shall write $T^\nu \prec T^\mu$ if the functor T^ν is dominated by T^μ. Consider now the functors $T^{\mu_{k,\ell,m}}$, $T^{\mu_{\ell,m}}$ and T^{μ_ℓ} from Examples 3, 4 and 7, respectively.

Proposition 4
Let T^μ be a functor on $\mathcal{FM}$ of the order (q, s, r), which is determined by an algebra homomorphism $\mu \colon A \to B$. Then we have:

$$\begin{aligned} T^{\mu_{k,\ell,m}} \prec T^{\mu_\ell} \prec T^\mu \qquad & \text{for any } \ell \geq k \leq m, \\ T^{\mu_{k,\ell,m}} \prec T^{\mu_{\ell,m}} \prec T^{\mu_\ell} \prec T^\mu \qquad & \text{for any } \ell \geq k \leq m,\ m \geq \ell. \end{aligned}$$

Proof. Clearly, all the functors $T^{\mu_{k,\ell,m}}$, $T^{\mu_{\ell,m}}$ and T^{μ_ℓ} are dominated by T^μ. First we prove $T^{\mu_{k,\ell,m}} \prec T^{\mu_\ell}$. Consider a diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \mu\ } & B \\ \Big\| & & \Big\downarrow f_1 \\ A & \xrightarrow{\ \mu_\ell\ } & B_\ell \\ f_2 \Big\downarrow & & \Big\downarrow f_3 \\ A_m & \xrightarrow{\ \mu_{k,\ell,m}\ } & B^\mu_{k,\ell} \end{array}$$

where f_1 and f_2 are algebra epimorphisms defined by $f_1(b) = b + B_B^{\ell+1}$ and $f_2(a) = a + N_A^{m+1}$. Write $J = \langle \mu(N_A)N_B^k, N_B^{\ell+1}\rangle$. The inclusion $N_B^{\ell+1} \subset J$ defines an epimorphism $f_3\colon B_\ell \to B/J$. Further, we have

$$\mu_{k,\ell,m}(f_2(a)) = \mu_{k,\ell,m}(a + N_A^{m+1}) = \mu(a) + J$$

and

$$f_3(\mu_\ell(a)) = f_3(\mu(a) + N_B^{\ell+1}) = \mu(a) + J.$$

Hence the diagram commutes and we have $T^{\mu_{k,\ell,m}} \prec T^{\mu_\ell}$. Using an analogous diagram chasing, we prove directly the remaining relations.

3. Underlying functors on fibered manifolds

In [2] we have introduced the concept of an underlying functor on $\mathcal{M}f$ and on $\mathcal{FM}_m$, which can be modified also for bundle functors on $\mathcal{FM}$. Let F be a bundle functor on $\mathcal{FM}$.

Definition
A bundle functor F_a^f is said to be the underlying functor of F with the fiber order a, if:

(1) F_a^f is dominated by F,

(2) The fiber order of F_a^f is a,

(3) Every functor $\widetilde{F}$ on $\mathcal{FM}$ with the fiber order a, which is dominated by F, is also dominated by F_a^f.

Roughly speaking, the underlying functor F_a^f is "the greatest" subordinated functor among all subordinated functors of F with the fiber order a. Replacing the fiber order with the base order (or with the total order), we obtain the underlying functor F_a^b with the base order a (or the underlying functor F_a^t with the total order a). Clearly, the concept of a subordinated functor is quite general. On the other hand, the definition of an underlying functor depends on the order. As the order of a bundle functor on $\mathcal{FM}$ depends on three integers, we have three types of underlying functors on $\mathcal{FM}$. We prove

Proposition 5
Let $F = T^\mu$, $\mu\colon A \to B$ be a product preserving functor on $\mathcal{FM}$ and let T^{μ_ℓ} be the functor from Example 7. Then the underlying functor of F with the fiber order ℓ is of the form $F_\ell^f = T^{\mu_\ell}$.

Proof. By Example 7, T^{μ_ℓ} is dominated by T^μ and has the fiber order ℓ. Consider an arbitrary functor T^ν, $\nu\colon C = A/I \to D = B/J$ of the fiber order ℓ, which is dominated by T^μ. Obviously, $\operatorname{ord}(D) = \ell$. By Lemma 1 there is an epimorphism $\varphi\colon B_\ell \to D$, $\varphi(b + N_B^{\ell+1}) = b + J$. Further, consider a diagram

$$\begin{array}{ccc} A & \xrightarrow{\mu_\ell} & B_\ell \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle \varphi} \\ C = A/I & \xrightarrow{\nu} & D = B/J \end{array}$$

where $f_1 \colon A \to A/I$ is an epimorphism defined by $f_1(a) = a + I$. We can write

$$\nu(f_1(a)) = \nu(a + I) = \mu(a) + J$$

and

$$\varphi(\mu_\ell(a)) = \varphi(\mu(a) + N_B^{\ell+1}) = \mu(a) + J.$$

So the diagram commutes and the functor T^ν is dominated by T^{μ_ℓ}.

Denote by $\mathcal{T}$ the class of functors T^ν, $\nu \colon C \to D$ of the order (k, ℓ, m), $\ell \geq k \leq m$, which are dominated by $F = T^\mu$ and satisfy the condition

$$\operatorname{ord}(C) \geq \operatorname{ord}(D). \tag{5}$$

For example, the functor $T^{\mu_{\ell,m}}$ from Example 4 is an element of $\mathcal{T}$. Further, if the algebra D is dominated by C, then (5) is true.

Lemma 2
Let T^ν be a functor of the order (k, ℓ, m) from the class $\mathcal{T}$. Then we have $\operatorname{ord}(C) = m \geq \ell$.

Proof. The relation $\operatorname{ord}(C) \geq \operatorname{ord}(D) = \ell \geq k$ implies $\operatorname{ord}(C) = m$.

Proposition 6
Let $F = T^\mu$, $\mu \colon A \to B$ be a product preserving functor on $\mathcal{FM}$ and let $T^{\mu_{\ell,m}}$ be the functor from Example 4. On the class $\mathcal{T}$ of subordinated functors of F, the underlying functor with the base order m is of the form $F_m^b = T^{\mu_{\ell,m}}$.

Proof. By Example 4, the functor $T^{\mu_{\ell,m}}$ has the order (q, ℓ, m) and is dominated by F. Consider now an arbitrary functor $T^\nu \in \mathcal{T}$, where $\nu \colon C \to D$. Then we have $C = A/I$ and $D = B/J$, where $\operatorname{ord}(D) = \ell$. Further, Lemma 2 implies $\operatorname{ord}(C) = m$. By Lemma 1, there are epimorphisms $\varphi_1 \colon A_m \to C$ and $\varphi_2 \colon B_\ell \to D$. We can write

$$\nu(\varphi_1(a + N_A^{m+1})) = \nu(a + I) = \mu(a) + J$$

and

$$\varphi_2(\mu_{\ell,m}(a + N_A^{m+1})) = \varphi_2(\mu(a) + N_B^{\ell+1}) = \mu(a) + J.$$

Thus, the functor T^ν is dominated by $T^{\mu_{\ell,m}}$.

Write $\mu\colon A \to B$, $\nu\colon C \to D$ and suppose that the functor T^ν is dominated by T^μ. Obviously, if μ is an epimorphism, then ν is an epimorphism too. This implies, that if μ is an epimorphism, then every functor T^ν of the order (k,ℓ,m), which is dominated by T^μ, is an element of the class $\mathcal{T}$. Thus, we have

Corollary 1
Let $F = T^\mu$, $\mu\colon A \to B$ be a product preserving functor on $\mathcal{FM}$. If μ is an epimorphism, then the underlying functor of F with the base order m is of the form $F^b_m = T^{\mu_{\ell,m}}$ for some $\ell \le m$.

4. Affine bundle structure on product preserving functors on $\mathcal{FM}$

Let $F = T^\mu$ be a functor with the fiber order b, which is determined by a homomorphism $\mu\colon A \to B$. By Proposition 5, the underlying functor with the fiber order $(b-1)$ is of the form $F^f_{b-1} = T^{\mu_{b-1}}$. We have

Proposition 7
$T^\mu Y \to T^{\mu_{b-1}} Y$ is an affine bundle, whose associated vector bundle is the pull back of $TY \otimes N^b_B$ over $T^{\mu_{b-1}}Y$.

Proof. By [4], $T^B Y \to T^{B_{b-1}} Y$ is an affine bundle, whose associated vector bundle is the pull back of $TY \otimes N^b_B$ over $T^{B_{b-1}}Y$. Consider now the expression of $T^\mu Y$ in the form (1). For $(U,V) \in T^A M \times T^B Y$ and $v \in TY \otimes N^b_B$ we can define the addition by

$$(U,V) + v := (U, V+v) \in T^A M \times T^B Y.$$

We have $T^B p\colon T^B Y \to T^B M$, so that $T^B p(V+v) \in T^B M$. As $T^B M \to T^{B_{b-1}} M$ is also an affine bundle, we can write $T^B p(V+v) = T^B p(V) + w$ for some $w \in TM \otimes N^b_B$. Clearly, w is of the form $w = (Tp \otimes \mathrm{id}_{N^b_B})(v)$. Further, $w = 0$ iff $v \in VY \otimes N^b_B$. In such a case we have $T^B p(V+v) = T^B p(V) = \mu_M(U)$. So for $(U,V) \in T^\mu Y$ and $v \in VY \otimes N^b_B$, the sum $(U,V)+v$ is an element of $T^\mu Y$.

As a particular case we obtain the result by I. Kolář and W.M. Mikulski from [8], who have proved that $T^{\mu_{q,s,r}} Y \to T^{\mu_{q,s-1,r}} Y$ is an affine bundle.

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On the differential first-order invariants for the non-splitting subgroups of the generalized Poincare group $P(1,4)$

Dedicated to Professor Andrzej Zajtz on his seventieth birthday

Abstract. The functional bases of the differential first-order invariants for all non-splitting subgroups of the generalized Poincaré group $P(1,4)$ are constructed. Some applications of the results obtained are considered.

Introduction

The developement of the theoretical physics has required various extensions of the four–dimensional Minkowski space and, correspondingly, various extensions of the Poincaré group $P(1,3)$. The natural extension of this group is the generalized Poincaré group $P(1,4)$. The group $P(1,4)$ is the group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. This group has many applications in theoretical and mathematical physics [1-3]. The group $P(1,4)$ has many subgroups used in theoretical physics [4-8]. Among these subgroups there are the Poincaré group $P(1,3)$ and the extended Galilei group $\widetilde{G}(1,3)$ (see also [9]). Thus, the results obtained with the help of the subgroup structure of the group $P(1,4)$ will be useful in relativistic and non-relativistic physics.

The papers [10-12] are devoted to the construction of the first-order differential invariants for the splitting subgroups [4, 5, 7] of the generalized Poincaré group $P(1,4)$.

The present paper is devoted to the construction of functional bases of the differential first-order invariants for the non-splitting subgroups [4, 6-8] of the group $P(1,4)$.

Our paper is organized as follows. In the first section we introduce some notation and results concerning the Lie algebra of the group $P(1,4)$ which we use in the following. Sections 2 and 3 present our main results.

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1. The Lie algebra of the group $P(1,4)$ and its non-conjugate subalgebras

The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ $(\mu,\nu = 0,1,2,3,4)$ and P'_μ $(\mu = 0,1,2,3,4)$, satisfying the commutation relations

$$\begin{aligned}
\left[P'_\mu, P'_\nu\right] &= 0,\\
\left[M'_{\mu\nu}, P'_\sigma\right] &= g_{\mu\sigma}P'_\nu - g_{\nu\sigma}P'_\mu,\\
\left[M'_{\mu\nu}, M'_{\rho\sigma}\right] &= g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\sigma}M'_{\mu\rho} - g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho},
\end{aligned}$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$.

In order to study the subgroup structure of the group $P(1,4)$ we used the method proposed in [13]. Continuous subgroups of the group $P(1,4)$ have been described in [4–8].

Further we will use the following basis elements:

$$\begin{aligned}
&G = M'_{40}, && L_1 = M'_{32}, \quad L_2 = -M'_{31}, \quad L_3 = M'_{21},\\
&P_a = M'_{4a} - M'_{a0}, && C_a = M'_{4a} + M'_{a0}, \quad (a = 1,2,3),\\
&X_0 = \frac{1}{2}(P'_0 - P'_4), && X_k = P'_k \quad (k = 1,2,3),\\
&X_4 = \frac{1}{2}(P'_0 + P'_4).
\end{aligned}$$

2. The differential first-order invariants of the non-splitting subgroups of the group $P(1,4)$

The group $P(1,4)$ acts on $M(1,3) \times R(u)$ (i.e., on the Cartesian product of the four-dimensional Minkowski space (of the independent variables x_0, x_1, x_2, x_3) and the number axis of the dependent variable u). The group $P(1,4)$ usually acts on $M(1,3) \times R(u)$ as a group generated by translations and rotations of this space.

Let

$$X = \sum_{i=0}^{3} \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u}$$

be one of the basis infinitesimal operators. The first prolongation of X has the form

$$X^{(1)} = X + \sum_{i=0}^{3}\left(\frac{\partial\eta}{\partial x_i} + \frac{\partial\eta}{\partial u}u_i - \sum_{j=0}^{3} u_j\frac{\partial\xi_j}{\partial x_i} - \sum_{j=0}^{3} u_i u_j\frac{\partial\xi_j}{\partial u}\right)\frac{\partial}{\partial u_i}.$$

Now, a function $J(x, u^{(1)})$ is a first-order differential invariant if

$$X^{(1)} \cdot J(x, u^{(1)}) = 0.$$

Here $u^{(1)} = (u, u_0, u_1, u_2, u_3)$ is an element of the first prolongation $R(u)^{(1)}$.

Let us consider the following representation of the Lie algebra of the group $P(1,4)$:

$$P'_0 = \frac{\partial}{\partial x_0}, \qquad P'_1 = -\frac{\partial}{\partial x_1}, \qquad P'_2 = -\frac{\partial}{\partial x_2}, \qquad P'_3 = -\frac{\partial}{\partial x_3},$$
$$P'_4 = -\frac{\partial}{\partial u}, \qquad M'_{\mu\nu} = -\left(x_\mu P'_\nu - x_\nu P'_\mu\right), \qquad x_4 \equiv u.$$

More details about this representation can be found in [14-16].

In the construction of the differential invariants it has turned out that different non-splitting subalgebras of the Lie algebra of the group $P(1,4)$ can have the same functional basis of the first-order differential invariants. Consequently, there is no one-to-one correspondence between non-conjugate subalgebras of the Lie algebra of the group $P(1,4)$ and corresponding differential invariants.

Definition 1
We call two subalgebras L^1 and L^2 of the Lie algebra of the group $P(1,4)$ equivalent if they have the same functional basis of the first-order differential invariants.

It is possible to prove that the relation of equivalence of subalgebras L^1 and L^2 given by Definition 1 is the set-theoretical equivalence relation. With respect to this equivalence relation, all non-splitting subalgebras of the Lie algebra of the group $P(1,4)$ split into classes of equivalent subalgebras. Each two different classes have different functional bases of the first-order differential invariants.

Definition 2
We call two functional bases of the first-order differential invariants of the non-splitting subalgebras of the Lie algebra of the group $P(1,4)$ equivalent if they belong to the equivalent subalgebras.

One of the results in this section can be formulated as follows.

Proposition
The non-splitting subgroups of the group $P(1,4)$ *have* 264 *non-equivalent functional bases of the first-order differential invariants.*

Proof. Here, we only give a sketch of the proof. Following the sketch, for the purpose of proving the Proposition, we have to use:

— the list of the non-splitting subalgebras of the Lie algebra of the group $P(1,4)$ [17];

— the general ranks of the matrices which contain coordinates of the one time prolonged basis elements of the subalgebras of the considered Lie algebra;

— theorem on number of invariants of the Lie group of the point transformations (see, for example, [18, 19]);

— Definition 1 and Definition 2.

For all non-splitting subgroups of the group $P(1,4)$ the functional bases of the first-order differential invariants have been constructed.

Below, for some of the non-splitting subalgebras of the Lie algebra of the group $P(1,4)$ we give their respective basis elements and corresponding functional basis of differential invariants.

One-dimensional subalgebras

1. $\langle L_3 + eG + \kappa_3 X_3,\ e > 0,\ \kappa_3 < 0\rangle$,

$$J_1 = (x_0^2 - u^2)^{\frac{1}{2}}, \qquad J_2 = (x_1^2 + x_2^2)^{\frac{1}{2}},$$

$$J_3 = \kappa_3 \ln(x_0 + u) - ex_3, \qquad J_4 = x_3 + \kappa_3 \arctan \frac{x_1}{x_2},$$

$$J_5 = \frac{x_1 u_2 - x_2 u_1}{x_1 u_1 + x_2 u_2}, \qquad J_6 = u_3 \frac{x_0 + u}{u_0 + 1},$$

$$J_7 = \frac{u_3^2}{u_0^2 - 1}, \qquad J_8 = \frac{u_1^2 + u_2^2}{u_3^2},$$

$$u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \qquad \mu = 0, 1, 2, 3;$$

2. $\langle P_3 + X_0\rangle$,

$$J_1 = x_1, \qquad J_2 = x_2,$$

$$J_3 = (x_0 + u)^2 - 2x_3, \qquad J_4 = x_0 - u + \frac{2}{3}(x_0 + u)^3 - 2x_3(x_0 + u),$$

$$J_5 = x_0 + u + \frac{u_3}{u_0 + 1}, \qquad J_6 = \frac{u_1}{u_2},$$

$$J_7 = \frac{u_1}{u_0 + 1}, \qquad J_8 = \frac{u_3^2}{(u_0 + 1)^2} + \frac{2}{u_0 + 1}.$$

Two-dimensional subalgebras

1. $\langle G,\ L_3 + dX_3,\ d < 0\rangle$,

$$J_1 = x_3 + d\arctan\frac{x_1}{x_2}, \qquad J_2 = (x_1^2 + x_2^2)^{\frac{1}{2}},$$
$$J_3 = (x_0^2 - u^2)^{\frac{1}{2}}, \qquad J_4 = (x_0 + u)^2\frac{1 - u_0}{u_0 + 1},$$
$$J_5 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}, \qquad J_6 = \frac{u_3^2}{u_0^2 - 1},$$
$$J_7 = \frac{u_1^2 + u_2^2}{u_3^2};$$

2. $\langle L_3 - X_4,\ P_3\rangle$,

$$J_1 = x_0 + u, \qquad J_2 = (x_1^2 + x_2^2)^{\frac{1}{2}},$$
$$J_3 = x_0^2 - x_3^2 - u^2 + (x_0 + u)\arctan\frac{x_1}{x_2}, \qquad J_4 = \frac{x_3}{x_0 + u} + \frac{u_3}{u_0 + 1},$$
$$J_5 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}, \qquad J_6 = \frac{u_1^2 + u_2^2}{(u_0 + 1)^2},$$
$$J_7 = \frac{u_3^2}{(u_0 + 1)^2} + \frac{2}{u_0 + 1}.$$

Three-dimensional subalgebras

1. $\langle G + a_1X_1 + a_3X_3,\ P_3,\ X_4,\ a_1 < 0,\ a_3 < 0\rangle$,

$$J_1 = x_2, \qquad J_2 = x_1 - a_1\ln(x_0 + u),$$
$$J_3 = x_3 - a_3\ln(x_0 + u) + u_3\frac{x_0 + u}{u_0 + 1}, \qquad J_4 = (x_0 + u)\frac{u_1}{u_0 + 1},$$
$$J_5 = \frac{u_1}{u_2}, \qquad J_6 = \frac{u_0^2 - u_3^2 - 1}{u_1^2};$$

2. $\langle L_3 - P_3 + \alpha_0X_0,\ X_3,\ X_4,\ \alpha_0 < 0\rangle$,

$$J_1 = (x_1^2 + x_2^2)^{\frac{1}{2}}, \qquad J_2 = \alpha_0\arctan\frac{x_1}{x_2} - x_0 - u,$$
$$J_3 = \arctan\frac{u_1}{u_2} - \frac{u_3}{u_0 + 1}, \qquad J_4 = x_0 + u - \alpha_0\frac{u_3}{u_0 + 1},$$
$$J_5 = \frac{u_1^2 + u_2^2}{(u_0 + 1)^2}, \qquad J_6 = \frac{u_3^2 + 2(u_0 + 1)}{(u_0 + 1)^2}.$$

Four-dimensional subalgebras

1. $\langle G + a_3X_3,\ L_3,\ P_3,\ X_4,\ a_3 < 0\rangle$,

$$J_1 = (x_1^2 + x_2^2)^{\frac{1}{2}}, \qquad J_2 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2},$$

$$J_3 = x_3 - a_3\ln(x_0 + u) + \frac{x_0 + u}{u_0 + 1}u_3, \qquad J_4 = (u_1^2 + u_2^2)\frac{(x_0 + u)^2}{(u_0 + 1)^2},$$

$$J_5 = \frac{u_0^2 - u_3^2 - 1}{u_1^2 + u_2^2};$$

2. $\langle L_3,\ P_1,\ P_2,\ P_3 + X_3\rangle$,

$$J_1 = x_0 + u,$$

$$J_2 = x_0^2 - x_1^2 - x_2^2 - u^2 - \frac{x_0 + u}{x_0 + u - 1}x_3^2,$$

$$J_3 = \frac{x_3}{x_0 + u - 1} + \frac{u_3}{u_0 + 1},$$

$$J_4 = \left(\frac{x_1}{x_0 + u} + \frac{u_1}{u_0 + 1}\right)^2 + \left(\frac{x_2}{x_0 + u} + \frac{u_2}{u_0 + 1}\right)^2,$$

$$J_5 = \frac{u_1^2 + u_2^2 + u_3^2 + 2(u_0 + 1)}{(u_0 + 1)^2}.$$

Five-dimensional subalgebras

1. $\langle G + a_2X_1,\ P_1,\ P_2,\ P_3,\ X_4,\ a_2 < 0\rangle$,

$$J_1 = x_1 + \frac{x_0 + u}{u_0 + 1}u_1 - a_2\ln(x_0 + u),$$

$$J_2 = x_2 + \frac{x_0 + u}{u_0 + 1}u_2,$$

$$J_3 = x_3 + (x_0 + u)\frac{u_3}{u_0 + 1},$$

$$J_4 = (u_0^2 - u_1^2 - u_2^2 - u_3^2 - 1)\frac{(x_0 + u)^2}{(u_0 + 1)^2};$$

2. $\langle L_3,\ P_1 + X_2,\ P_2 - X_1,\ X_3,\ X_4\rangle$,

$$J_1 = x_0 + u,$$

$$J_2 = \left(\frac{x_1}{x_0+u} + \frac{u_2}{(x_0+u)(u_0+1)} + \frac{u_1}{u_0+1}\right)^2 + \left(\frac{x_2}{x_0+u} - \frac{u_1}{(x_0+u)(u_0+1)} + \frac{u_2}{u_0+1}\right)^2,$$

$$J_3 = \frac{u_3}{u_0+1},$$

$$J_4 = \frac{u_1^2+u_2^2}{(u_0+1)^2} + \frac{2}{u_0+1}.$$

Six-dimensional subalgebras

1. $\langle G + aX_3,\ L_3 + dX_3,\ P_1,\ P_2,\ P_3,\ X_4,\ a < 0,\ d < 0\rangle$,

$$J_1 = (x_0+u)^2\left(\frac{u_1^2+u_2^2+u_3^2+2u_0+2}{(u_0+1)^2} - 1\right),$$

$$J_2 = \left(x_1 + \frac{x_0+u}{u_0+1}u_1\right)^2 + \left(x_2 + \frac{x_0+u}{u_0+1}u_2\right)^2,$$

$$J_3 = x_3 - a\ln(x_0+u) + (x_0+u)\frac{u_3}{u_0+1} + d\arctan\left(\frac{x_1(u_0+1)+u_1(x_0+u)}{x_2(u_0+1)+u_2(x_0+u)}\right);$$

2. $\langle P_1 + X_3,\ P_2,\ X_0,\ X_1,\ X_2,\ X_4\rangle$,

$$J_1 = \frac{u_1}{u_0+1} - x_3, \qquad J_2 = \frac{u_3}{u_0+1},$$

$$J_3 = \frac{u_1^2+u_2^2}{(u_0+1)^2} + \frac{2}{u_0+1}.$$

Seven-dimensional subalgebras

1. $\langle G + a_3X_3,\ L_3,\ P_1,\ P_2,\ X_1,\ X_2,\ X_4,\ a_3 < 0\rangle$,

$$J_1 = x_3 - a_3\ln(x_0+u), \qquad J_2 = (x_0+u)\frac{u_3}{u_0+1},$$

$$J_3 = \frac{u_0^2-u_1^2-u_2^2-1}{u_3^2};$$

2. $\langle L_3 - P_3 + \alpha_0X_0,\ P_1,\ P_2,\ X_1,\ X_2,\ X_3,\ X_4,\ \alpha_0 < 0\rangle$,

$$J_1 = x_0 + u - \alpha_0 \frac{u_3}{u_0+1}, \qquad J_2 = \frac{u_1^2+u_2^2+u_3^2+2(u_0+1)}{(u_0+1)^2}.$$

Eight-dimensional subalgebras

1. $\langle G + a_3X_3,\ L_3,\ P_1,\ P_2,\ X_0,\ X_1,\ X_2,\ X_4,\ a_3<0\rangle$,

$$J_1 = x_3 + a_3 \ln\left(\frac{u_3}{u_0+1}\right), \qquad J_2 = \frac{u_0^2-u_1^2-u_2^2-1}{u_3^2};$$

2. $\langle L_3 - X_0,\ P_1,\ P_2,\ P_3,\ X_1,\ X_2,\ X_3,\ X_4\rangle$,

$$J_1 = \frac{u_1^2+u_2^2+u_3^2+2(u_0+1)}{(u_0+1)^2}.$$

3. On some applications of the results obtained

The differential invariants of the local Lie groups of the point transformations play an important role in the group-analysis of differential equations (see, for example [18-28]). In particule, with the help of these invariants we can construct differential equations with non-trivial symmetry groups.

In our case the considered equations can be written in the following form (see, for example [18-20]):

$$F(J_1, J_2, \ldots, J_t) = 0,$$

where F is an arbitrary smooth function of its arguments, $\{J_1, J_2, \ldots, J_t\}$ is a functional basis of the first-order differential invariants of the non-splitting subgroups of the group $P(1,4)$.

Since the Lie algebra of the group $P(1,4)$ contains, as subalgebras, the Lie algebra of the Poincaré group $P(1,3)$ and the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ (see also [9]), the results obtained can be used in relativistic and non-relativistic physics.

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On conformally symmetric warped products

Dedicated to Professor Dr. Andrzej Zajtz on his seventieth birthday

Abstract. We prove the necessary and sufficient conditions for a warped product manifold to be conformally symmetric. Basing on these results we give two examples of such warped products.

1. Introduction

Let (M, g) and (M', g') be two Riemannian manifolds whose metrics need not be positive definite and let $f > 0$ be a smooth function on M. The warped product ([1], [11]) $\overline{M} = M \times_f M'$ is the product manifold $M \times M'$ furnished with the metric

$$\bar{g} = \pi^* g + (f \circ \pi)\sigma^* g',$$

where π and σ are the projections of $M \times M'$ onto M and M', respectively. M is called the base of $\overline{M} = M \times_f M'$, M' the fiber, and f the warping function. An n-dimensional $(n > 3)$ Riemannian manifold is said to be conformally symmetric ([2]) if its Weyl conformal curvature tensor

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}\,(g_{ij}S_{hk} - g_{ik}S_{hj} + g_{hk}S_{ij} - g_{hj}S_{ik}) + \frac{\kappa}{(n-1)(n-2)}\,(g_{hk}g_{ij} - g_{hj}g_{ik})$$

is parallel, i.e. $\nabla C = 0$. Such a manifold is said to be essentially conformally symmetric (shortly, e.c.s.) if it is neither conformally flat $(C = 0)$ nor locally symmetric $(\nabla R = 0)$. Properties and examples of e.c.s. manifolds can be found in [3]-[6] and [12].

In this paper we are concerned with e.c.s. warped products. Some necessary conditions for a warped product to be conformally symmetric can be found in [7] and [10].

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In Section 3 we prove many necessary conditions for a warped product to be an e.c.s. manifold. In Section 4 we state conditions which are also sufficient. Basing on these results we give two examples of e.c.s. warped products.

Throughout this paper, by a manifold we mean a connected paracompact manifold of class C^∞ or analytic. By abuse of notation concerning Riemannian manifolds we often write M instead of (M,g).

2. Preliminaries

Let $(\overline{M},\bar{g})$ be an n-dimensional $(n>3)$ warped product $M\times_f M'$ $(\dim M = q,\ 1\leqslant q<n,\ \dim M' = n-q = s)$. In a suitable product chart $x^1,\dots,x^n$ for $\overline{M}$ we have

$$\bar{g}_{ij}dx^i dx^j = g_{ab}dx^a dx^b + f\cdot g'_{\alpha\beta}dx^\alpha dx^\beta,$$

where $i,j=1,\dots,n$, $a,b=1,\dots,q$, $\alpha,\beta=q+1,\dots,n$, g_{ab} and f are functions of (x^a) only, and $g'_{\alpha\beta}$ are functions of (x^α) only.

We denote by Γ^a_{bc},R_{abcd}, S_{ab} and κ, the components of the Levi-Civita connection ∇, the Riemann-Christoffel curvature tensor R, the Ricci tensor S and the scalar curvature of (M,g), respectively. Moreover, when $\overline{\Omega}$ is a quantity formed with respect to $\bar{g}$, we denote by Ω' the similar quantity formed with respect to g'.

It is easy to show that the following relations hold (cf. [10])

$$\bar{\Gamma}^c_{ab} = \Gamma^c_{ab}\,, \qquad \bar{\Gamma}^\alpha_{b\gamma} = \frac{1}{2f}\delta^\alpha_\gamma f_b\,, \qquad \bar{\Gamma}^\alpha_{\beta\gamma} = \Gamma'^\alpha_{\beta\gamma}\,,$$

$$\bar{\Gamma}^a_{\beta\gamma} = -\frac{1}{2}g^{ad}f_d g'_{\beta\gamma}\,, \qquad \bar{\Gamma}^a_{b\gamma} = \bar{\Gamma}^\alpha_{bc} = 0,$$

where $f_b = \partial_b f$, $\partial_b = \frac{\partial}{\partial x^b}$.

In the sequel we will use the following notations

$$G_{abcd} = g_{ad}g_{bc} - g_{ac}g_{bd}\,,$$

$$T_{ab} = -\tfrac{1}{2f}(\nabla_b f_a - \tfrac{1}{2f}f_a f_b), \qquad tr(T) = g^{ab}T_{ab}\,,$$

$$Q = f((s-1)P - tr(T)), \qquad P = \tfrac{1}{4f^2}g^{ab}f_a f_b\,. \tag{1}$$

By an elementary calculation we can show that the only non-zero components of $\bar{R}$, $\bar{S}$ and $\bar{C}$ are those related to:

$$\bar{R}_{abcd} = R_{abcd}\,, \qquad \bar{R}_{a\beta c\delta} = fT_{ac}g'_{\beta\delta}\,, \qquad \bar{R}_{\alpha\beta\gamma\delta} = fR'_{\alpha\beta\gamma\delta} + f^2PG'_{\alpha\beta\gamma\delta}\,,$$

(2)
$$a)\quad \bar{S}_{ab} = S_{ab} - sT_{ab}\,,$$

$$b)\quad \bar{S}_{\alpha\beta} = S'_{\alpha\beta} + Qg'_{\alpha\beta}\,,$$

(3) $a)$ $\bar{C}_{abcd}$

$$= R_{abcd} - \frac{1}{n-2}(g_{ad}S_{bc} + g_{bc}S_{ad} - g_{ac}S_{bd} - g_{bd}S_{ac}) + \frac{s}{n-2}(g_{ad}T_{bc} + g_{bc}T_{ad} - g_{ac}T_{bd} - g_{bd}T_{ac}) + \frac{\bar{\kappa}}{(n-1)(n-2)}\, G_{abcd}\,,$$

$b)$ $\bar{C}_{\alpha\beta\gamma\delta}$

$$= f\left(R'_{\alpha\beta\gamma\delta} - \frac{1}{n-2}(g'_{\alpha\delta}S'_{\beta\gamma} + g'_{\beta\gamma}S'_{\alpha\delta} - g'_{\alpha\gamma}S'_{\beta\delta} - g'_{\beta\delta}S'_{\alpha\gamma})\right) + (f^2P - \frac{2fQ}{n-2} + \frac{f^2\bar{\kappa}}{(n-1)(n-2)}\; G'_{\alpha\beta\gamma\delta}\,,$$

$c)$ $\bar{C}_{a\beta c\delta}$

$$= \frac{1}{n-2}\left(fg'_{\beta\delta}(S_{ac} + (q-2)T_{ac}) + g_{ac}S'_{\beta\delta} + \left(Q - \frac{f\bar{\kappa}}{n-1}\right) g_{ac}g'_{\beta\delta}\right).$$

Moreover,

$$\bar{\kappa} = \kappa + \frac{\kappa'}{f} + s((s-1)P - 2tr(T)). \tag{4}$$

Similarly, by an elementary but lenghty calculation we can easily show that the only non-zero components of $\overline{\nabla}\bar{R}$ and $\overline{\nabla}\bar{S}$ are those related to:

(5)
$$a)\quad \overline{\nabla}_e\bar{R}_{abcd} = \nabla_e R_{abcd}\,,$$
$$b)\quad \overline{\nabla}_\varepsilon\bar{R}_{\alpha\beta\gamma\delta} = f\nabla'_\varepsilon R'_{\alpha\beta\gamma\delta}\,,$$
$$c)\quad \overline{\nabla}_e\bar{R}_{\alpha\beta\gamma\delta} = -f_eR'_{\alpha\beta\gamma\delta} + f^2(\partial_eP)G'_{\alpha\beta\gamma\delta}\,,$$
$$d)\quad \overline{\nabla}_\varepsilon\bar{R}_{\alpha\beta\gamma d} = -\frac{f_d}{2}R'_{\alpha\beta\gamma\varepsilon} + \frac{f^2}{2}(\partial_dP)G'_{\alpha\beta\gamma\varepsilon}\,,$$
$$e)\quad \overline{\nabla}_e\bar{R}_{a\beta c\delta} = f\nabla_eT_{ac}g'_{\beta\delta}\,,$$
$$f)\quad \overline{\nabla}_\varepsilon\bar{R}_{abc\delta} = \frac{1}{2}g'_{\varepsilon\delta}(f_aT_{bc} - f_bT_{ac}) + \frac{1}{2}f^dR_{abcd}g'_{\varepsilon\delta}\,,$$

(6)
$$a)\quad \overline{\nabla}_c\bar{S}_{ab} = \nabla_cS_{ab} - s\nabla_cT_{ab}\,,$$
$$b)\quad \overline{\nabla}_\gamma\bar{S}_{\alpha\beta} = \nabla'_\gamma S'_{\alpha\beta}\,,$$
$$c)\quad \overline{\nabla}_\gamma\bar{S}_{\alpha b} = -\frac{1}{2f}S'_{\alpha\gamma}f_b + \frac{1}{2}g'_{\alpha\gamma}(f^cS_{cb} - sf^cT_{cb} - \frac{Q}{f}f_b),$$

$$d)\quad \overline{\nabla}_c \bar{S}_{\alpha\beta} = (\partial_c Q) g'_{\alpha\beta} - \frac{f_c}{f}(S'_{\alpha\beta} + Q g'_{\alpha\beta}).$$

From the above formulas we immediately obtain

LEMMA 2.1
Let $\overline{M}$ be a warped product $M \times_f M'$ with vanishing scalar curvature $\bar{\kappa}$. Then $\overline{M}$ is conformally symmetric if and only if

$$(7)\quad a)\quad \overline{\nabla}_e \bar{R}_{abcd} = \frac{1}{n-2}(\bar{g}_{ad}\overline{\nabla}_e \bar{S}_{bc} + \bar{g}_{bc}\overline{\nabla}_c \bar{S}_{ad} - \bar{g}_{ac}\overline{\nabla}_e \bar{S}_{bd} - \bar{g}_{bd}\overline{\nabla}_e \bar{S}_{ac}),$$

$$b)\quad \overline{\nabla}_\varepsilon \bar{R}_{\alpha\beta\gamma\delta} = \frac{1}{n-2}(\bar{g}_{\alpha\delta}\overline{\nabla}_\varepsilon \bar{S}_{\beta\gamma} + \bar{g}_{\beta\gamma}\overline{\nabla}_\varepsilon \bar{S}_{\alpha\delta} - \bar{g}_{\alpha\gamma}\overline{\nabla}_\varepsilon \bar{S}_{\beta\delta} - \bar{g}_{\beta\delta}\overline{\nabla}_\varepsilon \bar{S}_{\alpha\gamma}),$$

$$c)\quad \overline{\nabla}_e \bar{R}_{\alpha\beta\gamma\delta} = \frac{1}{n-2}(\bar{g}_{\alpha\delta}\overline{\nabla}_e \bar{S}_{\beta\gamma} + \bar{g}_{\beta\gamma}\overline{\nabla}_e \bar{S}_{\alpha\delta} - \bar{g}_{\alpha\gamma}\overline{\nabla}_e \bar{S}_{\beta\delta} - \bar{g}_{\beta\delta}\overline{\nabla}_e \bar{S}_{\alpha\gamma}),$$

$$d)\quad \overline{\nabla}_\varepsilon \bar{R}_{\alpha\beta\gamma d} = \frac{1}{n-2}(\bar{g}_{\beta\gamma}\overline{\nabla}_\varepsilon \bar{S}_{\alpha d} - \bar{g}_{\alpha\gamma}\overline{\nabla}_\varepsilon \bar{S}_{\beta d}),$$

$$e)\quad \overline{\nabla}_e \bar{R}_{a\beta c\delta} = \frac{1}{n-2}(-\bar{g}_{ac}\overline{\nabla}_e \bar{S}_{\beta\delta} - \bar{g}_{\beta\delta}\overline{\nabla}_e \bar{S}_{ac}),$$

$$f)\quad \overline{\nabla}_\varepsilon \bar{R}_{abc\delta} = \frac{1}{n-2}(\bar{g}_{bc}\overline{\nabla}_\varepsilon \bar{S}_{a\delta} - \bar{g}_{ac}\overline{\nabla}_\varepsilon \bar{S}_{b\delta}),$$

$$g)\quad \overline{\nabla}_\varepsilon \bar{S}_{\beta\delta} = 0.$$

In the sequel we shall need the following properties of e.c.s. manifolds:

LEMMA 2.2 ([5], [6])
Every e.c.s. manifold $(\overline{M}, \bar{g})$ satisfies the relations:

$$\bar{\kappa} = 0, \tag{8}$$

$$\overline{\nabla}_k \bar{S}_{ij} = \overline{\nabla}_j \bar{S}_{ik}\,, \tag{9}$$

$$\bar{S}_{il}\bar{C}_{hmjk} + \bar{S}_{ij}\bar{C}_{hmkl} + \bar{S}_{ik}\bar{C}_{hmlj} = 0. \tag{10}$$

LEMMA 2.3 ([6])
Let $(\overline{M}, \bar{g})$ be an e.c.s. manifold. Then $\overline{M}$ admits a unique function $\bar{F}$ such that

$$\bar{F}\bar{C}_{hijk} = \bar{S}_{hk}\bar{S}_{ij} - \bar{S}_{hj}\bar{S}_{ik}. \tag{11}$$

$\bar{F}$ is said to be the fundamental function of $\overline{M}$. It is clear that $\bar{F}(x) = 0$ if and only if $\operatorname{rank} \bar{S}(x) \leqslant 1$.

Moreover we shall use the following fact

LEMMA 2.4 ([9], Theorem 1)
Let M be an n-dimensional Riemannian manifold. If B is a generalized curvature tensor satisfying $\nabla_m \nabla_l B_{hijk} = \nabla_l \nabla_m B_{hijk}$ and P is a vector field such that $w^r R_{rijk} = P_k g_{ij} - P_j g_{ik}$ for some vector field w, then

$$P_h \left(B_{lijk} - \frac{\kappa(B)}{n(n-1)} (g_{ij} g_{lk} - g_{ik} g_{lj}) \right) = 0,$$

where $\kappa(B) = B_{rijs} g^{rs} g^{ij}$.

3. Necessary conditions

LEMMA 3.1
If $\overline{M} = M \times_f M'$ is an e.c.s. manifold then $\dim M = q > 1$.

Proof. We shall use the following fact due to Kručkovič

THEOREM 3.1 ([11], p. 116)
A Riemannian space $\overline{V}^n$ admits a solution $k \neq constant$ of the equation

$$\overline{\nabla}_j \overline{\nabla}_i k = \phi \bar{g}_{ij} \tag{12}$$

such that grad k *is non-null vector field if and only if $\overline{V}^n$ is a warped product with one-dimensional base.*

Thus supposing that $q = 1$, we have (12). Differentiating (12) covariantly and alternating the resulting equation, by Ricci identity, we easily obtain

$$k_r \bar{R}^r_{ijk} = \overline{\nabla}_j \phi \bar{g}_{ik} - \overline{\nabla}_k \phi \bar{g}_{ij} .$$

Now, using Lemma 2.4 for $M = \overline{M}$, $B = C$ and $P = \operatorname{grad} \phi$, we have $\overline{\nabla}_j \phi = 0$. Thus $\overline{\nabla}_j \overline{\nabla}_i k = c \cdot \bar{g}_{ij}$, $c = constant$. We assert that $c = 0$. Suppose that $c \neq 0$. Then the manifold $\overline{M}$ admits a vector field v such that $\overline{\nabla}_j v_i = \bar{g}_{ij}$. This equation immediately implies $v_r \bar{R}^r_{hij} = 0$ and $v_r \bar{S}^r_j = 0$ and next $v_r \overline{\nabla}_l \bar{R}^r_{hij} = -\bar{R}_{lhij}$, $v_r \overline{\nabla}_l \bar{S}^r_h = -\bar{S}_{lh}$. Using now the second Bianchi identity, we have $v^r \overline{\nabla}_r \bar{R}_{ijhl} = -2\bar{R}_{ijhl}$ and $v^r \overline{\nabla}_r \bar{S}_{il} = -2\bar{S}_{il}$. Transvection of (9) with v^j leads to $\bar{S} = 0$, a contradiction. Thus the gradient of k is non-null parallel vector field. But in any e.c.s. manifold every parallel vector field must be isotropic ([5], Theorem 11 and [12]). This completes the proof.

PROPOSITION 3.1
Let $\overline{M} = M \times_f M'$ be an e.c.s. manifold. Then M' is of constant curvature. Moreover, if $\dim M' = s > 1$ *then*

$$s(s-1)((n-2) f^2 \partial_e P - f \partial_e Q + f_e Q) = f_e (q - s) \kappa' . \tag{13}$$

Proof. We can assume that $s > 1$. Using (7)c), (5)c) and (6)d), we have

$$\begin{aligned} &f_e((n-2)R'_{\alpha\beta\gamma\delta} - (g'_{\alpha\delta}S'_{\beta\gamma} + g'_{\beta\gamma}S'_{\alpha\delta} - g'_{\alpha\gamma}S'_{\beta\delta} - g'_{\beta\delta}S'_{\alpha\gamma})) \\ &\quad = ((n-2)f^2\partial_e P - f\partial_e Q + f_e Q)G'_{\alpha\beta\gamma\delta}\,. \end{aligned} \tag{14}$$

Contracting (14) with $g'^{\beta\gamma}$, we obtain

$$f_e(qS'_{\alpha\delta} - \kappa' g'_{\alpha\delta}) = ((n-2)f^2\partial_e P - f\partial_e Q + f_e Q)s(s-1). \tag{15}$$

Further contraction with $g'^{\alpha\delta}$ leads to (13). Substituting (13) into (15), we get $S'_{\alpha\delta} = \frac{\kappa'}{s}g'_{\alpha\delta}$ which together with (13) turns (14) into $R' = \frac{\kappa'}{s(s-1)}G'$. This completes the proof.

LEMMA 3.2
Let $\overline{M} = M \times_f M'$ be an e.c.s. manifold. Then $\bar{C}_{abcd} \neq 0$ at every point x of $\overline{M}$.

Proof. Suppose that there exists point $x \in \overline{M}$ at which $\bar{C}_{abcd} = 0$. Using (3)a), we have ($q > 1$ in virtue of Lemma 3.1)

$$(n-2)R_{abcd} = g_{ad}(S_{bc}-sT_{bc})+g_{bc}(S_{ad}-sT_{ad})-g_{ac}(S_{bd}-sT_{bd})-g_{bd}(S_{ac}-sT_{ac}).$$

Contracting this equation with g^{bc}, we get

$$s(S_{ad} + (q-2)T_{ad}) = \kappa - s \cdot tr(T) \tag{16}$$

and, after contraction with g^{ad},

$$(q-s)\kappa = 2s(q-1)tr(T). \tag{17}$$

Now substituting $S'_{\beta\delta} = \frac{\kappa'}{s}g'_{\beta\delta}$, which is an obvious consequence of Proposition 3.1, and (16) into (3)c) and using (17), we have $\bar{C}_{a\beta c\delta} = 0$. Finally, in the same way, using (3)a) and

$$\kappa f + \kappa' + sf((s-1)P - 2tr(T)) = 0 \tag{18}$$

which follows from (4) and (8), we easily obtain $\bar{C}_{\alpha\beta\gamma\delta} = 0$. Thus $\bar{C} = 0$ at x, a contradiction. This completes the proof.

LEMMA 3.3
Let $\overline{M} = M \times_f M'$ be an e.c.s. manifold. Then $\kappa = 0$, $\kappa' = 0$, $tr(T) = 0$, $Q = 0$ and if $s > 1$ also $P = 0$.

Proof. Using (10), (2) and (3), we have

$$\bar{S}_{\alpha\beta}\bar{C}_{abcd} = 0, \qquad \bar{S}_{ab}\bar{C}_{\alpha\beta\gamma\delta} = 0.$$

Thus, in virtue of Lemma 3.2, we get

$$\bar{C}_{\alpha\beta\gamma\delta} = 0, \tag{19}$$

$$\bar{S}_{\alpha\beta} = 0 \tag{20}$$

which, in view of (2)b) and Proposition 3.1, is equivalent to

$$\frac{\kappa'}{s} + Q = 0. \tag{21}$$

If $s = 1$, then $\kappa' = 0$ and $Q = 0$. Using now (1), we get $tr(T) = 0$. Applying now (18), we have $\kappa = 0$.

Consider now the case $s > 1$. Substituting (21) into (18), we obtain

$$\kappa = s \cdot tr(T). \tag{22}$$

Using (19), (21) and Proposition 3.1, we have

$$\frac{\kappa'}{s(s-1)} + fP = 0 \tag{23}$$

which implies $fP = constant$ and further $fP_e = -f_e P$. Substituting the last equality and (21) into (13) we get $(s-1)\kappa' = 0$. Thus $\kappa' = 0$, $P = 0$ (by (23)), $Q = 0$ (by (21)), $tr(T) = 0$ (by (1)) and $\kappa = 0$ (by (22)). This completes the proof.

Proposition 3.2
In every e.c.s. warped product $M \times_f M'$ the tensor $S + (q-2)T$ is parallel and the tensor T is a Codazzi tensor.

Proof. The first assertion is an immediate consequence of (7)e), (5)e), (6)a), (6)d) and

$$\overline{\nabla}_e \bar{S}_{\beta\delta} = 0 \tag{24}$$

which simply follows from $Q = 0$ and $\kappa' = 0$. Using (9) and (24), we have $\overline{\nabla}_\varepsilon \bar{S}_{a\delta} = 0$. Thus (7)f), in view of (5)f) takes the form

$$f^d R_{abcd} = f_b T_{ac} - f_a T_{bc}\,. \tag{25}$$

But this equation, via Ricci identity, is equivalent to $\nabla_c T_{ab} = \nabla_b T_{ac}$. This completes the proof.

Lemma 3.4
Let $\overline{M} = M \times_f M'$ be an e.c.s. manifold. Then $q = \dim M > 2$.

Proof. Suppose that $q = 2$. From $\kappa = 0$ and Proposition 3.2 we obtain $S = 0$, which yields, in view of Lemma 3.3 and Proposition 3.1, $\bar{C}_{a\beta c\delta} = 0$. Now the formula (3)a) reduces to

$$\bar{C}_{1212} = -g_{11}T_{22} - g_{22}T_{11}\,.$$

But $tr(T) = 0$, so $\bar{C}_{1212} = 0$. Taking (19) into account we see that $\bar{C} = 0$, a contradiction.

LEMMA 3.5
Let $\overline{M} = M \times_f M'$ be an e.c.s. manifold. If $S + (q-2)T = 0$ then $q > 3$.

Proof. In virtue of (19), (3)c) and Lemma 3.3, we have $\bar{C}_{\alpha\beta\gamma\delta} = 0 = \bar{C}_{a\beta c\delta}$ and it sufficies to show that if $q = 3$ then $\bar{C}_{abcd} = 0$. But substituting assumed equation into (3)a), in view of (8), we get $\bar{C}_{abcd} = C_{abcd} = 0$ $(q = 3)$.

PROPOSITION 3.3 (cf. [7])
If $\overline{M} = M \times_f M'$ is an e.c.s. manifold then M is conformally symmetric.

Proof. Using Proposition 3.2 and (6)a), we have $\overline{\nabla}_c\bar{S}_{ab} = \frac{n-2}{q-2}\nabla_c S_{ab}$ which together with (5)a) turns (7)a) into

$$\nabla_e R_{abcd} = \frac{1}{q-2}(g_{ad}\nabla_e S_{bc} + g_{bc}\nabla_e S_{ad} - g_{ac}\nabla_e S_{bd} - g_{bd}\nabla_e S_{ac}).$$

But this equation, in virtue of $\kappa = 0$, is equivalent to our assertion.

4. Main results

We are now in a position to prove main results of this paper.

PROPOSITION 4.1
Let $\overline{M} = M \times_f M'$ be an e.c.s. manifold. If $S + (q-2)T \neq 0$ then M is Ricci-recurrent conformally symmetric manifold and $\overline{M}$ is not Ricci-recurrent.

Proof. Using (11), in virtue of (20), we have

$$\bar{F} \cdot \bar{C}_{a\beta c\delta} = 0 \tag{26}$$

which in view of our assumption gives $\bar{F} = 0$, i.e., rank $\bar{S} \leq 1$. On the other hand, (10) implies

$$\bar{S}_{ab}\bar{C}_{c\beta d\delta} = \bar{S}_{ac}\bar{C}_{b\beta d\delta}$$

which can be written in the form

$$(S_{ab} - sT_{ab})(S_{cd} + (q-2)T_{cd}) = (S_{ac} - sT_{ac})(S_{bd} + (q-2)T_{bd}).$$

This means that rank of tensors $S - sT$ and $S + (q-2)T$ is equal to 1 and since $S + (q-2)T$ is parallel so

$$S + (q-2)T = e \cdot v \otimes v, \qquad |e| = 1$$

for some parallel vector field v. Consequently $S = \phi v \otimes v$, ϕ being a function and M is Ricci-recurrent.

The second part of our assertion follows, in a purely algebraic manner from (2), (6) and the above equalities.

THEOREM 4.1
Let $\overline{M}$ be a warped product $M \times_f M'$. Then the following conditions are equivalent:

(i) *$\overline{M}$ is Ricci-recurrent e.c.s. manifold,*

(ii) *M' is flat, M is Ricci-recurrent e.c.s. manifold, $S + (q-2)T = 0$ and* $\operatorname{grad} f$ *is null when* $\dim M' > 1$.

Proof. If $\overline{M}$ is Ricci-recurrent e.c.s. manifold then Proposition 4.1 leads to $S + (q-2)T = 0$. Thus $\bar{S}_{ab} = \frac{n-2}{q-2} S_{ab}$ and because $\bar{S}_{\alpha\beta} = 0$ so $\bar{S}$ and S are simultaneously recurrent and non-parallel. M cannot be conformally flat since $C_{abcd} = \bar{C}_{abcd} \neq 0$. Therefore M is Ricci-recurrent e.c.s. manifold and Proposition 3.1 and Lemma 3.3 imply $R' = 0$ and $P = 0$ if $s > 1$.

Now assume (ii). Since $T = -\frac{1}{q-2} S$, so $\kappa = 0$ implies $tr(T) = 0$ and $Q = 0$. S as well as T are Codazzi tensors. Thus we have (25) which implies $f^c S_{cb} = f^c T_{cb}$ and $f^c S_{cb} - s f^c T_{cb} = 0$. Therefore the only non-zero components of $\overline{\nabla}\bar{S}$ and $\overline{\nabla}\bar{R}$ are those related to $\overline{\nabla}_c \bar{S}_{ab}$, $\overline{\nabla}_e \bar{R}_{abcd}$, $\overline{\nabla}_e \bar{R}_{\alpha\beta c\delta}$ and $\overline{\nabla}_\varepsilon \bar{R}_{abc\delta}$. Using above facts we can easily see that all conditions (7) are satisfied. Thus $\overline{M}$ is conformally symmetric. As in the first part of this proof, we obtain $\bar{C}_{abcd} \neq 0$ and $\overline{\nabla}_c \bar{S}_{ab} \neq 0$ which imply that $\overline{M}$ is e.c.s. manifold. This completes the proof.

THEOREM 4.2
Let $\overline{M}$ be a warped product $M \times_f M'$. Then $\overline{M}$ is an e.c.s. manifold with $\bar{F} \neq 0$ if and only if M' is flat, M is e.c.s. manifold with $F \neq 0$, $S + (q-2)T = 0$ and $\operatorname{grad} f$ *is null when* $\dim M' > 1$.

Proof. If $\overline{M}$ is e.c.s. manifold with $\bar{F} \neq 0$ then (26) implies $\bar{C}_{\alpha\beta c\delta} = 0$ which leads to $S + (q-2)T = 0$. To prove remaining parts of this theorem, we use the same procedure as in the proof of Theorem 4.1.

Now we give two examples of e.c.s. warped products. The first warped product is Ricci-recurrent, the second one is not Ricci-recurrent. We shall use the following

THEOREM 4.3 ([12])
Let M denote the Euclidean q-space ($q \geqslant 4$) endowed with the metric g given by

$$g_{ab} dx^a dx^b = \Phi (dx^1)^2 + k_{\lambda\mu} dx^\lambda dx^\mu + 2 dx^1 dx^q, \tag{27}$$

$$\Phi = (Ak_{\lambda\mu} + a_{\lambda\mu})x^{\lambda}x^{\mu},$$

where $a, b = 1, \dots, q$ *and* $\lambda, \mu = 2, \dots, q-1$ *and* A *is a non-constant function of* x^1 *only,* $[k_{\lambda\mu}]$ *and* $[a_{\lambda\mu}]$ *are non-zero symmetric matrices such that* $[k_{\lambda\mu}]$ *is non-singular and* $k^{\lambda\mu}a_{\lambda\mu} = 0$*,* $[k^{\lambda\mu}]$ *being the reciprocal of* $[k_{\lambda\mu}]$*. Then* M *is an e.c.s. Ricci-recurrent Riemannian manifold.*

Example 4.1
Let M be the Euclidean q-space endowed with the metric g given by (27) with $A = \frac{2}{(x^1)^2}$.

The only Christoffel symbols not identically zero are

$$\Gamma^{\lambda}_{11} = -\frac{1}{2}k^{\lambda\omega}\partial_{\omega}\Phi, \qquad \Gamma^{q}_{11} = \frac{1}{2}\partial_1\Phi, \qquad \Gamma^{q}_{1\gamma} = \frac{1}{2}\partial_{\gamma}\Phi.$$

$S_{11} = (q-2)A$ and all other components of S are identically zero. Taking $f = f(x^1)$ we can easily show that the equation $S + (q-2)T = 0$ is equivalent to $f'' - \frac{(f')^2}{2f} = 2Af$, where $f' = \partial_1 f$. One can easily verify that the function $f(x^1) = (x^1)^4$ satisfies this equation and grad f is null ($g^{11} = 0$). Thus taking above described M and f, and a flat manifold M', via Theorem 4.1, we obtain Ricci-recurrent e.c.s. manifold $\overline{M} = M \times_f M'$.

Remark 4.1
The example satisfying assumptions of Theorem 4.1 can be found in [8]. However the authors of that paper were interested in conformally recurrent or birecurrent manifolds, but $k = \frac{1}{2}$ in (4.4) of [8] leads to a conformally symmetric warped product.

Example 4.2
Let M be as above with $A = \frac{2}{(x^1)^2} - c$, $c = constant$. In this metric the null parallel vector field is of the form $v_i = \delta^1_i$. Taking $f(x^1) = (x^1)^4$, we get

$$S + (q-2)T = c(q-2)v \otimes v.$$

By an elementary calculation we can show that if M' is flat then $\overline{M} = M \times_f M'$ is e.c.s. manifold which is not Ricci-recurrent.

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Locally conformal symplectic structures and their generalizations from the point of view of Lie algebroids

Dedicated to Andrzej Zajtz, with admiration, on his 70th birthday

Abstract. We study locally conformal symplectic structures and their generalizations from the point of view of transitive Lie algebroids. To consider l.c.s. structures and their generalizations we use Lie algebroids with trivial adjoint Lie algebra bundle $M \times \mathbb{R}$ and $M \times \mathfrak{g}$. We observe that important l.c.s's notions can be translated on the Lie algebroid's language. We generalize l.c.s. structures to $\mathfrak{g}$-l.c.s. structures in which we can consider an arbitrary finite dimensional Lie algebra $\mathfrak{g}$ instead of the commutative Lie algebra $\mathbb{R}$.

1. L.c.s. structures from the point of view of Lie algebroids

We study locally conformal symplectic structures and their generalizations from the point of view of transitive Lie algebroids. We recall that an l.c.s. structure on a manifold M is a pair (ω, Ω) of differentiable forms on M such that

(1) ω is a real closed 1-form on M,

(2) Ω is a real non-degenerated 2-form fulfilling the property

$$d\Omega = -\omega \wedge \Omega.$$

From the non-degeneracy of Ω it follows that M has even dimension.

By a *transitive Lie algebroid on a manifold* M ([16]) we mean a system $(A, [\![\cdot,\cdot]\!], \#_A)$ consisting of a vector bundle A over M and mappings

$$[\![\cdot,\cdot]\!] \colon \operatorname{Sec} A \times \operatorname{Sec} A \longrightarrow \operatorname{Sec} A, \qquad \#_A \colon A \longrightarrow TM,$$

such that

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(a) $(\mathrm{Sec}\, A, [\![\cdot,\cdot]\!])$ is a real Lie algebra,

(b) $\#_A$, called an *anchor*, is an epimorphism of vector bundles,

(c) $\mathrm{Sec}\,\#_A\colon \mathrm{Sec}\, A \longrightarrow \mathfrak{X}(M),\ \xi \longmapsto \#_A \circ \xi$, is a homomorphism of Lie algebras,

(d) $[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + (\#_A\circ\xi)(f)\cdot\eta,\ \ \xi,\eta\in \mathrm{Sec}\, A,\ f\in\Omega^0(M) = C^\infty(M)$.

The axiom (c) follows from the remaining ones, see [9], [1].

It follows that $\boldsymbol{g} := \ker \#_A$ is a LAB (Lie algebra bundle), called the *adjoint of* A. The Lie algebra $\boldsymbol{g}_x$ is called the structure Lie algebra at x. The exact sequence

$$0 \longrightarrow \boldsymbol{g} \longrightarrow A \xrightarrow{\#_A} TM \longrightarrow 0$$

is called the *Atiyah sequence* of A, while any splitting $\lambda\colon TM \longrightarrow A$, $\#_A\circ\lambda = \mathrm{id}_{TM}$, is a *connection* in A. The following geometric objects give rise to transitive Lie algebroids:

— Lie groupoids,
— principal fibre bundles,
— vector bundles,
— transversely complete foliations,
— nonclosed Lie subgroups.

Let us remark that differential groupoids (non-transitive, in general), Poisson and Jacobi manifolds as well as any infinitesimal action of a Lie algebra on a manifold produce nontransitive Lie algebroids. The image of the anchor is always an integrable singular (or regular) foliation ([17], [6]) and the restriction of the Lie algebroid to any leaf of this foliation is a transitive Lie algebroid.

To consider l.c.s. structures and their generalizations we use Lie algebroids with trivial adjoint Lie algebra bundle $\boldsymbol{g} = M\times\mathfrak{g}$.

From the general theorem concerning the form of any transitive Lie algebroids (Mackenzie [15], Kubarski [11]) we have:

Each transitive Lie algebroid on M *with a trivial adjoint bundle* $\boldsymbol{g}\cong M\times\mathbb{R}$ *is isomorphic to*

$$A = TM\times\mathbb{R}$$

with $\#_A = \mathrm{pr}_1\colon TM\times\mathbb{R}\longrightarrow TM$ *as the anchor and the bracket* $[\![\cdot,\cdot]\!]$ *in* $\mathrm{Sec}\, A$ *is defined via some flat covariant derivative* ∇ *in* $M\times\mathbb{R}$ *and a 2-form* $\Omega\in\Omega^2(M)$ *fulfilling the Bianchi identity* $\nabla\Omega = 0$ *in the following way*

$$[\![(X,f),(Y,g)]\!] = ([X,Y], \nabla_X g - \nabla_Y f - \Omega(X,Y)).$$

We recall that a covariant derivative ∇ in a vector bundle ξ determines a standard operator $d_\nabla\colon \Omega^*(M;\xi) \longrightarrow \Omega^*(M;\xi)$ and $d_\nabla\theta$ is sometimes denoted by $\nabla\theta$. If ∇ is flat then $(d_\nabla)^2 = 0$ and it determines the cohomology space $H_\nabla(M;\xi)$ in the obvious way.

Each flat covariant derivative in $\boldsymbol{g} = M \times \mathbb{R}$ is of the form

$$\nabla_X f = \partial_X f + \omega(X) \cdot f$$

where ω is a closed differentiable 1-form on M. Then the differential operator d_∇ is denoted rather by d_ω ([7], [8]). We have

$$d_\omega(\theta) = d\theta + \omega \wedge \theta$$

and write $H_\omega(M) := H_{d_\omega}(M)$.

The condition $\nabla\Omega = 0$ is then equivalent to $d\Omega = -\omega \wedge \Omega$.

Hence any transitive Lie algebroid with the trivial adjoint bundle $\boldsymbol{g} = M \times \mathbb{R}$ is determined by the following data:

$$\text{a closed 1-form } \omega \text{ and a 2-form } \Omega \text{ such that } d\Omega = -\omega \wedge \Omega. \qquad (\star)$$

The Lie algebroid obtained in this way will be denoted by

$$(TM \times \mathbb{R}, \omega, \Omega).$$

Lemma 1.1

A connection $\lambda\colon TM \longrightarrow TM \times \mathbb{R}$ in the Lie algebroid $A = (TM \times \mathbb{R}, \omega, \Omega)$ is of the form $\lambda(X) = (X, \eta(X))$ for a 1-form $\eta \in \Omega^1(M)$. The curvature form $\Omega^\lambda(X,Y) = [\![\lambda X, \lambda Y]\!] - \lambda[X,Y]$ of the connection λ is equal to

$$\Omega^\lambda = d_\omega(\eta) - \Omega = d\eta + \omega \wedge \eta - \Omega. \tag{1.1}$$

According to $(\star)$, the pair (ω, Ω) determining the above Lie algebroid is precisely a locally conformal symplectic structure (l.c.s. structure, for short) on the manifold M provided that the 2-form Ω is non-degenerate. Therefore our transitive Lie algebroids $TM \times \mathbb{R}$ determined by (ω, Ω) are natural generalizations of the locally conformal symplectic structures. For an l.c.s. structure (ω, Ω), following $(\star)$, the form Ω represents the cohomology class $[\Omega] \in H^2_\omega(M)$ which is called the *Lichnerowicz class* of the l.c.s. structure (ω, Ω) ([3]). If the 1-form ω is exact the l.c.s. structure is called *globally* conformal symplectic structure. The property that an l.c.s. structure is global can be equivalently expressed in the language of Lie algebroids ([10], [14]). For this purpose we recall that a transitive Lie algebroid $(A, [\![\cdot,\cdot]\!], \#_A)$ is called *invariantly oriented* ([13]) if there is specified a non-singular cross-section ε of the bundle $\bigwedge^n \boldsymbol{g}$, $\boldsymbol{g} := \ker \#_A$ and $n = \operatorname{rank} \boldsymbol{g}$, which is invariant with respect to the adjoint representation of A in $\bigwedge^n \boldsymbol{g}$, equivalently, if $\boldsymbol{g}$ is orientable and the modular class of the Lie algebroid is zero ([5], [14]). Let us remark that for a transitive Lie algebroid the modular class is equal to the characteristic class of the top-power of the adjoint representation ad_A. The structure Lie algebras $\boldsymbol{g}_x$ of the invariantly oriented Lie algebroid are unimodular.

A cross-section ε of the bundle $\bigwedge^n \boldsymbol{g}$ is invariant if and only if, in any open subset $U \subset M$ on which ε is of the form $\varepsilon_{|U} = (h_1 \wedge \ldots \wedge h_n)_{|U}$, $h_i \in \mathrm{Sec}\,\boldsymbol{g}$, we have, for all $\xi \in \mathrm{Sec}\, A$,

$$\sum_{i=1}^{n} (h_1 \wedge \ldots \wedge [\![\xi, h_i]\!] \wedge \ldots \wedge h_n)_{|U} = 0.$$

In the case $A = (TM \times \mathbb{R}, \omega, \Omega)$ we have $n = 1$ and $\boldsymbol{g} = M \times \mathbb{R}$ and a positive function $\varepsilon \in C^\infty(M) = \mathrm{Sec}(M \times \mathbb{R})$ is invariant if and only if ε is ∇-constant, $\nabla\varepsilon = 0$ ([13, Lemma 6.2.1]). The condition $\nabla\varepsilon = 0$ is equivalent to $\omega = d(-\ln(\varepsilon))$.

Theorem 1.1

Let (ω, Ω) be an l.c.s. structure on an arbitrary m-dimensional connected manifold (oriented or not). The following conditions are equivalent:

(a) *the l.c.s. structure (ω, Ω) is globally conformal symplectic structure (i.e., $[\omega] = 0$),*

(b) *the associated Lie algebroid $A = (TM \times \mathbb{R}, \omega, \Omega)$ is invariantly oriented,*

(c) $H^{m+1}_{\partial^{or}_A, c}(A; or(M)) \neq 0$,

(d) $H^{m+1}_{\partial^{or}_A, c}(A; or(M)) = \mathbb{R}$, *and the pairing*

$$H^j(A) \times H^{m+1-j}_{\partial^{or}_A, c}(A; or(M)) \longrightarrow H^{m+1}_{\partial^{or}_A, c}(A; or(M)) \cong \mathbb{R}$$

is non-degenerate, i.e., $H^j(A) \cong \left(H^{m+1-j}_{\partial^{or}_A, c}(A; or(M))\right)^*$.

Proof. (a) $\Longleftrightarrow$ (b) see [10], (b) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (d) see [14].

Remark 1.1

(1) For an orientable manifold M the conditions (c) and (d) are equal to:

(c′) $H^{m+1}_c(A) \neq 0$,

(d′) $H^{m+1}_c(A) = \mathbb{R}$, and the pairing

$$H^j(A) \times H^{m+1-j}_c(A) \longrightarrow H^{m+1}_c(A) \cong \mathbb{R}$$

is non-degenerate, i.e., $H^j(A) \cong (H^{m+1-j}_c(A))$.

(2) ∂_A^{or} is the canonical representation of A in the orientation bundle $or(M)$, $(\partial_A^{or})_\gamma(\sigma) = (\partial^{or})_{\#_A(\gamma)}(\sigma)$, $\gamma \in A$, $\sigma \in \Gamma(or(M))$. ∂^{or} is the canonical flat structure of the orientation bundle $or(M)$ ([4]).

(3) Each representation ∇ of a Lie algebroid A in a vector bundle ξ (i.e., a homomorphism of a Lie algebroid A in the Lie algebroid $A(\xi)$ of the vector bundle ξ ([12], [15])) determines a standard differential operator $d_\nabla \colon \Omega(A;\xi) \longrightarrow \Omega(A;\xi)$ and $H_\nabla(A;\xi)$ is the space of cohomology of the complex $(\Omega(A;\xi), d_\nabla)$. Local trivializations of $A(\mathfrak{f})$ are constucted in the following way: Let $\psi\colon U\times V \longrightarrow p^{-1}[U] = \mathfrak{f}_{|U}$ be a local trivialization of a vector bundle $\mathfrak{f}$; V is the typical fibre. Consider the trivial Lie algebroid $TU \times \operatorname{End}(V)$. For a cross-section $\sigma \in \operatorname{Sec}\mathfrak{f}$, denote by σ_ψ the V-valued function $U \ni x \longmapsto \psi_x^{-1}(\sigma(x)) \in V$. The mapping

$$\bar\psi\colon TU \times \operatorname{End}(V) \longrightarrow A(\mathfrak{f})_{|U}$$
$$\bar\psi(v,a)(\sigma) = \psi_x(v(\sigma_\psi) + a(\sigma_\psi(x))),$$

$(v \in T_xU,\ x\in U,\ a \in \operatorname{End}(V),\ \sigma \in \operatorname{Sec}\mathfrak{f})$ is an isomorphism of Lie algebroids ([12]).

(4) The associated Lie algebra bundle of the considered Lie algebroid $A = (TM\times\mathbb{R},\omega,\Omega)$ is the trivial line bundle $\boldsymbol{g} = M\times\mathbb{R}$. Therefore, the top group of cohomology $H^{m+1}_{\partial_A^{or},c}(A; or(M))$ can be written (analogously to real coefficients, see [10]) as follows

$$\begin{aligned} H^{m+1}_{\partial_A^{or},c}(A; or(M)) &= H^m_{d_{\partial^{-\omega}}\otimes\partial^{or}}(M; or(M)) \\ &= H^m_{(\partial^{or})^{-\omega}}(M; or(M)). \end{aligned}$$

Then the equivalence (a) $\Longleftrightarrow$ (c) follows trivially, since

$$H^m_{(\partial^{or})^{-\omega}}(M; or(M)) \neq 0 \iff [-\omega] = 0,$$

see [14].

Two l.c.s. structures (ω,Ω) and (ω',Ω') on a manifold M are called *conformally equivalent* if

$$\Omega' = \frac{1}{a}\Omega, \qquad \omega' = \omega + \frac{da}{a},$$

for a nowhere vanishing function a on M (non-singular for short).

If two l.c.s. structures (ω',Ω') and (ω,Ω) on a manifold M are conformally equivalent then the associated Lie algebroids $A' = (TM\times\mathbb{R},\omega',\Omega')$ and $(TM\times\mathbb{R},\omega,\Omega)$ are isomorphic via the mapping

$$H\colon (TM\times\mathbb{R},\omega',\Omega') \longrightarrow (TM\times\mathbb{R},\omega,\Omega)$$
$$H(X,f) = (X, a\cdot f)$$

where $a \in C^\infty(M)$ is a non-singular smooth function. The isomorphism $H\colon A' \longrightarrow A$ of the above form will be called a *conformal isomorphism.*

We must add that the general form of a homomorphism $H\colon TM \times \mathbb{R} \longrightarrow TM \times \mathbb{R}$ of vector bundles commuting with anchors $\#_A = pr_1$ is as follows

$$H(X,f) = H_{\eta,a}(X,f) := (X, \eta(X) + a \cdot f), \tag{$\star\star$}$$

for $\eta \in \Omega^1(M)$ and $a \in C^\infty(M)$.

Proposition 1.1

(A) *The following conditions are equivalent:*

(1) $H_{\eta,a}$ *is a homomorphism of Lie algebroids,*

(2) (a) $\nabla\eta = \Omega - a \cdot \Omega'$,
    (b) $\nabla_X(a \cdot f) = a \cdot \nabla'_X f$,

(3) (a) $d_\omega(\eta) = d\eta + \omega \wedge \eta = \Omega - a \cdot \Omega'$,
    (b) $a \cdot (\omega' - \omega) = da$.

The homomorphism $H_{\eta,a}$ is an isomorphism of Lie algebroids if and only if a is non-singular. Conditions (1), (2), (3) are then equivalent to

(4) (a) $\Omega' = \frac{1}{a} \cdot (\Omega - d_\omega(\eta))$,
    (b) $\omega' = \omega + d(\ln|a|)$.

(B) *For an arbitrary Lie algebroid $A' = (TM \times \mathbb{R}, \omega', \Omega')$ and data (η, a) where $\eta \in \Omega^1(M)$ and a is a non-singular function, the differential forms $\omega = \omega' - d(\ln|a|)$, $\Omega = a \cdot \Omega' + d_\omega(\eta)$ fulfil the condition $d\Omega = -\omega \wedge \Omega$, i.e., the data (ω, Ω) determines a Lie algebroid $A = (TM \times \mathbb{R}, \omega, \Omega)$ and $H_{\eta,a}\colon A' \longrightarrow A$ given by $(\star\star)$ is an isomorphism of Lie algebroids.*

Proof. Easy calculation.

Clearly $H_{\eta',a'} \circ H_{\eta,a} = H_{\eta' + a'\cdot\eta, a'\cdot a}$, $(H_{\eta,a})^{-1} = H_{-\frac{\eta}{a},\frac{1}{a}}$. In particular,

$$H_{\eta,a} = H_{\eta,1} \circ H_{0,a},$$

see the diagram

$$\begin{array}{ccc} A' = (TM \times \mathbb{R}, \omega', \Omega') & \xrightarrow{\quad H_{\eta,a} \quad} & (TM \times \mathbb{R}, \omega, \Omega) = A \\ & {\scriptstyle H_{0,a}} \searrow \qquad \swarrow {\scriptstyle H_{\eta,1}} & \\ & (TM \times \mathbb{R}, \omega, a \cdot \Omega') & \end{array}$$

It means that if A' is isomorphic to A then there exists a Lie algebroid $A'' = (TM \times \mathbb{R}, \omega, \Omega'')$, $\Omega'' = a \cdot \Omega'$, conformally isomorphic to A', i.e., such that $[A], [A''] \in Opext(TM, \nabla, M \times \mathbb{R})$ – the set of isomorphic classes of Lie algebroids having the same representation ∇ (a flat covariant derivative ∇).

Let (ω', Ω') and (ω, Ω) be l.c.s. structures. We observe that the isomorphism $H_{\eta,a}\colon A' \longrightarrow A$ given by $(\star\star)$ is equivalent to conformal equivalence of the associated l.c.s. structures if and only if $\eta = 0$.

How can we formulate the problem of existence of l.c.s. structures? We have the simple

Proposition 1.2

Any Lie algebroid $A' = (TM \times \mathbb{R}, \omega', \Omega')$ is isomorphic to $A = (TM \times \mathbb{R}, \omega, \Omega)$ with Ω non-degenerate (i.e. (ω, Ω) is an l.c.s. structure) if and only if there exists in A' a connection for which the curvature tensor is non-degenerate.

Proof. Let $H_{\eta,a}\colon A' \longrightarrow A$ be an isomorphism of Lie algebroids

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M \times \mathbb{R} & \longrightarrow & (TM \times \mathbb{R}, \omega', \Omega') & \underset{\lambda'}{\overset{\longleftarrow}{\longrightarrow}} & TM & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle H^+_{\eta,a}} & & \downarrow{\scriptstyle H_{\eta,a}} & & \downarrow & & \\ 0 & \longrightarrow & M \times \mathbb{R} & \longrightarrow & (TM \times \mathbb{R}, \omega, \Omega) & \underset{\lambda}{\overset{\longleftarrow}{\longrightarrow}} & TM & \longrightarrow & 0 \end{array} \tag{1.2}$$

$H^+_{\eta,a}(f) = a \cdot f$. For arbitrary connections λ' and λ in A' and A, respectively, such that $H_{\eta,a} \circ \lambda' = \lambda$ we have the following equality for curvature tensors

$$\Omega^{\lambda} = H^+_{\eta,a} \circ \Omega^{\lambda'}.$$

Therefore, if Ω is nondegenerate and λ' is a connection such that $H_{\eta,a} \circ \lambda' = \lambda$ where $\lambda(v) = (v, 0)$, then $\Omega^{\lambda} = -\Omega$ (see Lemma 1.1) and, clearly, $\Omega^{\lambda'}$ is non-degenerate.

Conversely, if $\lambda'(X) = (X, \eta(X))$ is any connection in A' such that $\Omega^{\lambda'}$ is non-degenerate, then $H_{-\eta,1}$ is an isomorphism of A' on $A := \big(TM \times \mathbb{R}, \omega', -\Omega^{\lambda'}\big)$ (see (1.1)) and $\big(\omega', -\Omega^{\lambda'}\big)$ is an l.c.s. structure.

So, the problem of existing of l.c.s. structures can be precisely formulated as follows:

Problem 1.1

We introduce into the class of pairs (ω, Ω) fulfilling $(\star)$, i.e., $d\Omega = -\omega \wedge \Omega$, the equivalence relation

r) $(\omega', \Omega') \approx (\omega, \Omega) \equiv$ the Lie algebroids $A' = (TM \times \mathbb{R}, \omega', \Omega')$ and $A = (TM \times \mathbb{R}, \omega, \Omega)$ are isomorphic, i.e., there exists $\eta \in \Omega^1(M)$ and $a \in C^\infty(M)$, $a(x) \neq 0$ for all $x \in M$, such that (4a), (4b) hold: (4a) $\Omega' = \frac{1}{a}(\Omega - d\eta - \omega \wedge \eta)$, (4b) $\omega' = \omega + \frac{da}{a}$.

Let $\dim M$ be even. We can ask: Does there in every (in given) equivalence class $[(\omega', \Omega')]$ exist (ω, Ω) being an l.c.s. structure; equivalently, does there in the Lie algebroid $A' = (TM \times \mathbb{R}, \omega', \Omega')$ exist a connection with non-degenerate curvature tensor, i.e., equivalently, does there exist a 1-form $\eta \in \Omega^1(M)$ such that $d\eta + \omega \wedge \eta - \Omega$ is non-degenerate.

This problem has a local solution, see Proposition 2.5 below for more general situations.

We must add that for a fixed closed form ω, i.e., a flat covariant derivative $\nabla_X f = \partial_X f + \omega(X) \cdot f$ in the trivial bundle $M \times \mathbb{R}$, the classification of Lie algebroids of the form $(TM \times \mathbb{R}, \omega, \cdot)$ up to isomorphism is as follows: for the class of isomorphic Lie algebroids $Opext(TM, \nabla, M \times \mathbb{R})$ we have ([15])

$$Opext(TM, \nabla, M \times \mathbb{R}) \cong H^2_\nabla(M; \mathbb{R}), \qquad [(TM \times \mathbb{R}, \omega, \Omega)] \longmapsto [\Omega].$$

A. Banyaga ([3]) gives examples of l.c.s. structures (ω, Ω) such that the Lichnerowicz class $[\Omega]$ is not trivial, $[\Omega] \neq 0$. For deformations and equivalence of l.c.s. structures see [2].

To sum up we see that important l.c.s's notions can be translated into the Lie algebroid's language. We have the following table:

l.c.s.	**Lie algebroid**
$(M, \omega, \Omega) \equiv$ ω is closed, $d\Omega = -\omega \wedge \Omega$.	$A = TM \times \mathbb{R}$ with anchor $\#_A = pr_1 \colon TM \times \mathbb{R} \longrightarrow TM$, with bracket $[\![(X, f), (Y, g)]\!] = ([X, Y], \nabla_X g - \nabla_Y f - \Omega(X, Y))$ where $\nabla_X g = \partial_X g + \omega(X) \cdot g$ ∇ is flat and $\nabla\Omega = 0$.
Globally c.s. $\equiv$ ω is exact.	A is invariantly oriented.
Two l.c.s. structures (ω', Ω') and (ω, Ω) on M are conformally equivalent $\equiv$ $\omega' = \omega + \frac{da}{a}$, $\Omega' = \frac{1}{a}\Omega$.	The corresponding Lie algebroids are isomorphic via $H_{0,a} \colon TM \times \mathbb{R} \longrightarrow TM \times \mathbb{R}$, $H(X, f) = (X, a \cdot f)$, $a \in C^\infty(M)$, $a(x) \neq 0$ for all x.

2. Generalizations: $\mathfrak{g}$-l.c.s. structures and Lie algebroids

We generalize l.c.s. structures to $\mathfrak{g}$-l.c.s. structures in which we can consider an arbitrary finite dimensional Lie algebra $\mathfrak{g}$ instead of the commutative Lie algebra $\mathbb{R}$. From the general theorem on the form of Lie algebroids, mentioned above, we have ([15], [11]):

THEOREM 2.1
Each transitive Lie algebroid with a trivial adjoint bundle of Lie algebras $M\times\mathfrak{g}$ *is isomorphic to* $TM\times\mathfrak{g}$ *with* $\#_A=\mathrm{pr}_1\colon TM\times\mathfrak{g}\longrightarrow TM$ *as the anchor and the bracket*

$$[\![(X,\sigma),(Y,\eta)]\!]=([X,Y],\nabla_X\eta-\nabla_Y\sigma+[\sigma,\eta]-\Omega(X,Y))$$

in Sec A *is defined via the following data* (∇,Ω)*: a covariant derivative* ∇ *in the trivial vector bundle* $M\times\mathfrak{g}$ *and a 2-form* $\Omega\in\Omega^2(M;\mathfrak{g})$ *fulfilling the conditions:*

(1) $R^{\nabla}_{X,Y}\sigma=-[\Omega(X,Y),\sigma]$, R^{∇} *being the curvature tensor of* ∇,

(2) $\nabla_X[\sigma,\eta]=[\nabla_X\sigma,\eta]+[\sigma,\nabla_X\eta]$, $\sigma,\eta\in C^\infty(M;\mathfrak{g})$,

(3) $\nabla\Omega=0$.

The Lie algebroid obtained in the above way via the data (∇,Ω) fulfilling (1)-(3) above will be denoted here by

$$(TM\times\mathfrak{g},\nabla,\Omega). \tag{2.3}$$

The form $-\Omega$ is the curvature form of the connection $\lambda\colon TM\longrightarrow TM\times\mathfrak{g}$, $\lambda(v)=(v,0)$, in this Lie algebroid $(TM\times\mathfrak{g},\nabla,\Omega)$.

$$0\longrightarrow M\times\mathfrak{g}\longrightarrow TM\times\mathfrak{g}\underset{\overset{\longleftarrow}{\lambda}}{\longrightarrow}TM\longrightarrow 0.$$

More generally, the curvature form of an arbitrary connection $\lambda(X)=(X,\eta(X))$, $\eta\in\Omega^1(M;\mathfrak{g})$, is given by

$$\Omega^{\lambda}(X,Y)=(\nabla\eta)(X,Y)+[\eta X,\eta Y]-\Omega(X,Y). \tag{2.4}$$

We write the covariant derivative ∇ in the trivial bundle $M\times\mathfrak{g}$ in the form

$$\nabla_X\sigma=\partial_X\sigma+\omega(X)(\sigma)$$

for a 1-form $\omega\in\Omega^1(M;\mathrm{End}\,\mathfrak{g})$. Then $\nabla\theta=d_\nabla\theta=d_{dR}\theta+\omega\wedge\theta$. The curvature tensor R^∇ of ∇ is equal to

$$R^{\nabla}_{X,Y}\sigma=d\omega(X,Y)(\sigma)+[\omega(X),\omega(Y)](\sigma).$$

Theorem 3.31, Chapter IV from [15] classifies all transitive Lie algebroids having a given coupling Ξ. For the Lie algebroid (2.3) we have,

$$\Xi\colon TM\longrightarrow \mathrm{OutDo}[(M\times\mathfrak{g})]=TM\times\mathrm{Der}(\mathfrak{g})/\mathrm{ad}(\mathfrak{g}),$$
$$\Xi(v)=(v,[a_v]),$$

where $a_v(\sigma) = \nabla_v \tilde{\sigma} - v(\tilde{\sigma})$, $\tilde{\sigma} \colon M \longrightarrow \mathfrak{g}$, $\tilde{\sigma}(x) \equiv \sigma \in \mathfrak{g}$,

$$Opext(TM, \Xi, M \times \mathfrak{g}) \cong H^2_{\rho^\Xi}(M, Z\mathfrak{g}) \tag{2.5}$$

where $Z\mathfrak{g}$ is the center of $\mathfrak{g}$ and $\rho^\Xi \colon TM \longrightarrow TM \times \mathrm{End}(Z\mathfrak{g})$ is the central representation $\rho^\Xi(v) = (v, a_v)$ for Ξ.

Proposition 2.1
The conditions (1)-(3) *characterizing the data* (∇, Ω) *determining the Lie algebroid* $(TM \times \mathfrak{g}, \nabla, \Omega)$ *can be expressed as follows*

— *the condition* (1) *is equivalent to*

$$d\omega(X,Y)(\sigma) + [\omega(X), \omega(Y)](\sigma) = -[\Omega(X,Y), \sigma],$$

— *the condition* (2) *is equivalent to* $\omega_x \in \mathrm{Der}(\mathfrak{g})$, *i.e.,* ω_x *is a differentiation of the Lie algebra* $\mathfrak{g}$,

— *the condition* (3) *is equivalent to*

$$d\Omega = -\omega \wedge \Omega$$

(the values of forms ω *and* Ω *are multiplied with respect to the 2-linear homomorphism* $\mathrm{End}\,\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, $(a, \sigma) \longmapsto a(\sigma)$.

Definition 2.1
The pair (∇, Ω) determining the above Lie algebroid $(TM \times \mathfrak{g}, \nabla, \Omega)$ will be called $\mathfrak{g}$-*locally conformal symplectic structure* ($\mathfrak{g}$-l.c.s. structure, for short) on the manifold provided that the 2-form Ω is *non-degenerate* in the following sense: for each point $x \in M$ the mapping

$$T_xM \longrightarrow L(T_xM, \mathfrak{g}), \qquad v \longmapsto \Omega_x(v, \cdot), \tag{2.6}$$

is a monomorphism.

It is easy to see that if the mapping (2.6) is a monomorphism at a point x then it is a monomorphism at every point near x.

We notice that if $\dim \mathfrak{g} \geq 2$ there is no dimensional obstructions to the existence of an non-degenerate tensors:

Lemma 2.1
For arbitrary vector spaces V *and* $\mathfrak{g}$ *such that* $\dim \mathfrak{g} \geq 2$ *there exists a 2-linear skew-symmetric non-degenerate tensor* $\Omega \in \Omega^2(V; \mathfrak{g})$.

Proof. Let $(e_1, \ldots, e_n)$ be a basis of $\mathfrak{g}$. If $\dim V$ is even, then there exists a real 2-linear skew-symmetric non-degenerate tensor, say Ω_0. The form $\Omega := \Omega_0 \cdot e_1 \in \Omega^2(V; \mathfrak{g})$ is non-degenerate. If $\dim V = 2k+1$ and $(v_1, \ldots, v_{2k+1})$ is a basis of V and $u^1, \ldots, u^{2k+1}$ is a dual basis, then put

$$\begin{aligned}\Omega_0 &= u^1 \wedge u^2 + \cdots + u^{2k-1} \wedge u^{2k},\\ \Omega_1 &= u^{2k} \wedge u^{2k+1}.\end{aligned}$$

The form $\Omega := \Omega_0 \cdot e_1 + \Omega_1 \cdot e_2$ is non-degenerate.

Definition 2.2
A $\mathfrak{g}$-l.c.s. structure is called *globally* conformal symplectic structure if the associated Lie algebroid $(TM \times \mathfrak{g}, \nabla, \Omega)$ is invariantly oriented.

Theorem 2.2
Let (∇, Ω) be a $\mathfrak{g}$-l.c.s. structure on an arbitrary m-dimensional connected manifold (oriented or not), $\dim \mathfrak{g} = n$. *Write $\nabla_X \sigma = \partial_X \sigma + \omega(X)(\sigma)$ for $\omega \in \Omega^1(M; \operatorname{End} \mathfrak{g})$. The following conditions are equivalent:*

(a) *The Lie algebroid $(TM \times \mathfrak{g}, \nabla, \Omega)$ is invariantly oriented (i.e., (∇, Ω) is a globally conformal symplectic structure),*

(b) $\mathfrak{g}$ *is unimodular and* $\operatorname{tr} \omega$ *is an exact form. [Let $e_1, \ldots, e_n$ be a basis of $\mathfrak{g}$. For a non-singular function $f \in C^\infty(M)$ the element $\varepsilon = f \cdot e_1 \wedge \ldots \wedge e_n$ is an invariant cross-section if and only if* $\operatorname{tr} \omega = d(-\ln |f|)$*],*

(c) *the modular class of $A = (TM \times \mathfrak{g}, \nabla, \Omega)$ is zero, $m_A = 0$,*

(d) $H^{m+n}_{\partial^{or}_A, c}(A; or(M)) \neq 0$,

(e) $H^{m+n}_{\partial^{or}_A, c}(A; or(M)) = \mathbb{R}$, *and the pairing*

$$H^j(A) \times H^{m+n-j}_{\partial^{or}_A, c}(A; or(M)) \longrightarrow H^{m+n}_{\partial^{or}_A, c}(A; or(M)) \cong \mathbb{R}$$

is non-degenerate, i.e., $H^j(A) \cong \left(H^{m+n-j}_{\partial^{or}_A, c}(A; or(M))\right)^*$.

Proof. (a) $\Longleftrightarrow$ (b) The very easy proof will be omitted. (a) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (d) $\Longleftrightarrow$ (e) see [14].

Theorem 2.3
If the Lie algebra $\mathfrak{g}$ is semisimple, then each $\mathfrak{g}$-l.c.s. structure is globally c.s. structure.

Proof. According to Theorem 7.2.3 from [11] (see independently (2.5)) for the trivial LAB $\boldsymbol{g} = M \times \mathfrak{g}$ there exists exactly one, up to isomorphism, a transitive Lie algebroid A with the adjoint LAB $\boldsymbol{g} = M \times \mathfrak{g}$. Therefore, A must be isomorphic to the trivial Lie algebroid $A = TM \times \mathfrak{g}$ with the data $(\partial, 0)$. This Lie algebroid is invariantly oriented: $\varepsilon(x) \equiv \varepsilon_o \in \bigwedge^n \mathfrak{g}$ is an invariant cros-section.

Let $(e_1, \ldots, e_n)$ be a basis of $\mathfrak{g}$ with the structure constants c_{ij}^k. The covariant derivative ∇ determines a matrix of 1-forms $\omega_i^j \in \Omega^1(M)$ by

$$\nabla_X e_i = \sum_j \omega_i^j(X) e_j \,.$$

Analogously we have a collection of 2-forms Ω^j by

$$\Omega_{X,Y} = \sum_j \Omega_{X,Y}^j e_j.$$

We interpret the data (1)-(3) concerning (∇, Ω) in the terms of the matrix ω_i^j and the collection Ω^j and the structure constants c_{ij}^k.

Proposition 2.2

(A) *The conditions* (1)-(3) *characterizing the data* (∇, Ω) *determining the Lie algebroid* $(TM \times \mathfrak{g}, \nabla, \Omega)$ *can be expressed as follows.*

— *The condition* (1) *is equivalent to*

$$-\sum_j \Omega_{X,Y}^j \cdot c_{j,i}^r = d\omega_i^r(X,Y) - \sum_j \left(\omega_i^j(X)\omega_j^r(Y) - \omega_i^j(Y)\omega_j^r(X)\right),$$

— *the condition* (2) *is equivalent to*

$$\sum_k c_{ij}^k \cdot \omega_k^r(X) = \sum_k \left(\omega_i^k(X) c_{kj}^r - \omega_j^k(X) c_{ki}^r\right),$$

— *the condition* (3) *is equivalent to* $d\Omega^j = -\sum_i \Omega^i \wedge \omega_i^j$.

(B) *For an abelian Lie algebra* $\mathfrak{g} = \mathbb{R}^n$ *(i.e.,* $c_{ij}^k = 0$*) the conditions above are equivalent to*

— $d\omega(X,Y) = -\omega(X) \circ \omega(Y) + \omega(Y) \circ \omega(X)$
(equivalently $d\omega_i^r(X,Y) = \sum_j \left(\omega_i^j(X)\omega_j^r(Y) - \omega_i^j(Y)\omega_j^r(X)\right)$*),*

— $d\Omega^j = -\sum_i \Omega^i \wedge \omega_i^j$.

Two $\mathfrak{g}$-l.c.s. structures (∇', Ω'), (∇, Ω) on a manifold M will be called *$\mathfrak{g}$-conformally equivalent* if the associated Lie algebroids are isomorphic via an isomorphism of the special form (called $\mathfrak{g}$-*conformal*) $H(X, \sigma) = (X, a(\sigma))$ for some mapping $a\colon M \longrightarrow \operatorname{Aut}(\mathfrak{g})$. Then the equivalent relations between the data (∇, Ω) and (∇', Ω') are as follows:

— $\Omega' = a^{-1} \circ \Omega$,

— $a \circ \nabla'_X(\sigma) = \nabla_X(a \circ \sigma)$.

We use the notation $a \circ \sigma$ for the cross-section defined by $(a \circ \sigma)_x = a_x(\sigma_x)$.

Writing ∇' and ∇ with using 1-forms $\omega', \omega \in \Omega^1(M; \operatorname{End}\mathfrak{g})$ (as above) the last condition can be equivalently written in the form

$$\omega(X) \circ a = -\partial_X a + a \circ \omega'(X).$$

In the terms of the matrices $\omega_i'^j$ and ω_i^j this condition is equivalent to

$$\sum_j \omega_i'^j(X) \cdot a_j^k - \sum_j a_i^j \cdot \omega_j^k(X) = \partial_X(a_i^k).$$

The general form of a homomorphism $H\colon TM \times \mathfrak{g} \longrightarrow TM \times \mathfrak{g}$ commuting with anchors pr_1 is as follows

$$H(X, \sigma) = H_{\eta, a}(X, \sigma) = (X, \eta(X) + a \circ \sigma) \tag{2.7}$$

for $\eta \in \Omega^1(M; \mathfrak{g})$, $a \in C^\infty(M, \operatorname{End}\mathfrak{g})$. Consider two Lie algebroids

$$A' = (TM \times \mathfrak{g}, \nabla', \Omega') \qquad \text{and} \qquad A = (TM \times \mathfrak{g}, \nabla, \Omega)$$

Proposition 2.3

The following conditions are equivalent.

(1) *H is a homomorphism of Lie algebroids $H\colon A' \longrightarrow A$,*

(2) (a) *a_x is a homomorphism of Lie algebras,*

(b) $(\nabla\eta)(X, Y) + [\eta(X), \eta(Y)] = (\Omega - a\Omega')(X, Y)$,

(c) $a \circ \nabla'_X \sigma = \nabla_X(a \circ \sigma) + [\eta(X), a \circ \sigma]$,

(3) *For the basis $e_1, \ldots, e_n$ and the matrix a_i^j defined by $a(e_i) = \sum_j a_i^j(e_j)$*

(a) *a_x is a homomorphism of Lie algebras,*

(b) $d\eta^k(X, Y) - \left(\sum_i \eta^i \wedge \omega_i^k\right)(X, Y) + \sum_{i,j} \eta^i(X) \cdot \eta^j(Y) \cdot c_{ij}^k$
$= \left(\Omega^k - \sum_i \Omega'^i \cdot a_i^k\right)(X, Y)$,

(c) $\sum_j \omega_i'^j(X) \cdot a_j^k = \sum_j a_i^j \cdot \omega_j^k(X) + \partial_X a_i^k + \sum_{j,s} \eta^j(X) \cdot a_i^s \cdot c_{js}^k$.

The homomorphism $H_{\eta,a}$ is an isomorphism of Lie algebroids if and only if a_x is an isomorphism of Lie algebras.

Proof. Straightforward calculations.

If (∇', Ω') and (∇, Ω) are $\mathfrak{g}$-l.c.s. structures and A' and A are corresponding Lie algebroids, then the isomorphism $H_{\eta,a}$ given by (2.7) is equivalent to conformal equivalence of the associated $\mathfrak{g}$-l.c.s. structures (∇', Ω') and (∇, Ω) if and only if $\eta = 0$.

Analogously, we can put the problem of existence of l.c.s. structures. We have firstly the simple

PROPOSITION 2.4
Any Lie algebroid $A' = (TM \times \mathfrak{g}, \nabla', \Omega')$ is isomorphic to $A = (TM \times \mathfrak{g}, \nabla, \Omega)$ with Ω non-degenerate (i.e., (∇, Ω) is a $\mathfrak{g}$-l.c.s. structure) if and only if there exists in A' a connection for which the curvature tensor is non-degenerate.

PROBLEM 2.1
We introduce into the class of pairs (∇, Ω) fulfilling (1)-(3) from Theorem 2.1, the equivalence relation

rg) $(\nabla', \Omega') \approx (\nabla, \Omega) \equiv$ the Lie algebroids $A' = (TM \times \mathfrak{g}, \nabla', \Omega')$ and $A = (TM \times \mathfrak{g}, \nabla, \Omega)$ are isomorphic,

i.e., there exist $\eta \in \Omega^1(M; \mathfrak{g})$, $a \in C^\infty(M, \operatorname{Aut}\mathfrak{g})$ such that (2b) and (2c), from Proposition 2.3 holds: $(\nabla\eta)(X,Y) + [\eta(X), \eta(Y)] = (\Omega - a\Omega')(X,Y)$ and $a \circ \nabla'_X \sigma = \nabla_X(a \circ \sigma) + [\eta(X), a \circ \sigma]$.

We can ask: does there in every (in given) equivalence class $[(\nabla', \Omega')]$ exist (∇, Ω) being a $\mathfrak{g}$-l.c.s. structure; equivalently, does there in the Lie algebroid $A' = (TM \times \mathfrak{g}, \nabla', \Omega')$ exist a connection with non-degenerate curvature tensor, i.e., equivalently, does there exists a 1-form $\eta \in \Omega^1(M; \mathfrak{g})$ such that the 2-form $(\nabla\eta)(X,Y) + [\eta X, \eta Y] - \Omega(X,Y)$ is a non-degenerate.

For $\mathfrak{g} = \mathbb{R}$ we obtain Problem 1.1 and we need to assume that $\dim M$ is even.

PROPOSITION 2.5
The above problem has a local solution.

Proof. Let $a\colon T_{x_0}M \times T_{x_0}M \longrightarrow \mathfrak{g}$ be an arbitrary non-degenerate 2-linear skew-symmetric tensor (for $\dim\mathfrak{g} \geq 2$ see Lemma 2.1). We can locally extend $\Omega_{x_0} + a$ to a closed 2-form Φ and find by the Poincaré lemma a 1-form η such that $d\eta = \Phi$; therefore that $(d\eta)_{x_0} = \Omega_{x_0} + a$. Slightly modifying η we can assume that $\eta_{x_0} = 0$, indeed, locally there is a closed 1-form θ such that $\theta_{x_0} = \eta_{x_0}$, so $\eta - \theta$ is zero at x_0 and $d(\eta - \theta)_{x_0} = (d\eta)_{x_0}$. Clearly $(\nabla\eta)_{x_0}(X,Y) + [\eta_{x_0}X, \eta_{x_0}Y] - \Omega_{x_0}(X,Y) = a(X,Y)$ so the curvature tensor Ω^λ of the connection $\lambda(X) = (X, \eta(X))$, see (2.4), is a non-degenerate near x_0.

PROBLEM 2.2
It would be interesting to investigate the group of all compactly supported diffeomorphisms of M that preserve the $\mathfrak{g}$-l.c.s. structure up to $\mathfrak{g}$-conformal equivalence (analogously as it was given for usual l.c.s. structures by Haller and Rybicki in [8]).

Let us remark that two extreme cases: (1) $\mathfrak{g}$ commutative (for example $\mathfrak{g} = \mathbb{R}$) and (2) $\mathfrak{g}$ semisimple, are quite different. In the second case all Lie algebroids of the form $(TM \times \mathfrak{g}, \nabla, \Omega)$ (i.e., with the trivial adjoint Lie algebra $M \times \mathfrak{g}$) are isomorphic, clearly to the trivial one $TM \times \mathfrak{g}$ with the structure given by the data $(\partial, 0)$. Let us remark that not each isomorphism is $\mathfrak{g}$-conformal. This Lie algebroid is invariantly oriented.

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On the rate of convergence theorem for the alternate Poisson integrals for Hermite and Laguerre expansions

Dedicated to Professor Andrzej Zajtz on his 70th birthday

Abstract. The aim of this paper is the study of a rate of convergence of alternate Poisson integrals for Hermite and Laguerre expansions. We state some estimates of the rate of convergence of these integrals using the classical moduli of continuity.

1. Introduction

Let $L^p(\exp(-z^2))$, $p \geq 1$ denote the set of functions f defined on $\mathbb{R} = (-\infty, \infty)$ such that

$$\int_{-\infty}^{\infty} |f(t)|^p \exp\left(-t^2\right) dt < \infty \qquad \text{if } 1 \leq p < \infty$$

and f is bounded a.e. on $\mathbb{R}$ if $p = \infty$.

Muckenhoupt in [2] studied Poisson integrals and alternate Poisson integrals for Hermite polynomial expansions. He considered the Poisson integral $A(f)(r,x)$ of a function $f \in L^p(\exp(-z^2))$ for Hermite expansions defined by

$$A(f)(r,x) = A(f;r,x) = \int_{-\infty}^{\infty} P(r,x,z) f(z) \exp\left(-z^2\right) dz, \quad 0 < r < 1,\ x > 0,$$

where

$$\begin{aligned} P(r,x,z) &= \sum_{n=0}^{\infty} \frac{r^n H_n(x) H_n(z)}{\sqrt{\pi}\, 2^n n!} \\ &= \frac{1}{\sqrt{\pi(1-r^2)}} \exp\left(\frac{-r^2x^2 + 2rxz - r^2z^2}{1-r^2}\right) \end{aligned}$$

and H_n is the nth Hermite polynomial, $n = 0, 1, \ldots$.

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The alternate Poisson integral is defined by

$$\begin{aligned} F(f)(x,y) = F(f;x,y) &= \int_0^1 T(x,r)A(f;r,y)\,dr \\ &= \int_{-\infty}^{\infty} \left(\int_0^1 T(x,r)P(r,y,z)\,dr \right) f(z)\exp(-z^2)\,dz, \\ & \qquad\qquad\qquad\qquad\qquad\qquad x>0,\ y\in\mathbb{R}, \end{aligned} \tag{1}$$

where

$$T(x,r) = \frac{x\exp\left(\frac{x^2}{2\ln r}\right)}{(2\pi)^{\frac{1}{2}} r(-\ln r)^{\frac{3}{2}}}.$$

Muckenhoupt obtained the following result.

If $f \in L^p(\exp(-z^2))$, then $F(f;x,\cdot) \in L^p(\exp(-z^2))$ for $x>0$ and:

(a) $\|F(f;x,\cdot)\|_p \le \|f(\cdot)\|_p,\ \ 1\le p\le\infty,$

(b) $\|F(f;x,\cdot) - f(\cdot)\|_p \to 0$ as $x\to 0^+$ for $1\le p<\infty$,

(c) $\lim_{x\to 0^+} F(f;x,y) = f(y)$ almost everywhere, $1\le p\le\infty$,

(d) $\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} - 2y\frac{\partial F}{\partial y} = 0$ in $\Omega = \{(x,y):\ x>0,\ y\in\mathbb{R}\}$,

where $\|\ \|_p$ denotes the norm in $L^p\left(\exp(-z^2)\right)$.

By $L^p(z^\alpha\exp(-z))$, $p\ge 1$, $\alpha>-1$ we denote the set of functions f defined on $\mathbb{R}_+ = [0,\infty)$ such that

$$\int_0^\infty |f(t)|^p t^\alpha \exp(-t)\ dt < \infty \qquad \text{if } 1\le p<\infty$$

and f is bounded a.e. on $\mathbb{R}_+$ if $p=\infty$. In [2] the Poisson integral of a function $f\in L^p(z^\alpha\exp(-z))$ is also considered. This integral is defined by

$$B(f)(r,x) = B(f;r,x) = \int_0^\infty K(r,x,z)f(z)z^\alpha\exp(-z)\,dz, \quad 0<r<1,\ x\ge 0$$

with the Poisson kernel

$$\begin{aligned} K(r,x,z) &= \sum_{n=0}^\infty \frac{r^n n!}{\Gamma(n+\alpha+1)} L_n^\alpha(x)L_n^\alpha(z) \\ &= \frac{(rxz)^{-\frac{\alpha}{2}}}{1-r}\exp\left(\frac{-r(x+z)}{1-r}\right) I_\alpha\left(\frac{2(rxz)^{\frac{1}{2}}}{1-r}\right), \end{aligned}$$

where L_n^α is the nth Laguerre polynomial, $n = 0, 1, \ldots$ and I_α is the modified Bessel function ([1]),

$$I_\alpha(s) = \sum_{n=0}^{\infty} \frac{s^{\alpha+2n}}{2^{\alpha+2n} n!\Gamma(\alpha+n+1)}.$$

We define the alternate Poisson integral by

$$\begin{aligned} G(f)(x,y) = G(f;x,y) &= \int_0^1 U(x,r)B(f;r,y)\,dr \\ &= \int_0^\infty \left(\int_0^1 U(x,r)K(r,y,z)\,dr\right) f(z)z^\alpha \exp(-z)\,dz, \\ &\qquad x>0,\ y\geq 0, \end{aligned}$$

where

$$U(x,r) = T\left(\frac{x}{\sqrt{2}}, r\right).$$

It was proved in [2] that if $f \in L^p\,(z^\alpha \exp(-z))$, then $G(f;x,\cdot) \in L^p(z^\alpha \exp(-z))$ for $x > 0$ and:

(a) $\|G(f;x,\cdot)\|_p \leq \|f(\cdot)\|_p,\ 1 \leq p \leq \infty,$

(b) $\|G(f;x,\cdot) - f(\cdot)\|_p \to 0$ as $x \to 0^+$ for $1 \leq p < \infty,$

(c) $\lim_{x\to 0^+} G(f;x,y) = f(y)$ almost everywhere in $[0,\infty)$, $1 \leq p \leq \infty,$

(d) $\frac{\partial^2 G}{\partial x^2} + y\frac{\partial^2 G}{\partial y^2} + (\alpha+1-y)\frac{\partial G}{\partial y} = 0$ in $\Omega = \{(x,y):\ x>0,\ y\geq 0\}$.

The symbol $\|\ \|_p$ is used here to denote the norm in $L^p(z^\alpha \exp(-z))$.

This note contains some estimates of the rate of convergence of the alternate Poisson integrals $F(f)$, $G(f)$. We state these estimates using the moduli of continuity, severally for $F(f)$ and $G(f)$.

2. Auxiliary results

In this section we shall give some properties of the above operators, which we shall apply to the proofs of the main theorems.

First we prove

Lemma 1

Let $x > 0$. For each $y \in \mathbb{R}$ the following equalities hold

$$F(1;x,y) = 1,$$

$$\begin{aligned} F(z-y;x,y) &= y\left(\exp(-\sqrt{2}x)-1\right), \\ F((z-y)^2;x,y) &= y^2\left(1-2\exp(-\sqrt{2}x)+\exp(-2x)\right) \\ &\quad + \frac{1}{2}\left(1-\exp(-2x)\right). \end{aligned}$$

Proof. Using equality (3.9) in [2]

$$\int_0^1 T(x,r)r^n\,dr = \exp\left(-(2n)^{\frac{1}{2}}x\right), \qquad n=0,1,\dots$$

and [3, Lemma 2.3]

$$\begin{aligned} A(z-x;r,x) &= -x(1-r), \\ A((z-x)^2;r,x) &= (1-r)\left(x^2(1-r)+\frac{1}{2}(r+1)\right) \end{aligned}$$

we obtain from (1) and by elementary calculations the assertion of Lemma 1.

Similarly, using the formula established in [2]

$$\int_0^1 U(x,r)r^n\,dr = \exp(-\sqrt{n}x), \qquad n=0,1,\dots$$

and [3, Lemma 2.4]

$$\begin{aligned} B(z-x;r,x) &= (1-r)(1+\alpha-x), \\ B((z-x)^2;r,x) &= (1-r)\big(x^2(1-r)+2(\alpha+2)rx-2(\alpha+1)x \\ &\quad + (\alpha+2)(\alpha+1)(1-r)\big), \end{aligned}$$

where $\alpha > -1$, we can prove

Lemma 2
Let $x>0$ and $\alpha > -1$. For each $y \in \mathbb{R}_+$

$$\begin{aligned} G(1;x,y) &= 1, \\ G(z-y;x,y) &= (1+\alpha-y)\left(1-\exp(-x)\right), \\ G((z-y)^2;x,y) &= y^2\left(1-2\exp(-x)+\exp(-\sqrt{2}x)\right) \\ &\quad + 2(\alpha+2)y\left(\exp(-x)-\exp(-\sqrt{2}x)\right) \\ &\quad + 2(\alpha+1)y\left(\exp(-x)-1\right) \\ &\quad + (\alpha+2)(\alpha+1)\left(1-2\exp(-x)+\exp(-\sqrt{2}x)\right) \end{aligned}$$

hold.

3. Rate of convergence

In this part we shall give some estimates of the rate of convergence of the integrals $F(f)$ and $G(f)$. We shall use the classical modulus of continuity defined by

$$\omega(f,\delta) = \sup_{\substack{0\le t\le \delta \\ x\,\in\, Q}} |f(x+t)-f(x)|,$$

where $Q=\mathbb{R}$ or $Q=\mathbb{R}_+$, respectively.

Let $C(Q)$ be the set of all continuous functions on $Q=\mathbb{R}$ or $Q=\mathbb{R}_+$. By $C^1(Q)$ we denote, for $Q=\mathbb{R}$ or $Q=\mathbb{R}_+$, the set of all continuously differentiable functions on Q.

Theorem 1
Let $f\in C(\mathbb{R})\cap L^p(\exp(-z^2))$. Then

$$|F(f;x,y)-f(y)| \le 3\,\omega\left(f,\mu_x(y)\right)$$

for $x>0$ and $y\in\mathbb{R}$, where

$$\mu_x(y) = \left(y^2\left(1-2\exp(-\sqrt{2}x)+\exp(-2x)\right)+\frac{1}{2}\left(1-\exp(-2x)\right)\right)^{\frac{1}{2}}.$$

Proof. First we suppose that f is continuously differentiable on $\mathbb{R}$. We have

$$f(z) = f(y) + \int_y^z f'(\tau)\,d\tau.$$

Hence by equality (3.7) in [2]

$$\int_{-\infty}^{\infty} P(r,y,z)\exp(-z^2)\,dz = 1,$$

Definition 1, Lemma 1 and the Hölder inequality, we obtain

$$\begin{aligned}
&|F(f;x,y)-f(y)| \\
&\quad = \left|\int_0^1 T(x,r)\left(A(f;r,y)-f(y)\right)dr\right| \\
&\quad \le \int_0^1 T(x,r)\,|A(f;r,y)-f(y)|\,dr \\
&\quad \le \int_0^1 T(x,r)\left(\int_{-\infty}^{\infty} P(r,y,z)\exp(-z^2)\,|f(z)-f(y)|\;dz\right)dr \\
&\quad \le \sup_{z\,\in\,\mathbb{R}}|f'(z)|\int_{-\infty}^{\infty}\left(\int_0^1 T(x,r)P(r,y,z)\,dr\right)\exp(-z^2)|z-y|\,dz
\end{aligned}$$

$$\begin{aligned}&\leq \sup_{z\in\mathbb{R}} |f'(z)|\,(F(\varphi;x,y))^{\frac{1}{2}}\,(F(1;x,y))^{\frac{1}{2}}\\&= \sup_{z\in\mathbb{R}} |f'(z)|\,\mu_x(y)\end{aligned}$$

for $x > 0$, $y \in \mathbb{R}$, where $\varphi(z) = (z-y)^2$.

Let $f \in C(\mathbb{R}) \cap L^p(\exp(-z^2))$. We have

$$\begin{aligned}f(y) - f_\delta(y) &= \frac{1}{\delta}\int_0^\delta (f(y) - f(y+\tau))\,d\tau,\\ f'_\delta(y) &= \frac{1}{\delta}\left[f(y+\delta) - f(y)\right],\end{aligned}$$

where

$$f_\delta(y) = \frac{1}{\delta}\int_0^\delta f(y+\tau)\,d\tau, \qquad \delta > 0,\ y \in \mathbb{R}.$$

This implies that f_δ is continuously differentiable on $\mathbb{R}$. Moreover

$$\sup_{z\in\mathbb{R}} |f(z) - f_\delta(z)| \leq \omega(f,\delta), \qquad \sup_{z\in\mathbb{R}} |f'_\delta(z)| \leq \delta^{-1}\omega(f,\delta). \tag{2}$$

Observe that

$$|F(f;x,y) - f(y)| \leq |F(f-f_\delta;x,y)| + |F(f_\delta;x,y) - f_\delta(y)| + |f_\delta(y) - f(y)|.$$

From (2) and the first part of this proof we get

$$|F(f_\delta;x,y) - f_\delta(y)| \leq \delta^{-1}\omega(f,\delta)\,\mu_x(y).$$

By (2) we have

$$\begin{aligned}|F(f-f_\delta;x,y)| &\leq F(|f-f_\delta|;x,y) \leq \sup_{z\in\mathbb{R}} |f(z) - f_\delta(z)| \leq \omega(f,\delta),\\ |f_\delta(y) - f(y)| &\leq \omega(f,\delta).\end{aligned}$$

Hence

$$|F(f;x,y) - f(y)| \leq 2\,\omega(f,\delta) + \frac{1}{\delta}\,\omega(f,\delta)\,\mu_x(y)$$

for $x > 0$, $y \in \mathbb{R}$ and $\delta > 0$.

Setting $\delta = \mu_x(y)$ we obtain the assertion of Theorem 1.

Similarly we can prove the following theorem for the operator $G(f)$.

Theorem 2

Let $f \in C(\mathbb{R}_+) \cap L^p(z^\alpha \exp(-z))$. Then

$$|G(f;x,y) - f(y)| \leq 3\,\omega\left(f, \mu_{\alpha,x}(y)\right)$$

for $x>0$ and $y\geq 0$, where

$$\begin{aligned}\mu_{\alpha,x}(y) = &\Big(y^2\left(1-2\exp(-x)+\exp(-\sqrt{2}x)\right)\\ &+2(\alpha+2)y\left(\exp(-x)-\exp(-\sqrt{2}x)\right)\\ &+2(\alpha+1)y\left(\exp(-x)-1\right)\\ &+(\alpha+2)(\alpha+1)\left(1-2\exp(-x)+\exp(-\sqrt{2}x)\right)\Big)^{\frac{1}{2}}.\end{aligned}$$

Now we prove

THEOREM 3
If $f\in C^1(\mathbb{R})\cap L^p(\exp(-z^2))$, then

$$|F(f;x,y)-f(y)|\leq|f'(y)|\left|y\left(\exp(-\sqrt{2}x)-1\right)\right|+2\,\mu_x(y)\,\omega\left(f',\mu_x(y)\right)$$

for $x>0$ and $y\in\mathbb{R}$.

Proof. Let $f\in C^1(\mathbb{R})\cap L^p(\exp(-z^2))$ and $\psi(z)=|z-y|$. Observe that

$$f(z)-f(y)=f'(y)\,(z-y)+\int_y^z\left(f'(s)-f'(y)\right)ds \tag{3}$$

and

$$\left|\int_y^z\left(f'(s)-f'(y)\right)ds\right|\leq\left|\int_y^z\left|f'(s)-f'(y)\right|ds\right|\leq\psi(z)\,\omega\left(f',\psi(z)\right).$$

Let $\delta>0$. We have

$$\omega\left(f',\psi(z)\right)\leq\left(1+\delta^{-1}\psi(z)\right)\omega\left(f',\delta\right).$$

Hence we get

$$\left|\int_y^z\left(f'(s)-f'(y)\right)ds\right|\leq\psi(z)\,\omega\left(f',\delta\right)+\delta^{-1}\,\psi^2(z)\,\omega\left(f',\delta\right). \tag{4}$$

Applying the Hölder inequality and Lemma 1 we obtain

$$F(\psi;x,y)\leq\left(F\left(\psi^2;x,y\right)\right)^{\frac{1}{2}}\left(F\left(1;x,y\right)\right)^{\frac{1}{2}}=\mu_x(y). \tag{5}$$

From (3), (4), (5) and Lemma 1 we have

$$\begin{aligned}&|F(f;x,y)-f(y)|\\ &\quad=\ \left|F\left(f'(y)(z-y);x,y\right)\right|+\left|F\left(\int_y^z\left(f'(s)-f'(y)\right)ds;x,y\right)\right|\\ &\quad\leq\ |f'(y)|\,|F(z-y;x,y)|+\omega\left(f',\delta\right)F(\psi;x,y)+\frac{1}{\delta}\,\omega\left(f',\delta\right)F\left(\psi^2;x,y\right)\\ &\quad\leq\ |f'(y)|\left|y\left(1-\exp(-\sqrt{2}x)\right)\right|+\omega\left(f',\delta\right)\,\mu_x(y)+\frac{1}{\delta}\,\omega\left(f',\delta\right)\,\mu_x^2(y)\end{aligned}$$

for $x > 0$, $y \in \mathbb{R}$ and $\delta > 0$.

Setting $\delta = \mu_x(y)$ completes the proof of Theorem 3.

For $G(f)$ we obtain a similar result.

THEOREM 4
If $f \in C^1(\mathbb{R}_+) \cap L^p(z^\alpha \exp(-z))$, then

$$\begin{aligned} &|G(f;x,y) - f(y)| \\ &\qquad \le\ |f'(y)|\,|(1+\alpha-y)\,(1-\exp(-x))| + 2\,\mu_{\alpha,x}(y)\,\omega\,(f', \mu_{\alpha,x}(y)) \end{aligned}$$

for $x > 0$ and $y \in \mathbb{R}_+$.

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Higher order jet prolongations type gauge natural bundles over vector bundles

Dedicated to Professor Andrzej Zajtz on the occasion of his 70th birthday with respect and gratitude

Abstract. Let $r \geq 3$ and $m \geq 2$ be natural numbers and E be a vector bundle with m-dimensional basis. We find all gauge natural bundles "similar" to the r-jet prolongation bundle $J^r E$ of E. We also find all gauge natural bundles "similar" to the vector r-tangent bundle $(J^r_{fl}(E, \mathbb{R})_0)^*$ of E.

Introduction

Natural bundles over m-manifolds were introduced by Nijenhuis [31] as a modern approach to geometric objects [1]. The most important results in the theory of natural bundles over m-manifolds is the Palais-Terng finite order theorem [32] and the Epstein-Thurston regularity theorem [7]. The sharp estimation of the order of natural bundles over m-manifolds was obtained by Zajtz [38]. From the order theorem follows that natural bundles over m-manifolds are associated to the principal bundles of repers of higher orders.

Natural bundles on some other categories over manifolds (e.g. all manifolds, fibered manifolds, principal fiber bundles, vector bundles, manifolds with structures) were introduced in [17]. Some sharp order estimations for some such natural bundles can be found in [17] and [22]. A natural bundle over all manifolds of infinite order can be found in [24]. Classifications of some type natural bundles over some categories over manifolds can be found in [35], [37], [3], [12], [23], [25], [26], [30], [16], [4], [6], [27], [18], [28].

Natural bundles make possible to precise define geometric constructions. Investigations, applications and classifications of natural operators (canonical constructions) on sections of natural bundles (geometric objects) were (and actually are) fundamental in the differential geometry. There are over 200 papers on this subject, e.g. [2], [3], [8]-[11], [13]-[15] [19]-[21], [33], [34], [36] and the fundamental monograph of Kolář, Michor and Slovák [17].

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The present paper is a next contribution to the theory of natural bundles. Let us recall the following definition (see for ex. [17], [27]).

Let $F\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ be a covariant functor from the category $\mathcal{VB}_m$ of vector bundles with m-dimensional bases and their vector bundle maps with local diffeomorphisms as base maps. Let $B_{\mathcal{VB}m}\colon \mathcal{VB}_m \longrightarrow \mathcal{M}f_m$ be the base functor.

A gauge natural bundle over $\mathcal{VB}_m$ (called also a gauge bundle functor on $\mathcal{VB}_m$) is a functor F as above satisfying:

(i) (**Base preservation**) $B_{\mathcal{VB}m} \circ F = B_{\mathcal{VB}m}$. Hence the induced projections form a functor transformation $\pi\colon F \longrightarrow B_{\mathcal{VB}m}$.

(ii) (**Localization**) For every inclusion of an open vector subbundle $i_{E|U}\colon E|U \longrightarrow E$, $F(E|U)$ is the restriction $\pi^{-1}(U)$ of $\pi\colon FE \longrightarrow B_{\mathcal{VB}m}(E)$ over U and $Fi_{E|U}$ is the inclusion $\pi^{-1}(U) \longrightarrow FE$.

(iii) (**Regularity**) F transforms smoothly parametrized systems of $\mathcal{VB}_m$-morphisms into smoothly parametrized systems of $\mathcal{VB}_m$-morphisms.

A gauge natural bundle F over $\mathcal{VB}_m$ is *fiber product preserving* if for every fiber product projections $E_1 \overset{pr_1}{\longleftarrow} E_1 \times_M E_2 \overset{pr_2}{\longrightarrow} E_2$ in the category $\mathcal{VB}_m$ $FE_1 \overset{Fpr_1}{\longleftarrow} F(E_1 \times_M E_2) \overset{Fpr_2}{\longrightarrow} FE_2$ are fiber product projections in the category $\mathcal{VB}_m$. In other words $F(E_1 \times_M E_2) = F(E_1) \times_M F(E_2)$ modulo the corestriction of (Fpr_1, Fpr_2).

All fiber product preserving gauge natural bundles over $\mathcal{VB}_m$ are described in [28]. The most important example of the ones is the r-jet prolongation gauge natural bundle $J^r\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$. Another example is the (described in I.1) vector (r)-tangent gauge natural bundle $T^{(r)fl}\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$. Fiber product preserving gauge natural bundles on $\mathcal{VB}_m$ with "similar to $T^{(r)fl}$ and J^r constructions" (Definitions 1 and 2) are also called vector r-tangent gauge natural bundles and r-jet prolongation gauge natural bundles over $\mathcal{VB}_m$, respectively.

Roughly speaking, the main results are the following classification theorems.

Theorem A
Let $m \geq 2$ and $r \geq 3$ be integers. Up to isomorphism there are only two vector r-tangent gauge natural bundles over $\mathcal{VB}_m$.

Theorem B
Let $m \geq 2$ and $r \geq 3$ be integers. Up to isomorphism there are only three r-jet prolongation gauge natural bundles over $\mathcal{VB}_m$.

The above theorems will be detailed formulated later (Theorems 1 and 2 of the present paper). In particular, the two vector r-tangent gauge natural bundles over $\mathcal{VB}_m$ and the three r-jet prolongation gauge natural bundles over $\mathcal{VB}_m$ will be precisely constructed.

All manifolds and maps are assumed to be of class C^∞. Manifolds are assumed to be finite dimensional.

I. The vector r-tangent gauge natural bundles over $\mathcal{VB}_m$

1. The vector (r)-tangent gauge natural bundle over $\mathcal{VB}_m$

Given a $\mathcal{VB}_m$-object $p\colon E \longrightarrow M$, the vector (r)-tangent bundle $T^{(r)fl}E$ of E is the vector bundle

$$T^{(r)fl}E = (J^r_{fl}(E,\mathbb{R})_0)^*$$

over M, where

$$J^r_{fl}(E,\mathbb{R})_0 = \{j^r_x\gamma \mid \ \gamma\colon E \longrightarrow \mathbb{R} \text{ is fiber linear}, \ \gamma_x = 0, \ x \in M\}.$$

Every $\mathcal{VB}_m$-map $f\colon E_1 \longrightarrow E_2$ covering $\underline{f}\colon M_1 \longrightarrow M_2$ induces a vector bundle map $T^{(r)fl}f\colon T^{(r)fl}E_1 \longrightarrow T^{(r)fl}E_2$ covering $\underline{f}$ such that

$$< T^{(r)fl}f(\omega), j^r_{\underline{f}(x)}\xi > = < \omega, j^r_x(\xi \circ f) >,$$

$\omega \in T^{(r)fl}_x E_1$, $j^r_{\underline{f}(x)}\xi \in J^r_{fl}(E_2,\mathbb{R})_0$, $x \in M_1$.

The correspondence $T^{(r)fl}\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ is a fiber product preserving gauge natural bundle.

2. Another construction of $T^{(r)fl}$

Let $p\colon E \longrightarrow M$ be a $\mathcal{VB}_m$-object. For any m-manifold M and $x \in M$ we have a canonical unital associative algebra homomorphism $t^{(r)}_x\colon J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}((J^r_x(M,\mathbb{R})_0)^*)$ given by

$$t^{(r)}_x(j^r_x\gamma)(\omega)(j^r_x\eta) = \omega(j^r_x(\gamma\eta)),$$

$j^r_x\eta \in J^r_x(M,\mathbb{R})_0$, $j^r_x\gamma \in J^r_x(M,\mathbb{R})$, $\omega \in (J^r_x(M,\mathbb{R})_0)^*$. We have a vector bundle

$$\tilde{T}^{(r)fl}E = \bigcup_{x\in M} \mathrm{Hom}_{t^{(r)}_x}(J^r\mathcal{C}^{\infty,fl}_x(E), (J^r_x(M,\mathbb{R})_0)^*)$$

over M. Here $\mathrm{Hom}_{t^{(r)}_x}(J^r\mathcal{C}^{\infty,fl}_x(E), (J^r_x(M,\mathbb{R})_0)^*)$ is the vector space of all module homomorphisms over $t^{(r)}_x\colon J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}((J^r_x(M,\mathbb{R})_0)^*)$ from the (free) $J^r_x(M,\mathbb{R})$-module $J^r\mathcal{C}^{\infty,fl}_x(E)$ of r-jets at x of germs at x of fiber linear maps $E \longrightarrow \mathbb{R}$ into the $\mathrm{gl}((J^r_x(M,\mathbb{R})_0)^*)$-module $(J^r_x(M,\mathbb{R})_0)^*$. Every

$\mathcal{VB}_m$-map $f\colon E_1 \longrightarrow E_2$ covering $\underline{f}\colon M_1 \longrightarrow M_2$ induces a vector bundle map $\tilde{T}^{(r)fl}f\colon \tilde{T}^{(r)fl}E_1 \longrightarrow \tilde{T}^{(r)fl}E_2$ covering $\underline{f}$ such that

$$\tilde{T}^{(r)fl}f(\Phi)(j^r_{\underline{f}(x)}\xi)(j^r_{\underline{f}(x)}\gamma) = \Phi(j^r_x(\xi \circ f))(j^r_x(\gamma \circ \underline{f})),$$

$\Phi \in \mathrm{Hom}_{t^{(r)}_x}(J^r\mathcal{C}^{\infty,fl}_x(E_1), (J^r_x(M_1,\mathbb{R})_0)^*)$, $x \in M_1$, $j^r_{\underline{f}(x)}\xi \in J^r\mathcal{C}^{\infty,fl}_{\underline{f}(x)}(E_2)$, $j^r_{\underline{f}(x)}\gamma \in J^r(M_2,\mathbb{R})_0$.

The correspondence $\tilde{T}^{(r)fl}\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ is a fiber product preserving gauge natural bundle.

LEMMA 1
We have a natural isomorphism $T^{(r)fl} \tilde{=} \tilde{T}^{(r)fl}$

Proof. Define a base preserving vector bundle map $\Theta\colon T^{(r)fl}E \longrightarrow \tilde{T}^{(r)fl}E$ by

$$\Theta(\omega)(j^r_x\xi)(j^r_x\gamma) = T^{(r)fl}(\xi)(\omega)(j^r_x\gamma) \in \mathbb{R},$$

$\omega \in T^{(r)fl}_x E$, $j^r_x\xi \in J^r\mathcal{C}^{\infty,fl}_x(E)$, $j^r_x\gamma \in J^r_x(M,\mathbb{R})_0$, $x \in M$, where $\xi\colon E \longrightarrow \mathbb{R}$ is considered as $\mathcal{VB}_{m,n}$-map $\xi\colon E \longrightarrow M \times \mathbb{R}$, $\xi(v) = (p(v), \xi(v))$, $v \in E$ and where $\gamma\colon M \longrightarrow \mathbb{R}$ is considered as the fiber linear map $\gamma\colon M \times \mathbb{R} \longrightarrow \mathbb{R}$, $\gamma(m,t) = \gamma(m)t$, $m \in M$, $t \in \mathbb{R}$. Using trivializations of E it is easy to see that Θ is a vector bundle isomorphism.

3. The vector $[r]$-tangent gauge natural bundle over $\mathcal{VB}_m$

Given a $\mathcal{VB}_m$-object $p\colon E \longrightarrow M$ we have the vector bundle

$$T^{[r]fl}E = E \otimes (J^r(M,\mathbb{R})_0)^*$$

over M. Every $\mathcal{VB}_m$-map $f\colon E_1 \longrightarrow E_2$ covering $\underline{f}\colon M_1 \longrightarrow M_2$ induces (in obvious way) a vector bundle map $T^{[r]fl}f\colon T^{[r]fl}E_1 \longrightarrow T^{[r]fl}E_2$ covering $\underline{f}$.

The correspondence $T^{[r]fl}\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ is a fiber product preserving gauge natural bundle.

4. Another construction of $T^{[r]fl}$

For any m-manifold M and $x \in M$ we have a canonical unital associative algebra homomorphism $t^{[r]}_x\colon J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}((J^r_x(M,\mathbb{R})_0)^*)$ given by

$$t^{[r]}_x(j^r_x\gamma)(\omega) = \gamma(x)\omega,$$

$j^r_x\gamma \in J^r_x(M,\mathbb{R})$, $\omega \in (J^r_x(M,\mathbb{R})_0)^*$. Using $t^{[r]}_x$ instead of $t^{(r)}_x$ we can construct similarly as in Section 2 a fiber product preserving gauge natural bundle $\tilde{T}^{[r]fl}\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$.

LEMMA 2
We have a natural isomorphism $T^{[r]fl} \cong \tilde{T}^{[r]fl}$.

Proof. The proof is quite similar to the one of Lemma 1.

5. $T^{(r)fl} \neq T^{[r]fl}$

LEMMA 3
The gauge natural bundle $T^{(r)fl}\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *and* $T^{[r]fl}\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *are not isomorphic.*

Proof. Let $f\colon \mathbb{R}^{m,n} \longrightarrow \mathbb{R}^{m,n}$ be a $\mathcal{VB}_{m,n}$-map given by

$$f(x,y) = (x, y + x^1 y), \qquad x = (x^1, \dots, x^m) \in \mathbb{R}^m,\ y \in \mathbb{R}^n.$$

Clearly $T_0^{[r]fl} f = \mathrm{id}$ and $T_0^{(r)fl} f \neq \mathrm{id}$. That is why there is no base preserving vector bundle isomorphism $\Theta\colon T^{(r)fl}\mathbb{R}^{m,n} \longrightarrow T^{[r]fl}\mathbb{R}^{m,n}$ commuting with f.

6. The vector r-tangent gauge natural bundles over $\mathcal{VB}_m$

DEFINITION 1
We say that a fiber product preserving gauge natural bundle $F\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ is a vector r-tangent gauge natural bundle if modulo isomorphism F can be constructed similarly as $\tilde{T}^{(r)fl}$ by using eventually another canonical unital associative algebra homomorphism $t_x\colon J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}((J^r_x(M,\mathbb{R})_0)^*)$ instead of $t^{(r)}_x$.

We have the following classification theorem.

THEOREM 1
Let $m \geq 2$ *and* $r \geq 3$ *be integers. Up to isomorphism* $T^{(r)fl}\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *and* $T^{[r]fl}\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *are only vector* r*-tangent gauge natural bundles over* $\mathcal{VB}_m$.

Proof. Suppose that for any m-manifold M and $x \in M$

$$t_x\colon J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}((J^r_x(M,\mathbb{R})_0)^*)$$

is a canonical unital associative algebra homomorphism. By the invariance of t_0 with respect to $(\tau^1 x^1, \dots, \tau^m x^m)$ for $\tau^i \neq 0$ we deduce that

$$t_0(j^r_0 x^1)((j^r_0 x^\alpha)^*)(j^r_0 x^2) = 0$$

for $\alpha \neq (1,1,0,\dots,0)$. Then we can write

$$t_0(j^r_0 x^1)(\omega)(j^r_0 x^2) = a\omega(j^r_0(x^1 x^2))$$

for all $\omega \in (J^r_0(\mathbb{R}^m,\mathbb{R})_0)^*$, where $a = t_0(j^r_0 x^1)((j^r_0(x^1x^2))^*)(j^r_0 x^2)$. Then by the invariance of t_0 with respect to 0-preserving diffeomorphisms we have

$$t_0(j^r_0\gamma)(\omega)(j^r_0\eta) = a\omega(j^r_0(\gamma\eta))$$

for all $\omega \in (J^r_0(\mathbb{R}^m,\mathbb{R})_0)^*$ and all $j^r_0\gamma, j^r_0\eta \in J^r_0(\mathbb{R}^m,\mathbb{R})_0$. But t_0 is an algebra homomorphism. Then

$$\begin{aligned} a\omega(j^r_0((x^1)^2x^2)) &= t_0(j^r_0((x^1)^2))(\omega)(j^r_0 x^2) \\ &= t_0(j^r_0 x^1)(t_0(j^r_0 x^1)(\omega))(j^r_0 x^2) \\ &= at_0(j^r_0 x^1)(\omega)(j^r_0(x^1x^2)) \\ &= a^2\omega(j^r_0((x^1)^2x^2)). \end{aligned}$$

If $r \geq 3$ then $a^2 = a$, i.e. $a = 1$ or $a = 0$. Hence $t_x = t^{(r)}_x$ or $t_x = t^{[r]}_x$.

II. The r-jet prolongation gauge natural bundles over $\mathcal{VB}_m$

7. The (usual) r-jet prolongation gauge natural bundle J^r over $\mathcal{VB}_m$

Given a $\mathcal{VB}_m$-object $p\colon E \longrightarrow M$ the (usual) r-jet prolongation J^rE of E is a vector bundle

$$J^rE = \{j^r_x\sigma \mid \ \sigma \text{ is a local section of } E\ ,\ x \in M\}$$

over M. Every $\mathcal{VB}_m$-map $f\colon E_1 \longrightarrow E_2$ covering $\underline{f}\colon M_1 \longrightarrow M_2$ induces a vector bundle map $J^rf\colon J^rE_1 \longrightarrow J^rE_2$ covering $\underline{f}$ such that

$$J^rf(j^r_x\sigma) = j^r_{\underline{f}(x)}(f\circ\sigma\circ\underline{f}^{-1}), \qquad j^r_x\sigma \in J^rE_1\,.$$

The functor $J^r\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ is a fiber product preserving gauge natural bundle.

8. Another construction of J^r

Let $p\colon E \longrightarrow M$ be a $\mathcal{VB}_m$-object. For any m-manifold M and $x \in M$ we have a canonical unital associative algebra homomorphism $t^{r,m}_x\colon J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}(J^r_x(M,\mathbb{R}))$ given by

$$t^{r,m}_x(j^r_x\eta)(j^r_x\gamma) = j^r_x(\gamma\eta),$$

$j^r_x\eta \in J^r_x(M,\mathbb{R})$, $j^r_x\gamma \in J^r_x(M,\mathbb{R})$. We have a vector bundle

$$\tilde{\tilde{J}}^rE = \bigcup_{x\in M} \mathrm{Hom}_{t^{r,m}_x}(J^r\mathcal{C}^{\infty,fl}_x(E), J^r_x(M,\mathbb{R}))$$

over M. Here $\mathrm{Hom}_{t_x^{r,m}}(J^r\mathcal{C}_x^{\infty,fl}(E), J_x^r(M,\mathbb{R}))$ is the vector space of all module homomorphisms over $t_x^{r,m}\colon J_x^r(M,\mathbb{R}) \longrightarrow \mathrm{gl}(J_x^r(M,\mathbb{R}))$ from the (free) $J_x^r(M,\mathbb{R})$-module $J^r\mathcal{C}_x^{\infty,fl}(E)$ of r-jets at x of germs at x of fiber linear maps $E \longrightarrow \mathbb{R}$ into the $\mathrm{gl}(J_x^r(M,\mathbb{R}))$-module $J_x^r(M,\mathbb{R})$. Every $\mathcal{VB}_m$-map $f\colon E_1 \longrightarrow E_2$ covering $\underline{f}\colon M_1 \longrightarrow M_2$ induces a vector bundle map $\tilde{\tilde{J}}^r f\colon \tilde{\tilde{J}}^r E_1 \longrightarrow \tilde{\tilde{J}}^r E_2$ covering $\underline{f}$ such that

$$\tilde{\tilde{J}}^r f(\Phi)(j^r_{\underline{f}(x)}\xi) = J^r(\underline{f}, \mathrm{id}_{\mathbb{R}}) \circ \Phi(j_x^r(\xi \circ f)),$$

$\Phi \in \mathrm{Hom}_{t_x^{r,m}}(J^r\mathcal{C}_x^{\infty,fl}(E_1), J_x^r(M_1,\mathbb{R}))$, $x \in M_1$, $j^r_{\underline{f}(x)}\xi \in J^r\mathcal{C}^{\infty,fl}_{\underline{f}(x)}(E_2)$.

The correspondence $\tilde{\tilde{J}}^r\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ is a fiber product preserving gauge natural bundle.

Lemma 4
We have a natural isomorphism $J^r \tilde{=} \tilde{\tilde{J}}^r$.

Proof. The proof is similar to the one of Lemma 1. We define an isomorphism $\Theta\colon J^rE \longrightarrow \tilde{\tilde{J}}^r E$ by

$$\Theta(j_x^r\sigma)(j_x^r\xi) = j_x^r(\xi \circ \sigma),$$

$j_x^r\sigma \in J_x^rE$, $j_x^r\xi \in J^r\mathcal{C}_x^{\infty,fl}(E)$.

9. The vertical r-jet prolongation gauge natural bundle J_v^r over $\mathcal{VB}_m$

Given a $\mathcal{VB}_m$-object $p\colon E \longrightarrow M$, the vertical r-jet prolongation J_v^rE of E is a vector bundle

$$J_v^rE = \{j_x^r\sigma \mid \ \sigma\colon M \longrightarrow E_x\,,\ x \in M\}$$

over M. Every $\mathcal{VB}_m$-map $f\colon E_1 \longrightarrow E_2$ covering $\underline{f}\colon M_1 \longrightarrow M_2$ induces a vector bundle map $J_v^rf\colon J_v^rE_1 \longrightarrow J_v^rE_2$ covering $\underline{f}$ such that

$$J_v^rf(j_x^r\sigma) = j^r_{\underline{f}(x)}(f \circ \sigma \circ \underline{f}^{-1})\,, \qquad j_x^r\sigma \in J_v^rE_1\,.$$

The functor $J_v^r\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ is a fiber product preserving gauge natural bundle.

10. Another construction of J_v^r

For any m-manifold M and $x \in M$ we have a canonical unital associative algebra homomorphism $t_x^{r,m,o}\colon J_x^r(M,\mathbb{R}) \longrightarrow \mathrm{gl}(J_x^r(M,\mathbb{R}))$ given by

$$t_x^{r,m,o}(j_x^r\eta)(j_x^r\gamma) = \eta(x)j_x^r\gamma\,,$$

$j^r_x\gamma \in J^r_x(M,\mathbb{R})$, $j^r_x\eta \in J^r_x(M,\mathbb{R})$. Using $t^{r,m,o}_x$ instead of $t^{r,m}_x$ we can construct similarly as in Section 9 a fiber product preserving gauge natural bundle $\tilde{\tilde{J}}^r_v : \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$.

LEMMA 5
We have a natural isomorphism $J^r_v \tilde{=} \tilde{\tilde{J}}^r_v$.

Proof. The proof is quite similar to the one of Lemma 4.

11. The $[r]$-jet prolongation gauge natural bundle $J^{[r]}$ over $\mathcal{VB}_m$

For any m-manifold M and $x \in M$ we have a canonical unital associative algebra homomorphism $t^{[r],m}_x : J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}(J^r_x(M,\mathbb{R}))$ given by

$$t^{[r],m}_x(j^r_x\eta)(j^r_x\gamma) = j^r_x(\eta\gamma) - \gamma(x)j^r_x\eta + \eta(x)\gamma(x)j^r_x 1\,,$$

$j^r_x\gamma \in J^r_x(M,\mathbb{R})$, $j^x\eta \in J^r_x(M,\mathbb{R})$. Using $t^{[r],m}_x$ instead of $t^{r,m}_x$ we can construct similarly as in Section 9 a fiber product preserving gauge natural bundle $J^{[r]} : \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$.

12. J^r, J^r_v and $J^{[r]}$ are not equivalent

LEMMA 6
The gauge natural bundles $J^r : \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *and* $J^r_v : \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *are not isomorphic.*

Proof. Let $f : \mathbb{R}^{m,n} \longrightarrow \mathbb{R}^{m,n}$ be a $\mathcal{VB}_{m,n}$-map given by

$$f(x,y) = (x, y + x^1 y), \qquad x = (x^1,\dots,x^m) \in \mathbb{R}^m,\ y \in \mathbb{R}^n.$$

Clearly $J^r_0 f \neq \mathrm{id}$ and $(J^r_v)_0 f = \mathrm{id}$. That is why there is no base preserving vector bundle isomorphism $\Theta : J^r\mathbb{R}^{m,n} \longrightarrow J^r_v\mathbb{R}^{m,n}$ commuting with f.

LEMMA 7
The gauge natural bundles $J^{[r]} : \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *and* $J^r_v : \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *are not isomorphic.*

Proof. The proof is quite similar to the one of Lemma 6.

LEMMA 8
Let $m \geq 2$. *The gauge natural bundles* $J^{[r]} : \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *and* $J^r : \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *are not isomorphic.*

Proof. It follows from some result from [29]. More precisely, the vector space of all $\mathcal{VB}_{m,n}$-natural affinors on J^rE is one dimensional and the vector space of all $\mathcal{VB}_{m,n}$-natural affinors on $J^{[r]}E$ is two dimensional.

13. The r-jet prolongation gauge natural bundles over $\mathcal{VB}_m$

Definition 2
We say that a fiber product preserving gauge natural bundle $F\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ is a r-jet prolongation gauge natural bundle if modulo isomorphism F can be constructed similarly as $\tilde{\tilde{J}}^r$ by using eventually another canonical unital associative algebra homomorphism $t_x\colon J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}(J^r_x(M,\mathbb{R}))$ instead of $t^{r,m}_x$.

We have the following classification theorem.

Theorem 2
Let $m \geq 2$ and $r \geq 3$ be integers. Up to isomorphism $J^r\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$, $J^r_v\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ and $J^{[r]}\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ are only r-jet prolongation gauge natural bundles over $\mathcal{VB}_m$.

14. A preparation

Lemma 9
For any $\alpha \in \mathbb{R}$ we have a canonical unital algebra homomorphism

$$t^{<\alpha>}_x\colon J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}(J^r_x(M,\mathbb{R}))$$

given by

$$t^{<\alpha>}_x(j^r_x\gamma)(j^r_x\eta) = j^r_x(\gamma\,\eta) + \alpha\eta(x)j^r_x\gamma - \alpha\eta(x)\gamma(x)j^r_x 1\,,$$

where $j^r_x\eta, j^r_x\gamma \in J^r_x(M,\mathbb{R})$, $\dim(M) = m$, $x \in M$. If $\alpha \notin \{-1,0\}$ we have a canonical isomorphism $a_x\colon J^r_x(M,\mathbb{R}) \longrightarrow J^r_x(M,\mathbb{R})$ given by

$$a_x(j^r_x\gamma) = -\frac{\alpha+1}{\alpha} j^r_x\gamma + \gamma(x)j^r_x 1$$

such that $t^{<\alpha>}_x(j^r_x\eta)\circ a_x = a_x \circ t^{r,m}_x(j^r_x\eta)$ for any $j^r_x\eta \in J^r_x(M,\mathbb{R})$. Roughly speaking, for $\alpha \notin \{-1,0\}$ homomorphism $t^{<\alpha>}_x$ is canonically isomorphic to $t^{r,m}_x$.

Proof. It is easy to verify.

15. Proof of Theorem 2

Let for any m-manifold M and $x \in M$

$$t_x\colon J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}(J^r_x(M,\mathbb{R}))$$

be a canonical unital associative algebra homomorphism. Clearly, t_x is uniquely determined by the values

$$t_0(j^r_0\gamma)(j^r_0\eta) \in J^r_0(\mathbb{R}^m,\mathbb{R})$$

for any $j_0^r\gamma, j_0^r\eta \in J_0^r(\mathbb{R}^m, \mathbb{R})$ with $\gamma(0) = 0$. By the invariance of t_0 with respect to 0-preserving diffeomorphisms and the rank theorem and $m \geq 2$ we can assume that

$$j_0^r\gamma = j_0^r x^1$$

and

$$j_0^r\eta = j_0^r 1 \quad \text{or} \quad j_0^r\eta = j_0^r x^2.$$

By the invariance of t_0 with respect to $(\tau_1 x^1, \dots, \tau_m x^m)$ for $\tau_1, \dots, \tau_m \in \mathbb{R}_+$ we deduce that

$$t_0(j_0^r x^1)(j_0^r x^2) = \sigma j_0^r(x^1 x^2)$$

and

$$t(j_0^r x^1)(j_0^r 1) = \rho j_0^r x^1$$

for some $\sigma, \rho \in \mathbb{R}$. Then by the invariance of t_0 with respect to 0-preserving diffeomorphisms we have

$$t_0(j_0^r\xi)(j_0^r\eta) = \sigma j_0^r(\xi\eta)$$

and

$$t_0(j_0^r\xi)(j_0^r 1) = \rho j_0^r\xi$$

for all $j_0^r\xi \in J_0^r(\mathbb{R}^m, \mathbb{R})$ and $j_0^r\eta \in J_0^r(\mathbb{R}^m, \mathbb{R})$ and with $\xi(0) = \eta(0) = 0$. Then, since t_0 is an unital homomorphism, it is easy to verify (using the assumptions on r, m) that $\sigma = \rho = 0$ or $\sigma = 1$ and ρ is arbitrary. Therefore $t_x = t_x^{r,m,o}$ or $t_x = t_x^{<\alpha>}$ for $\alpha = \rho - 1$.

Using Lemma 9 we end the proof.

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Abstract Lie groups and locally compact topological groups

Dedicated to Professor Andrzej Zajtz
on the occasion of his 70th birthday

Abstract. We introduce a notion of abstract Lie group by means of the mapping which plays the role of the evolution operator. We show some basic properties of such groups very similar to the fundamentals of the infinite dimensional Lie theory. Next we give remarkable examples of abstract Lie groups which are not necessarily usual Lie groups. In particular, by making use of Yamabe theorem we prove that any locally compact topological group admits the structure of abstract Lie group and that the Lie algebra and the exponential mapping of it coincide with those determined by the Lie group structure.

1. Introduction

An infinite dimensional Lie group G with its Lie algebra $\mathfrak{g}$ is called regular if there is a bijective evolution mapping

$$\mathrm{Evol}^r_G\colon \mathcal{C}^\infty(\mathbb{R},\mathfrak{g}) \longrightarrow \mathcal{C}^\infty((\mathbb{R},0),(G,e))$$

such that its evaluation at $1\in\mathbb{R}$ is smooth. This notion has been introduced by J. Milnor [10] (see also [8]). The right logarithmic derivative δ^r_G is then the inverse of Evol^r_G. Notice that one can use equivalently the left evolution mapping and the left logarithmic derivative to define the regularity. Next the exponential mapping of a regular Lie group $\exp\colon\mathfrak{g}\longrightarrow G$ is given by

$$\exp(X) = \mathrm{Evol}^r_G(X)(1).$$

In particular, $\exp(tX) = \mathrm{Evol}^r_G(X)(t)$ and clearly $\exp(t{+}s)X = \exp tX \exp sX$. Let us mention that all known Lie groups are regular (cf. [8]). It seems that

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the role of the evolution mapping in the infinite dimensional Lie theory is so important as the role of the exponential mapping in the finite dimensional case.

In this paper we propose a notion generalizing regular Lie groups, namely the notion of abstract Lie groups. In this concept the smooth structure is defined by means of the family of smooth curves $\mathcal{S}^\infty(\mathbb{R}, G)$, and the $\mathcal{S}^\infty(\mathbb{R}, G)$ is defined by the evolution mapping Evol^r_G. The notion of abstract Lie groups generalizes regular Lie groups and is motivated by important examples, cf. Section 3. It is significant that basic properties of infinite dimensional Lie groups can be derived from our definition and new interpretations of the inheritance property, the integrability of Lie subalgebras, and the quotient structures are possible. Here we give only some introductory facts and we omit some proofs as a presentation of the whole setting is beyond the scope of this paper and will be given elsewhere.

We would like to indicate that there are other abstract settings of the infinite dimensional Lie theory. Examples are the following:

— Diffeological groups due to Souriau [16]. A smooth structure is there defined by establishing sets of local smooth mappings from $\mathbb{R}^n$ to G, $n = 1, 2, \ldots$, and by imposing some conditions on them. It is possible to define the tangent space T_eG, but a Lie algebra structure can be given on some subspace $\mathfrak{g}$ of T_eG only. Then one introduces the exponential mapping on $\mathfrak{g}$.

— The concept of generalized Lie groups in the sense of Omori [11]. The definition is based on a continuous mapping $\exp\colon G \longrightarrow \mathfrak{g}$ between a topological metric group G and a topological Lie algebra $\mathfrak{g}$ with several technical conditions which mimic essential properties of the exponential map. In particular, it is possible to distiguish the set of differentiable curves.

— Another category of generalized Lie groups was proposed by Chen and Yoh [3]. A clue point is there a description of the Lie algebra $\mathrm{Hom}(\mathbb{R}, G)$ for a topological group. However this framework concerns mainly finite dimensional groups.

Note that all these concepts do not use the regularity and, in view of them, the Lie group structure is inherited by any (closed) subgroup. Consequently it seems that they are not enough refined to give a satisfactory abstract description of the infinite dimensional Lie theory.

As a general framework we will use the concept of smooth structure defined by a family of smooth curves. Let us mention that there exist in the literature similar settings as well as calculi by means of smooth curves, e.g. an interesting calculus of flows on convenient manifolds by A. Zajtz [20].

In this paper we wish to illustrate the introduced concept in the case of LP-groups, i.e. the groups which can be expressed as the projective limts of an inverse system of finite dimensional Lie groups. We recall basic properties of such groups and endow them with the structure of an abstract Lie group. Consequently, in view of Yamabe theorem, we show that any connected locally compact topological group carries the structure of an abstract Lie group.

For simplicity we will confine our considerations to the $\mathcal{C}^\infty$ smooth category.

2. Smooth spaces

First we introduce the category of smooth spaces.

Definition 2.1
A *smooth space* is a set X endowed with a subset $\mathcal{S}^\infty(\mathbb{R}, X) \subset \mathrm{Map}(\mathbb{R}, X)$ such that constant mappings are in $\mathcal{S}^\infty(\mathbb{R}, X)$ and $c \circ f \in \mathcal{S}^\infty(\mathbb{R}, X)$ whenever $f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ is a smooth reparametrization and $c \in \mathcal{S}^\infty(\mathbb{R}, X)$.

The set $\mathcal{S}^\infty(\mathbb{R}, X)$ is called a *smooth structure* on X. If Y is another smooth space endowed with a smooth structure $\mathcal{S}^\infty(\mathbb{R}, Y)$ then a mapping $f: X \longrightarrow Y$ is said to be *smooth* (or a *morphism of smooth structures*) if $f \circ c \in \mathcal{S}^\infty(\mathbb{R}, Y)$ whenever $c \in \mathcal{S}^\infty(\mathbb{R}, X)$. It is clear that the composition of smooth maps is smooth. We write $f_*: \mathcal{S}^\infty(\mathbb{R}, X) \longrightarrow \mathcal{S}^\infty(\mathbb{R}, Y)$ for the induced map. Note that in view of Boman theorem [1] this concept extends the usual concept of smoothness.

Any smooth space X is equipped with a natural topology, namely the final topology of $\mathcal{S}^\infty(\mathbb{R}, X)$.

Proposition 2.2
For any smooth spaces X, Y, Z one has

$$\mathcal{S}^\infty(X, \mathcal{S}^\infty(Y, Z)) \cong \mathcal{S}^\infty(X \times Y, Z),$$

i.e., the category of smooth spaces is cartesian closed.

Proof. We endow $\mathcal{S}^\infty(Y, X)$ with a smooth structure as follows:

$$c \in \mathcal{S}^\infty(\mathbb{R}, \mathcal{S}^\infty(Y, Z)) \iff \hat{c} \in \mathcal{S}^\infty(\mathbb{R} \times Y, Z),$$

where $\hat{c}(x, y) = c(x)(y)$. The bijective mapping

$$\alpha: \mathcal{S}^\infty(X, \mathcal{S}^\infty(Y, Z)) \ni c \longmapsto \hat{c} \in \mathcal{S}^\infty(X \times Y, Z)$$

is smooth. In fact, for every $\varphi \in \mathcal{S}^\infty(\mathbb{R}, \mathcal{S}^\infty(X, \mathcal{S}^\infty(Y, Z)))$ one has $\hat{\varphi} \in \mathcal{S}^\infty(\mathbb{R} \times X, \mathcal{S}^\infty(Y, Z))$ and $\alpha \circ \hat{\varphi} \in \mathcal{S}^\infty(\mathbb{R} \times X \times Y, Z)$. Consequently $\alpha_*(\varphi) \in \mathcal{S}^\infty(\mathbb{R}, \mathcal{S}^\infty(X \times Y, Z))$. Analogously α^{-1} is smooth as well.

Proposition 2.3
The set of all smooth maps from $\mathbb{R}$ *to* X *coincides with* $\mathcal{S}^\infty(\mathbb{R}, X)$*, i.e.,*

$$\mathcal{S}^\infty(\mathbb{R}, X) = \{c : \mathbb{R} \longrightarrow X \text{ smooth}\}.$$

Proof. For every smooth curve $c \in \mathcal{S}^\infty(\mathbb{R}, X)$ and $f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ the curve $c_*(f) = c \circ f$ is smooth. Conversely, let $c: \mathbb{R} \longrightarrow X$ be smooth. Then $c = c_*(\mathrm{id}_\mathbb{R}) \in \mathcal{S}^\infty(\mathbb{R}, X)$ by definition.

Proposition 2.4
Let $\{X_\alpha, p_\alpha^\beta\}$ *be an inverse system of smooth spaces. Then its projective limit is naturally endowed with a smooth structure such that*

$$\mathcal{S}^\infty(\mathbb{R}, X) = \lim\operatorname{proj} \mathcal{S}^\infty(\mathbb{R}, X_\alpha)$$

is the projective limit of the inverse system $\{\mathcal{S}^\infty(\mathbb{R}, X_\alpha), p_{\alpha *}^\beta\}$.

Proof. By definition $c = \{c_\alpha\} \in \mathcal{S}^\infty(\mathbb{R}, X)$ if and only if for every $\alpha \in A$ $c_\alpha \in \mathcal{S}^\infty(\mathbb{R}, X_\alpha)$ and $(p_\alpha^\beta)_*(c_\beta) = c_\alpha$ for $\alpha \leq \beta$.

Let $c(t) = \{c_\alpha(t)\} = \{c_\alpha\} \in X$ be a constant curve. Then it is smooth as it is componentwise smooth. Likewise, $\mathcal{S}^\infty(\mathbb{R}, X)$ is closed with respect to a smooth reparametrization if it has this property componentwise. Thus $\mathcal{S}^\infty(\mathbb{R}, X)$ is a smooth structure on X.

Next for every $c = \{c_\alpha\} \in \mathcal{S}^\infty(\mathbb{R}, X)$ we see that

$$(p_\alpha^\beta)_*(c_\beta)(t) = p_\alpha^\beta(c_\beta(t)) = c_\alpha(t)$$

so, by definition, $\lim\operatorname{proj} \mathcal{S}^\infty(\mathbb{R}, X_\alpha) = \mathcal{S}^\infty(\mathbb{R}, X)$.

3. Smooth groups and abstract Lie groups

Let G be a group with the multiplication $\mu: G \times G \ni (g, h) \longmapsto gh \in G$ and the inversion $\nu: G \ni g \longmapsto g^{-1} \in G$. If, in addition, G is a smooth space and μ and ν are smooth as morphisms of smooth structures it is called a *smooth group*. Here the product $G \times G$ is given a smooth structure in the obvious way.

Let G be a smooth group. Now we formulate some conditions for G.

($G1$) Let $\mathfrak{g}$ be a sequentially complete locally convex topological vector space (s.c.l.c.t.v.s. for short), cf. [10], [11]. Assume that there exists a bijective mapping (called the *right evolution operator*)

$$\mathrm{Evol}_G^r: \mathcal{C}^\infty(\mathbb{R}, \mathfrak{g}) \longrightarrow \mathcal{S}_e^\infty(\mathbb{R}, G),$$

where $\mathcal{S}_e^\infty(\mathbb{R}, G)$ is a space of all mappings from $\mathbb{R}$ to G sending 0 to e. Note that we have the natural right action of G onto the space $\mathrm{Map}(\mathbb{R}, G)$. We assume that the union of orbits of mappings from $\mathcal{S}_e^\infty(\mathbb{R},G)$ is equal to $\mathcal{S}^\infty(\mathbb{R},G)$ and that any translation by $g \in G$ is smooth.

Observe that one can use the convenient vector spaces [8] instead of s.c.l.c.t. v.s.

The inverse of Evol_G^r will be denoted by δ_G^r. The exponential mapping $\exp\colon \mathfrak{g} \subset \mathcal{C}^\infty(\mathbb{R},\mathfrak{g}) \longrightarrow G$ is given by $\exp(X) = \mathrm{Evol}_G^r(X)(1)$. Clearly $\exp((t+s)X) = \exp(tX)\,\exp(sX)$.

$(G2)$ One has

$$\mathrm{Evol}_G^r(\mathfrak{g}) \subset \Lambda(G)$$

where $\Lambda(G) := \{\varphi\colon \mathbb{R} \longrightarrow G \mid \ \forall\, t,s \in \mathbb{R} \quad \varphi(t+s) = \varphi(t)\varphi(s)\}$ is the set of one-parameter subgroups of G. Moreover we assume that

$$\Lambda_0(G) := \mathrm{Evol}_G^r(\mathfrak{g})$$

is the totality of smooth one-parameter subgroups of G.

As a consequence, every smooth homeomorphism $f\colon G \longrightarrow H$ induces the tangent map $Tf\colon \Lambda_0(G) \longrightarrow \Lambda_0(H)$.

$(G3)$ The set $\mathcal{S}_e^\infty(\mathbb{R},G)$ admits a cone structure

$$\mathbb{R} \times \mathcal{S}_e^\infty(\mathbb{R},G) \ni (\lambda, f) \longmapsto f^\lambda \in \mathcal{S}_e^\infty(\mathbb{R},G),$$

where $f^\lambda(t) := f(\lambda t)$. We assume that for any $f \in \mathcal{S}_e^\infty(\mathbb{R},G)$ and $\lambda \in \mathbb{R}$ we have $\delta_G^r(f^\lambda) = \lambda\delta_G^r(f)$.

Putting $\lambda = 0$ we get that the constant e belongs to $\mathcal{S}_e^\infty(\mathbb{R},G)$ and $\delta_G^r(e) = 0$.

Let $\mathrm{conj}_g\colon G \ni g \longmapsto hgh^{-1} \in G$ be the conjugation by $g \in G$. Assume that

$$\mathrm{Ad}(g) = T\mathrm{conj}_g.$$

Then we have

$$g(\exp(tX))g^{-1} = \exp(t\mathrm{Ad}(g)X),$$

where $\mathfrak{g}$ is identified with $\Lambda_0(G)$. Clearly for every $g,h \in G$ one has $\mathrm{Ad}(gh) = \mathrm{Ad}(g)\mathrm{Ad}(h)$.

$(G4)$ For any $g \in G$ the mapping $\mathrm{Ad}(g) : \mathfrak{g} \longrightarrow \mathfrak{g}$ is smooth.

$(G5)$ For every $f,g \in \mathcal{S}_e^\infty(\mathbb{R},G)$ we have

$$\delta_G^r(fg) = \delta_G^r f + \mathrm{Ad}(f).\delta_G^r g.$$

Here the dot denotes the componentwise action.

$(G6)$ The map

$$\mathrm{ev}_1 \circ \mathrm{Evol}_G^r\colon \mathcal{C}^\infty(\mathbb{R},\mathfrak{g}) \longrightarrow G,$$

where ev_1 is the evaluation at 1 mapping, is smooth as a morphism of smooth structures.

Definition 3.1
A smooth group G is said to be an *abstract Lie group* if the conditions $(G1)$-$(G6)$ are satisfied.

Remark 3.2
Equivalently we can use the left evolution and left logarithmic derivative operators. Then the formula $(G4)$ is replaced by

$$\delta^l_G(fg) = \delta^l_G g + \mathrm{Ad}(g^{-1})\, \delta^l_G f.$$

Let us mention some examples of abstract Lie groups.

Example 3.3
Any regular Lie group [8] satisfies all the above conditions. In particular, they hold for any finite dimensional Lie group.

The following fact is well known.

Proposition 3.4
Let G be a group in the above example, and $\mathfrak{g} = \mathrm{Lie}(G)$. Then $\mathrm{Evol}^r_G(X)(t) = g(t)$ if and only if $g(0) = e$ and

$$\frac{\partial}{\partial t} g(t) = T_e(\mu^{g(t)})X(t),$$

where $\mu^g(x) = \mu(x, g)$, and μ is the multiplication of G.

As immediate consequences we have the following

Corollary 3.5
Under the assumptions of Proposition 3.4, for any smooth reparametrization $f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ and $X \in \mathcal{C}(\mathbb{R}, \mathfrak{g})$

$$\mathrm{Evol}^r_G(X)(f(t)) = \mathrm{Evol}^r_G(f'.(X \circ f))(t).\mathrm{Evol}^r_G(X)(f(0)).$$

Corollary 3.6
Let G, H be as in Proposition 3.4. If $\varphi\colon G \longrightarrow H$ is a smooth homomorphism then the diagram

$$\begin{array}{ccc} \mathcal{C}^\infty(\mathbb{R}, \mathfrak{g}) & \xrightarrow{(T\varphi)_*} & \mathcal{C}(\mathbb{R}, \mathfrak{h}) \\ \downarrow{\scriptstyle \mathrm{Evol}^r_G} & & \downarrow{\scriptstyle \mathrm{Evol}^r_H} \\ \mathcal{S}^\infty_e(\mathbb{R}, G) & \xrightarrow{\varphi_*} & \mathcal{S}^\infty_e(\mathbb{R}, H) \end{array}$$

commutes, where $\mathfrak{h} = \mathrm{Lie}(H)$.

Example 3.7
The following situation often arise. Given a regular Lie group G, there is a closed subgroup $H \subset G$ and a Lie algebra $\mathfrak{h}$ such that smooth curves with values in $\mathfrak{h}$ are sent bijectively by Evol^r_G to isotopies with values in H. E.g. this takes place if $G = \mathrm{Diff}(M)$, $\mathfrak{g} = \mathfrak{X}_c(M)$, H is the group of automorphisms of a geometric structure on M, and $\mathfrak{h}$ is the Lie algebra of infinitesimal automorphisms of this structure. Unfortunately, such a bijection does not yield a Lie group structure on H and usually it is very difficult to introduce a Lie group structure on H as the possible construction of such a structure involves a deep insight into the geometry determined by H. However, a common intuition is that such a situation is not bad, and sometimes by abuse one states that H is a Lie group with the Lie algebra $\mathfrak{h}$. The second named author formalized this intuition in [16] by introducing the concept of a *weak Lie subgroup.*

Let us mention only some examples of weak Lie subgroups:

(i) Let $(M, \mathcal{F})$ be foliated manifold and $G = \mathrm{Diff}(M)$. Then it was shown in [14] that the subgroup of all leaf preserving diffeomorphisms H is a regular Lie group if $\mathcal{F}$ is a regular foliation. On the other hand, if $\mathcal{F}$ is singular ([18]) then H carries the structure of weak Lie subgroup but it is hopeless to expect that H would admit the usual Lie group structure.

(ii) If $G = \mathrm{Diff}(M)$ and H is the automorphism group of either singular Poisson manifold, or Jacobi manifold, or cosymplectic manifold then H admits the structure of weak Lie subgroup but it would be difficult or impossible to find the usual Lie group structure on H.

(iii) A broad class of groups in differential geometry constitute strict groups. Recall that G is a strict group if it is a subgroup of the group of all smooth bisections of some Lie groupoid.

Recently, it has been shown that for any Lie groupoid its group of all smooth bisections possesses a regular Lie group structure, cf. [15] and references therein. Several subgroups of this group are examples of weak Lie subgroups.

Example 3.8
It is well known that $\mathrm{Diff}(M)$ does not admit a (usual) Lie group structure whenever M is a manifold of infinite dimension (cf. [8]). However, under natural assumption the evolution operator exists in this case (cf. [11]) and, consequently, it is possible to endow $\mathrm{Diff}(M)$ with the structure of an abstract Lie group.

Example 3.9
It is well known that the quotient of an infinite dimensional Lie group by its normal subgroup need not inherit the Lie group structure. When we consider the structure in the abstract sense the situation is better. It can be shown that the quotient of an abstract Lie group by its normal subgroup satisfies $(G1)$, $(G3)$-$(G5)$ and the first assertion of $(G2)$.

Example 3.10
The basic examples in this paper are LP-groups and locally compact topological groups, cf. Sections 5 and 6.

4. Basic properties of abstract Lie groups

Let $f\colon G \longrightarrow H$ be a smooth homomorphism, where G and H are abstract Lie groups modelled on s.c.l.c.t.v. spaces $\mathfrak{g}$ and $\mathfrak{h}$, resp. Then, in view of $(G2)$, for any $X \in \mathfrak{g}$ the mapping

$$\mathbb{R} \ni t \longmapsto f(\exp_G(tX)) \in H$$

is a smooth one-parameter subgroup of H. There exists a unique $Y \in \mathfrak{h}$ such that $f(\exp_G(tX)) = \exp_H(tY)$. We define $Tf(X) := Y$. It is a linear mapping and we have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\;Tf\;} & \mathfrak{h} \\ \big\downarrow{\scriptstyle \exp_G} & & \big\downarrow{\scriptstyle \exp_H} \\ G & \xrightarrow{\;f\;} & H \end{array}$$

For all $X \in \mathfrak{g}$ we have the following formulae

$$\mathrm{Ad}(\exp tX)X = X \tag{4.1}$$

and

$$\mathrm{Ad}(g \exp tX)Y = \mathrm{Ad}(\exp t(\mathrm{Ad}(g)X))\mathrm{Ad}(g)Y. \tag{4.2}$$

Indeed,

$$\begin{aligned} &g \exp tX \exp tY (\exp tX)^{-1} g^{-1} \\ &\quad = g \exp tX g^{-1} g \exp tY g^{-1} (g \exp tX g^{-1})^{-1} \\ &\quad = \exp t(\mathrm{Ad}(g)X) \exp t(\mathrm{Ad}(g)Y)(\exp t(\mathrm{Ad}(g)X))^{-1} \end{aligned}$$

As usual, we let $\mathrm{ad} := T\mathrm{Ad}$. Then $\mathrm{ad}_X(Y) = [X, Y] = \frac{\partial}{\partial t}|_{t=0}\mathrm{Ad}(\exp tX)Y$.

Proposition 4.1
The bracket $[\,,\,]$ is a Lie algebra bracket on $\mathfrak{g}$.

Proof. By $(G4)$ $[\,,\,]$ is linear in the second variable since $\mathrm{ad} = T\mathrm{Ad}$.
Next

$$\begin{aligned} [X + Y, Z] &= \frac{\partial}{\partial t}|_{t=0}\mathrm{Ad}(\exp tX \exp tY)Z \\ &= \frac{\partial}{\partial t}|_{t=0}\mathrm{Ad}(\exp tX)\mathrm{Ad}(\exp tY)Z \\ &= [X, Z] + [Y, Z] \end{aligned}$$

The skew-symmetry follows by (4.1). The Jacobi identity we get by (4.2)

$$\begin{aligned}\frac{\partial}{\partial t}|_{t=0}\mathrm{Ad}(g\exp tX)Y &= \frac{\partial}{\partial t}|_{t=0}\mathrm{Ad}(\exp t(\mathrm{Ad}(g)X))\mathrm{Ad}(g)Y\\ &= [\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y],\end{aligned}$$

where we use the condition ($G4$).

The following can be shown by a standard argument

Proposition 4.2
If (G, Evol^r_G) and (H, Evol^r_H) are abstract Lie groups, and $\varphi : G \longrightarrow H$ be a smooth homomorphism. Then $T\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a Lie algebra homomorphism.

Corollary 4.3
The s.c.l.c.t.v.s. $\mathfrak{g}$ in Definition 3.1 is uniquely defined and admits a uniquely defined Lie algebra structure.

From now on $\mathfrak{g}$ will be called the Lie algebra of G and will be denoted by $\mathrm{Lie}(G)$.

Is seems that contrary to other abstract settings of Lie theory, our setting enables to introduce an appropriate notion of Lie subgroups in infinite dimension. This notion is strictly connected with the evolution operator.

Definition 4.4
A subgroup $H \subset G$ is said to be a *Lie subgroup* if there exists a closed subspace $\mathfrak{h} \subset \mathfrak{g}$ such that

$$\mathrm{Evol}^r_G|_{\mathcal{C}^\infty(\mathbb{R},\mathfrak{h})} : \mathcal{C}^\infty(\mathbb{R}, \mathfrak{h}) \longrightarrow \mathcal{S}^\infty_e(\mathbb{R}, G) \cap \mathrm{Map}_e(\mathbb{R}, H)$$

is a bijection. Then we set

$$\mathrm{Evol}^r_H := \mathrm{Evol}^r_G|_{\mathcal{C}^\infty(\mathbb{R},\mathfrak{h})} \quad \text{and} \quad \mathcal{S}^\infty_e(\mathbb{R}, H) := \mathcal{S}^\infty(\mathbb{R}, G) \cap \mathrm{Map}_e(\mathbb{R}, H).$$

Consequently we have

$$X \in \mathfrak{h} \iff \exp tX \in \mathcal{S}^\infty_e(\mathbb{R}, H) \tag{4.3}$$

i.e. $\Lambda_0(H) = \Lambda_0(G) \cap \mathcal{S}^\infty_e(\mathbb{R}, H)$.

As a consequence one can prove the following

Proposition 4.5
If $H \subset G$ is a Lie subgroup then (H, Evol^r_H) is an abstract Lie group itself. Furthermore, if H_i is a Lie subgroup of G_i, $i = 1, 2$, and $\varphi : G_1 \longrightarrow G_2$ is a morphism with $\varphi(H_1) \subset H_2$ then $\varphi|_{H_1} : H_1 \longrightarrow H_2$ is also a morphism and $T(\varphi|_{H_1}) = T\varphi|_{\mathfrak{h}}$.

Since a celebrated paper on non-enlargibility by van Est and Korthagen [5] it is well known that the third Lie theorem is, in general, no longer true in the infinite dimensional case. However, there are several important infinite dimensional generalizations, e.g. [4], [13], [19], [7]. In our abstract setting this theorem holds under natural conditions on the Lie algebra.

Let us formulate two conditions related to the integrability in the abstract sense of a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

(A) If $g \in \mathcal{S}_e^\infty(\mathbb{R}, G)$ and for any $t \in \mathbb{R}$ there exists $h \in \mathcal{S}_e^\infty(\mathbb{R}, G)$ with $\delta_G^r(h) \in \mathcal{C}(\mathbb{R}, \mathfrak{h})$ and $h_t = g_t$ then $\delta_G^r(g) \in \mathcal{C}(\mathbb{R}, \mathfrak{h})$. In other words, any G-isotopy with values in H is an H-isotopy, where the group H is defined as in the theorem below.

(B) For any $g \in \mathcal{S}_e^\infty(\mathbb{R}, G)$ with $\delta_G^r(g) \in \mathcal{C}(\mathbb{R}, \mathfrak{h})$ one has $\mathrm{Ad}(g)\mathfrak{h} \subset \mathfrak{h}$.

The following is proved in [16].

Theorem 4.6
Let $\mathfrak{h}$ be a Lie subalgebra satisfying the conditions (A) and (B). Define

$$H = \{h \in G: \ h = g_1, \ \textit{where } g \in \mathcal{S}_e^\infty(\mathbb{R}, G), \ \delta_G^r(g) \in \mathcal{C}(\mathbb{R}, \mathfrak{h})\}.$$

Then H is a weak Lie subgroup of G with the Lie algebra $\mathfrak{h}$.

Remark 4.7
The condition (B) is closely related to the notion of sigular foliations [18] whenever $G = \mathrm{Diff}(M)$. Namely it says that the distribution defined by $\mathfrak{h} \subset \mathfrak{X}_c(M)$ integrates to a singular foliation.

Example 4.8
Let ω be a symplectic structure on M, and let $\sharp^\omega\colon \Omega^1(M) \longrightarrow \mathfrak{X}(M)$ be the corresponding musical isomorphism.

We set ($\mathcal{L}$ is the Lie derivative)

$$L(M, \omega) = \{X \in \mathfrak{X}_c(M): \ \mathcal{L}_X \omega = 0\}$$
$$G(M, \omega) = \{f \in \mathrm{Diff}(M): \ f^*\omega = \omega\}.$$

Then $L(M, \omega)$ is a Lie subalgebra of $\mathfrak{X}_c(M)$, and $G(M, \omega)$ is its Lie group in view of a well known result by Weinstein [19].

A smooth path of diffeomorphisms $f(t)$ satisfying $\delta_{\mathrm{Diff}(M)}^r(f)(t) = X(t) \in L(M, \omega)$ for any t is called a symplectic isotopy.

A symplectic isotopy $f(t)$ is *Hamiltonian* (or *exact*) if $\delta_{\mathrm{Diff}(M)}^r(f)(t) = X(t) \in L^*(M, \omega)$ for each t. Here $L^*(M, \omega)$ stands for the Lie algebra Hamiltonian vector fields, i.e. $X \in L^*(M, \omega)$ iff $\iota(X)\omega$ is exact, ι being the insertion.

A diffeomorphism f of M is called *Hamiltonian* if there exists a Hamiltonian isotopy $f(t)$ such that $f(0) = \mathrm{id}$ and $f(1) = f$. The totality of all Hamiltonian diffeomorphisms is denoted by $G^*(M, \omega)$.

It can be shown that $G^*(M, \omega)$ is a normal subgroup of $G(M, \omega)$.

Observe that $G^*(M, \omega)$ is a Lie subgroup in the abstract sense. It is a usual Lie group if and only if the group of periods of ω (i.e. the image by the flux homomorphism of the first homotopy group of $G(M, \omega)$) is discrete.

Analogous statements still hold for locally conformal symplectic structures, cf. [7], in view of the existence the flux homomorphism and other invariants, and for regular Poisson manifolds, cf. [14]. For singular Poisson manifolds all groups in question admit only the structure of abstract Lie group.

5. LP-groups and LP-Lie algebras

In this section we recall the concept of LP-groups which can serve as an example of abstract Lie groups.

A topological group G is called an *LP-group* if it is the projective limit of finite dimensional Lie groups, i.e. there exists an inverse system of finite dimensional Lie groups $\{G_\alpha, p_\alpha^\beta\}$ such that $G = \lim\mathrm{proj}\,\{G_\alpha, p_\alpha^\beta\}$. Here the index set A is directed and p_α^β for $\alpha \leq \beta$ denotes a continuous and open epimorphism, which maps the Lie group G_β onto G_α. In particular, one has $p_\alpha^\beta \circ p_\beta^\gamma = p_\alpha^\gamma$, if $\alpha \leq \beta \leq \gamma$.

For every $\alpha \in A$ there exists a canonical continuous epimorphism $p_\alpha \colon G \longrightarrow G_\alpha$ and it holds $p_\alpha^\beta \circ p_\beta = p_\beta$, if $\alpha \leq \beta$. The mappings p_α for $\alpha \in A$ can be proved to be open.

Let us denote $\mathfrak{g}_\alpha = \mathrm{Lie}(G_\alpha)$ and let $\exp_\alpha \colon \mathfrak{g}_\alpha \longrightarrow G_\alpha$ be the corresponding exponential map. Then there exists a unique epimorphism $Tp_\alpha^\beta \colon \mathfrak{g}_\beta \longrightarrow \mathfrak{g}_\alpha$ such that $\exp_\beta \circ Tp_\alpha^\beta = p_\alpha^\beta \circ \exp_\alpha$ for $\alpha \leq \beta$. If $\alpha \leq \beta \leq \gamma$, then the equation $p_\alpha^\beta \circ p_\beta^\gamma = p_\alpha^\gamma$ implies $Tp_\alpha^\beta \circ Tp_\beta^\gamma = Tp_\alpha^\gamma$ and $\{\mathfrak{g}_\alpha, Tp_\alpha^\beta\}$ is an inverse system of finite dimensional Lie algebras defining a topological Lie algebra $\mathfrak{g} = \lim\mathrm{proj}\,\{\mathfrak{g}_\alpha, Tp_\alpha^\beta\}$.

A topological Lie algebra is called an *LP-Lie algebra* if it is the projective limit of an inverse system of finite dimensional Lie algebras. To every LP-group G corresponds an LP-Lie algebra $\mathfrak{g}$, which we call the *Lie algebra of the LP-group* G and denote by $\mathfrak{g} = \mathrm{Lie}(G)$.

It is well known [9], [6] that the Lie algebra $\mathrm{Lie}(G)$ is independent of an inverse system representing the LP-group G as the projective limit of Lie groups. It follows from the categorical properties that there exists a unique continuous lift $\exp\colon \mathfrak{g} \longrightarrow G$ of the mappings $\exp_\alpha \colon \mathfrak{g}_\alpha \longrightarrow G_\alpha$ for $\alpha \in A$ such that for every $\alpha \in A$ $\exp_\alpha \circ Tp_\alpha = p_\alpha \circ \exp$ holds.

Moreover, for any $\alpha \leq \beta$ the following diagram

$$\begin{array}{ccccc}
\mathfrak{g}_\alpha & \xleftarrow{Tp_\alpha^\beta} & \mathfrak{g}_\beta & \xleftarrow{Tp_\alpha} & \mathfrak{g} \\
\downarrow \scriptstyle{\exp_\alpha} & & \downarrow \scriptstyle{\exp_\beta} & & \downarrow \scriptstyle{\exp} \\
G_\alpha & \xleftarrow{p_\alpha^\beta} & G_\beta & \xleftarrow{p_\alpha} & G
\end{array}$$

is commutative.

In order to define explicitly exp, let us assume that G is represented as a closed subgroup of the direct product $\prod G_\alpha$ and that $\mathfrak{g}$ is represented as a closed subalgebra of the direct product $\prod \mathfrak{g}_\alpha$. Then we can define the lifted map exp in the following way.

Given $X \in \mathfrak{g}$, let $\{X_\alpha = Tp_\alpha(X)\}$ be the corresponding element in $\prod \mathfrak{g}_\alpha$. Then one has $X_\alpha = Tp_\alpha^\beta(X_\beta)$, if $\alpha \leq \beta$, and $\exp X$ is the element of G which corresponds to $\{\exp_\alpha X_\alpha\}$ of $\prod G_\alpha$.

If $\alpha \leq \beta$ then $\exp_\alpha X_\alpha = \exp_\alpha(Tp_\alpha^\beta(X_\beta)) = p_\alpha^\beta(\exp_\beta X_\beta)$ holds and this implies the existence of $\exp X$. Using the commutativity of the above diagram we get the continuity of exp since the mappings p_α are open and the mappings Tp_α and $\exp_\alpha$ are continuous.

Every $\varphi \in \Lambda(G)$ defines an element $\varphi_\alpha = p_\alpha \circ \varphi \in \Lambda(G_\alpha)$ for every $\alpha \in A$. Since G_α is a Lie group there exists a unique element $X_\alpha \in \mathfrak{g}_\alpha$ such that $\varphi_\alpha(t) = \exp_\alpha tX_\alpha$. Then we have

$$\begin{aligned}
\exp_\alpha tX_\alpha &= \varphi_\alpha(t) = p_\alpha \circ \varphi(t) = p_\alpha^\beta \circ p_\beta \circ \varphi(t) \\
&= p_\alpha^\beta(\varphi_\beta(t)) = p_\alpha^\beta(\exp_\beta(tX_\alpha)) \\
&= \exp_\alpha(tTp_\alpha^\beta(X_\beta))
\end{aligned}$$

for any $\alpha \leq \beta$. This implies that $X_\alpha = Tp_\alpha^\beta(X_\beta)$, if $\alpha \leq \beta$. Consequently the family $\{X_\alpha\}$ defines an element $X \in \mathfrak{g}$ such that $\varphi(t) = \exp(tX)$, $t \in \mathbb{R}$.

On the other hand every $X \in \mathfrak{g}$ defines a one-parameter subgroup $\varphi^X(t) = \exp(tX)$ of the LP-group G. Thus we have proved that every $\varphi \in \Lambda(\mathbb{R}, G)$ is defined by an element $X \in \mathfrak{g}$.

Now we will prove the injectivity of exp. Let us assume that

$$\exp(tX_1) = \exp(tX_2), \qquad X_1, X_2 \in \mathfrak{g}.$$

It follows that

$$\exp_\alpha(tTp_\alpha(X_1)) = p_\alpha(\exp(tX_1)) = p_\alpha(\exp(tX_2)) = \exp_\alpha(tTp_\alpha(X_2))$$

and hence $Tp_\alpha X_1 = Tp_\alpha X_2$ for every $\alpha \in A$. This implies $X_1 = X_2$ and the considered mapping is injective.

Proposition 5.1

For any LP-group G, $\Lambda(G)$ is a Lie algebra and the exponential mapping

$$\exp_G\colon \mathfrak{g} \ni X \longmapsto \varphi^X \in \Lambda(G)$$

is an isomorphism of Lie algebras.

Proof (see also [2]). We have already shown that the this mapping is bijective. It remains to prove that it a homomorphism as well. We use the Trotter formulae. Assume $\varphi_1, \varphi_2 \in \Lambda(G)$. Then for every $\alpha \in A$ holds

$$\begin{aligned}(p_\alpha \circ \varphi_1 + p_\alpha \circ \varphi_2)(t) &= \lim_{k\to\infty} \left(p_\alpha \circ \varphi_1 \left(\frac{t}{k}\right) p_\alpha \circ \left(\frac{t}{k}\right)\right)^k \\ &= \lim_{k\to\infty} p_\alpha \left(\varphi_1 \left(\frac{t}{k}\right) \varphi_2 \left(\frac{t}{k}\right)\right)^k,\end{aligned}$$

and

$$\begin{aligned}&[p_\alpha \circ \varphi_1, p_\alpha \circ \varphi_2](t^2) \\ &\quad= \lim_{k\to\infty} \left(p_\alpha \circ \varphi_1 \left(\frac{t}{k}\right) p_\alpha \circ \varphi_2 \left(\frac{t}{k}\right) p_\alpha \circ \varphi_1 \left(-\frac{t}{k}\right) p_\alpha \circ \varphi_2 \left(-\frac{t}{k}\right)\right)^{k^2} \\ &\quad= \lim_{k\to\infty} p_\alpha \left(\varphi_1 \left(\frac{t}{k}\right) \varphi_2 \left(\frac{t}{k}\right) \varphi_1 \left(-\frac{t}{k}\right) \varphi_2 \left(-\frac{t}{k}\right)\right)^{k^2}.\end{aligned}$$

This implies the existence of the limits

$$\lim_{k\to\infty} \left(\varphi_1 \left(\frac{t}{k}\right) \varphi_2 \left(\frac{t}{k}\right)\right)^k,$$

$$\lim_{k\to\infty} p_\alpha \left(\varphi_1 \left(\frac{t}{k}\right) \varphi_2 \left(\frac{t}{k}\right) \varphi_1 \left(-\frac{t}{k}\right) \varphi_2 \left(-\frac{t}{k}\right)\right)^{k^2}.$$

It is true that $p_\alpha \circ \varphi_i(t) = \exp_\alpha(t(X_i)_\alpha)$ for $i = 1, 2$ and every $\alpha \in A$, so

$$\begin{aligned}p_\alpha \circ (\varphi_1 + \varphi_2)(t) &= p_\alpha \left(\lim_{k\to\infty} \left(\varphi_1 \left(\frac{t}{k}\right) \varphi_2 \left(\frac{t}{k}\right)\right)^k\right) \\ &= \lim_{k\to\infty} p_\alpha \left(\varphi_1 \left(\frac{t}{k}\right) \varphi_2 \left(\frac{t}{k}\right)\right)^k \\ &= (p_\alpha \circ \varphi_1 + p_\alpha \circ \varphi_2)(t) \\ &= \exp_\alpha t((X_1)_\alpha + (X_2)_\alpha)\end{aligned}$$

for every $\alpha \in A$.

Moreover

$$\begin{aligned}p_\alpha \circ [\varphi_1, \varphi_2](t^2) &= p_\alpha \left(\lim_{k\to\infty} p_\alpha \left(\varphi_1 \left(\frac{t}{k}\right) \varphi_2 \left(\frac{t}{k}\right) \varphi_1 \left(-\frac{t}{k}\right) \varphi_2 \left(-\frac{t}{k}\right)\right)^{k^2}\right) \\ &= \lim_{k\to\infty} p_\alpha \left(\varphi_1 \left(\frac{t}{k}\right) \varphi_2 \left(\frac{t}{k}\right) \varphi_1 \left(-\frac{t}{k}\right) \varphi_2 \left(-\frac{t}{k}\right)\right)^{k^2} \\ &= [p_\alpha \circ \varphi_1, p_\alpha \circ \varphi_2](t^2) \\ &= \exp_\alpha t^2[(X_1)_\alpha, (X_2)_\alpha].\end{aligned}$$

Let $X_i = \{(X_i)_\alpha\} \in \mathfrak{g}$, $i = 1, 2$. Then $\varphi_i(x) = \exp(tX_i)$, and we have conditions

$$(\varphi_1 + \varphi_2)(t) = \exp(t(X_1 + X_2)),$$
$$[\varphi_1, \varphi_2](t) = \exp(t[X_1, X_2]).$$

We get that the mapping $\exp_G$ is an isomorphism.

Finally let us mention the existence of a universal covering group for any LP-group. Let $\mathfrak{g}$ denote an arbitrary LP-Lie algebra and $\{\mathfrak{g}_\alpha, q_\alpha^\beta\}$ an inverse system of finite dimensional Lie algebras such that $\mathfrak{g} = \lim\operatorname{proj}\{\mathfrak{g}_\alpha, q_\alpha^\beta\}$.

Let G_α be a unique connected and simply connected Lie group corresponding to $\mathfrak{g}_\alpha$. There exists a continuous and open epimorphism $\tilde{p}_\alpha^\beta \colon \tilde{G}_\beta \longrightarrow \tilde{G}_\alpha$, if $\alpha \leq \beta$ such that $T\tilde{p}_\alpha^\beta = q_\alpha^\beta$.

We have that $\{\tilde{G}_\alpha, \tilde{p}_\alpha^\beta\}$ is an inverse system of Lie groups and it defines an LP-group $\tilde{G} = \lim\operatorname{proj}\{\tilde{G}_\alpha, \tilde{p}_\alpha^\beta\}$.

It can be proven that for any LP-Lie algebra $\mathfrak{g}$ corresponds a unique simply connected LP-group $\tilde{G}$ such that $\mathfrak{g} = \mathrm{Lie}(\tilde{G})$ defined in this way [2]. The group $\tilde{G} = \mathrm{Lie}(\mathfrak{g})$ is called the *universal LP-group corresponding to the LP-Lie algebra* $\mathfrak{g}$.

Starting with a connected LP-group $G = \lim\operatorname{proj}\{G_\alpha, Tp_\alpha^\beta\}$ we get its Lie algebra $\mathfrak{g} = \lim\operatorname{proj}\{\mathfrak{g}_\alpha, Tp_\alpha^\beta\}$, which is an LP-Lie algebra and hence defines a universal LP-group $\tilde{G} = \lim\operatorname{proj}\{\tilde{G}_\alpha, \tilde{p}_\alpha^\beta\}$. The Lie group $\tilde{G}_\alpha$ is the universal covering group of the connected Lie group G_α for every $\alpha \in A$. Let $\tilde{q}_\alpha \colon \tilde{G}_\alpha \longrightarrow G_\alpha$ denote the covering epimorphism. There exists a lift $\tilde{q} \colon \tilde{G} \longrightarrow G$ of the covering maps $\tilde{q}_\alpha$, $\alpha \in A$, satisfying the equations $p_\alpha \circ \tilde{q} = \tilde{q}_\alpha \circ \tilde{p}_\alpha$ and the following diagram

$$\begin{array}{ccccc}
\tilde{G}_\alpha & \xleftarrow{\ \tilde{p}_\alpha^\beta\ } & \tilde{G}_\beta & \xleftarrow{\ \tilde{p}_\alpha\ } & \tilde{G} \\
\downarrow{\scriptstyle \tilde{q}_\alpha} & & \downarrow{\scriptstyle \tilde{q}_\beta} & & \downarrow{\scriptstyle \tilde{q}} \\
G_\alpha & \xleftarrow{\ p_\alpha^\beta\ } & G_\beta & \xleftarrow{\ p_\alpha\ } & G
\end{array}$$

is commutative for any $\alpha \leq \beta$.

One can show that the mapping $\tilde{q}$ is a continuous homomorphism which maps $\tilde{G}$ onto a dense algebraic subgroup G_0 of G, but in general $\tilde{q}$ is not a covering mapping and even it need not be surjective.

However, in view of [2], the mapping $\tilde{q}$ is a continuous epimorphism if G is an arcwise connected LP-group.

6. LP-groups as abstract Lie groups

PROPOSITION 6.1
The projective limit of an inverse system of groups carries a natural group structure. Moreover, if $G = \lim\operatorname{proj}\{G_\alpha, p_\alpha^\beta\}$ *is the projective limit of smooth groups with smooth homomorphisms* p_α^β*, then* G *is a smooth group and the canonical projections* $p_\alpha: G \longrightarrow G_\alpha$ *are smooth.*

Proof. It is obvious that

$$\mu_\alpha \circ (p_\alpha^\beta \times p_\alpha^\beta) = p_\alpha^\beta \circ \mu_\beta,$$

$$\nu_\alpha \circ p_\alpha^\beta = p_\alpha^\beta \circ \nu_\beta.$$

Taking $\mu(g_1, g_2) = \{\mu_\alpha((g_1)_\alpha, (g_2)_\alpha)\}$, $e := \{e_\alpha\}$ and $\{f_\alpha\}^{-1} := \{f_\alpha^{-1}\}$ we see that G is a group.

Next, to prove the second assertion we have to show that the mappings $\mu: G \times G \longrightarrow G$ and $\nu: G \longrightarrow G$ are smooth. Let $f, g \in \mathcal{S}^\infty(\mathbb{R}, G)$. Then $\mu(f, g) := \{\mu_\alpha(f_\alpha, g_\alpha)\}$, $\nu(f) := \{f_\alpha^{-1}\}$, where μ_α is the multiplication of G_α. Consequently, $\mu(f, g) \in \mathcal{S}^\infty(\mathbb{R}, G)$ by definition, and μ is smooth. Likewise, the inversion ν is smooth.

LEMMA 6.2
Let X_α*,* Y_α *be smooth spaces,* $\alpha \in A$*, and let* $X = \lim\operatorname{proj}\{X_\alpha, p_\alpha^\beta\}$*,* $Y = \lim\operatorname{proj}\{Y_\alpha, q_\alpha^\beta\}$ *be the corresponding smooth spaces (cf. Proposition 2.4). If for any* α*,* $f_\alpha: X_\alpha \longrightarrow Y_\alpha$ *is smooth such that* $q_\alpha^\beta \circ f_\beta = f_\alpha \circ p_\alpha^\beta$ *for any* $\alpha \le \beta$*, then* $f = \{f_\alpha\}$ *is smooth.*

Proof. Let $c = \{c_\alpha\} \in \mathcal{S}^\infty(\mathbb{R}, X)$. Then clearly $f \circ c = \{f_\alpha \circ c_\alpha\} \in \mathcal{S}^\infty(\mathbb{R}, Y)$ and the lemma follows.

We have the following main result.

THEOREM 6.3
Let G *be an LP-group. Then* G *admits the structure of an abstract Lie group. Moreover, the Lie algebra of* G *coincides with the Lie algebra defined by the abstract Lie group structure of* G*, and* $\exp_G$ *coincides with that defined by* Evol_G^r*.*

Proof. Let $G = \lim\operatorname{proj} G_\alpha$, where $\{G_\alpha, p_\alpha^\beta\}$ is an inverse system of finite dimensional Lie groups, be an LP-group.

Let $\mathfrak{g} = \lim\operatorname{proj}\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \mathrm{Lie}(G_\alpha)$ is the Lie algebra of G_α. Let us define Evol_G^r as the projective limit of the corresponding inverse system of the mappings $\mathrm{Evol}_{G_\alpha}^r$,

$$\mathrm{Evol}_G^r := \lim\operatorname{proj}\mathrm{Evol}_{G_\alpha}^r .$$

This definition is correct since for any $\alpha \le \beta$ we have that the following diagram

$$\begin{array}{ccc}
\mathcal{C}^\infty(\mathbb{R},\mathfrak{g}_\alpha) & \xleftarrow{(Tp^\beta_\alpha)_*} & \mathcal{C}^\infty(\mathbb{R},\mathfrak{g}_\beta) \\
\downarrow{\scriptstyle \mathrm{Evol}^r_{G_\alpha}} & & \downarrow{\scriptstyle \mathrm{Evol}^r_{G_\beta}} \\
\mathcal{S}^\infty_e(\mathbb{R},G_\alpha) & \xleftarrow{(p^\beta_\alpha)_*} & \mathcal{S}^\infty_e(\mathbb{R},G_\beta)
\end{array}$$

commutes in view of Corollary 3.6.

Evol^r_G is bijective. In fact, this follows by the bijectivity of $\mathrm{Evol}^r_{G_\alpha}$ for any $\alpha \in A$ and the general property of projective limits. Thus we have showed that the set of smooth curves $\mathcal{S}^\infty_e(\mathbb{R},G)$ is defined by the bijective map Evol^r_G.

In order to show that $(G1)$ is satisfied it suffices to check that

$$\mathcal{S}^\infty(\mathbb{R},G) = \bigcup_{g\in G} \mathcal{S}^\infty_e(\mathbb{R},G).g$$

is indeed a smooth structure. That all constants belong to $\mathcal{S}^\infty(\mathbb{R},G)$ follows from $(G2)$ proved below. Next let $g \in \mathcal{S}^\infty(\mathbb{R},G)$ and $f \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$. By using the translation by $g(0)^{-1} \in G$ and the condition $(G1)$ we may assume that $g \in \mathcal{S}^\infty_e(\mathbb{R},G)$, i.e. $g = \mathrm{Evol}^r_G(X)$. Next in view of Corollary 3.5 one has

$$\mathrm{Evol}^r_G(X)(f(t)) = \mathrm{Evol}^r_G(f'.(X\circ f))(t).\mathrm{Evol}^r_G(X)(f(0)).$$

Consequently, $g\circ f = \mathrm{Evol}^r_G(X)\circ f \in \mathcal{S}^\infty(\mathbb{R},G)$. Thus $(G1)$ is fulfilled.

Specifically, the following diagram

$$\begin{array}{ccccc}
\mathcal{C}^\infty(\mathbb{R},\mathfrak{g}_\alpha) & \xleftarrow{(Tp^\beta_\alpha)_*} & \mathcal{C}^\infty(\mathbb{R},\mathfrak{g}_\beta) & \xleftarrow{(Tp_\beta)_*} & \mathcal{C}^\infty(\mathbb{R},\mathfrak{g}) \\
\downarrow{\scriptstyle \mathrm{Evol}^r_{G_\alpha}} & & \downarrow{\scriptstyle \mathrm{Evol}^r_{G_\beta}} & & \downarrow{\scriptstyle \mathrm{Evol}^r_{G}} \\
\mathcal{S}^\infty_e(\mathbb{R},G_\alpha) & \xleftarrow{(p^\beta_\alpha)_*} & \mathcal{S}^\infty_e(\mathbb{R},G_\beta) & \xleftarrow{(p_\beta)_*} & \mathcal{S}^\infty_e(\mathbb{R},G)
\end{array}$$

is commutative for any $\alpha \le \beta$.

To check $(G3)$ let $f \in \mathcal{S}^\infty_e(\mathbb{R},G)$ and let $f^\lambda \in \mathrm{Map}(\mathbb{R},G)$ is given by $f^\lambda(t) := f(\lambda t) = \{f_\alpha(\lambda t)\}$ for $\lambda \in \mathbb{R}$. Then, due to $(G1)$, $f^\lambda \in \mathcal{S}^\infty_e(\mathbb{R},G)$ and we have

$$\delta^r_G(f^\lambda) = \{\delta^r_{G_\alpha}(f^\lambda_\alpha)\} = \{\lambda\delta^r_{G_\alpha}(f_\alpha)\} = \lambda\{\delta^r_{G_\alpha}(f_\alpha)\} = \lambda\delta^r_G(f).$$

Next the condition $(G4)$ follows from Lemma 6.2 as for $g = \{g_\alpha\} \in G$ one has $\mathrm{Ad}_G(g) = \{\mathrm{Ad}_{G_\alpha}(g_\alpha)\}$.

The condition $(G5)$ holds by the componentwise computation

$$\begin{aligned}
\delta^r_G(fg) &= \{\delta^r_{G_\alpha}(f_\alpha g_\alpha)\} = \{\delta^r_{G_\alpha}(f_\alpha)\} + \{\mathrm{Ad}_{G_\alpha}(f_\alpha)\,\delta^r_{G_\alpha}(g_\alpha)\} \\
&= \{\delta^r_{G_\alpha}(f_\alpha)\} + \{\mathrm{Ad}_{G_\alpha}(f_\alpha)\}\{\delta^r_{G_\alpha}(g_\alpha)\} \\
&= \delta^r_G(f) + \mathrm{Ad}_G(f)\,\delta^r_G(g).
\end{aligned}$$

In order to show $(G6)$ let us observe that the following diagram

$$\begin{array}{ccccc}
\mathcal{S}^\infty(\mathbb{R}, G_\alpha) & \xleftarrow{(p^\beta_\alpha)_*} & \mathcal{S}^\infty(\mathbb{R}, G_\beta) & \xleftarrow{(p_\beta)_*} & \mathcal{S}^\infty(\mathbb{R}, G) \\
\downarrow \mathrm{ev}^\alpha_1 & & \downarrow \mathrm{ev}^\beta_1 & & \downarrow \mathrm{ev}_1 \\
G_\alpha & \xleftarrow{p^\beta_\alpha} & G_\beta & \xleftarrow{p_\beta} & G
\end{array}$$

is commutative, where $\mathrm{ev}_1 := \lim\mathrm{proj}\,\mathrm{ev}^\alpha_1$, since

$$\begin{aligned} p_\beta \circ \mathrm{ev}_1(\varphi) &= p_\beta(\{\mathrm{ev}^\alpha_1(\varphi_\alpha)\}) = \mathrm{ev}^\beta_1(\varphi_\beta) = \varphi_\beta(1) = p_{\beta_*}(\varphi(1)) \\ &= \mathrm{ev}^\beta_1(p_{\beta_*}(\varphi)). \end{aligned}$$

Hence

$$\mathrm{ev}_1 \circ \mathrm{Evol}^r_G(\{f_\alpha\}) = \{\mathrm{ev}^\alpha_1 \circ \mathrm{Evol}^r_{G_\alpha}(f_\alpha)\} \in G,$$

and in view of Lemma 6.2 the condition follows.

Finally we have to prove $(G2)$. The first assertion is fulfilled by the definition of Evol^r_G. Now let $\varphi \in \Lambda_0(G)$. Then $\varphi = \{\varphi_\alpha\}$ and $\varphi_\alpha \in \Lambda_0(G_\alpha)$. For any $\alpha \leq \beta$ one has $\varphi_\alpha = p^\beta_\alpha \circ \varphi_\beta$. If $\varphi_\alpha = \exp_{G_\alpha}(X_\alpha)$ then for $X = \{X_\alpha\}$ we have

$$\frac{d}{dt}|_0\, p^\beta_\alpha(\exp_{G_\beta}(tX_\beta)) = Tp^\beta_\alpha(X_\beta).$$

On the other hand

$$\frac{d}{dt}|_0\, \exp_{G_\alpha}(tTp^\beta_\alpha(X_\beta)) = T\exp_{G_\alpha}.Tp^\beta_\alpha(X_\beta) = Tp^\beta_\alpha(X_\beta).$$

and

$$p^\beta_\alpha(\exp_{G_\beta}(0)) = e_\alpha = \exp_{G_\alpha}(0).$$

So we have

$$p^\beta_\alpha(\exp_{G_\beta}(tX_\beta)) = \exp_{G_\alpha}(tTp^\beta_\alpha(X_\beta))$$

if $\alpha \leq \beta$. Consequently, we have $X = \{X_\alpha\} \in \mathfrak{g} = \lim\mathrm{proj}\,\mathfrak{g}_\alpha$ and $\exp_G(X) = \varphi$ as required. $(G2)$ is then proved.

From the above considerations it is clear that $\mathfrak{g} = \lim\mathrm{proj}\,\mathfrak{g}_\alpha = \mathrm{Lie}(G)$. Moreover for $X \in \mathfrak{g}$

$$\exp_G(X) = \mathrm{Evol}^r_G(X)(1) = \{\mathrm{Evol}^r_{G_\alpha}(X_\alpha)(1)\} = \{\exp_{G_\alpha}(X_\alpha)\}$$

so we have $\exp_G = \lim\mathrm{proj}\,\exp_{G_\alpha}$.

From the above proof it follows the following

Corollary 6.4
If G is an LP-group then $\Lambda(G) = \Lambda_0(G)$.

Finally we have

Theorem 6.5
Let G be a connected locally compact topological group. Then G admits the structure of an abstract Lie group. Furthermore its Lie algebra and the exponential mapping are the same as those determined by the abstract Lie group structure.

Proof. In fact, by Yamabe theorem such a G can be expressed as the projective limit of an inverse system of finite dimensional Lie groups. Moreover, one defines $\mathfrak{g} = \mathrm{Lie}(G)$ and $\exp_G$ independently of the inverse system chosen, so Theorem 6.3 applies to G.

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Sur la fonction de choix et la fonction d'indice

Dédié à M. le Professeur Andrzej Zajtz, mon Collegue,
à l'occasion de son 70 ème anniversaire

Résumé. On considère quelques problèmes au sujet des solutions des équations fonctionnelles conditionnelles

$$F(\mathbf{x}) \cap F(\mathbf{y}) \neq \emptyset \implies F(\mathbf{xy}) = F(\mathbf{x}) \cap F(\mathbf{y}), \qquad (*)$$

où $F\colon E^* := \bigcup_{n=1}^{\infty} E^n \longrightarrow 2^E \setminus \{\emptyset\}$, E est un ensemble arbitraire et pour chaque $\mathbf{x} = (x_1, \dots, x_n)$ et $\mathbf{y} = (y_1, \dots, y_m)$ de E^*, $\mathbf{xy} = (x_1, \dots, x_n, y_1, \dots, y_m)$, et

$$f(a) \cdot f(b) \neq \underline{0} \implies f(a+b) = f(a) \cdot f(b), \qquad (**)$$

où

$$f\colon A(p) = \{(a_1, \dots, a_p) \in A^p : \ a_k \geq 0\} \setminus \{\underline{0}\} \longrightarrow B(p),$$

$A, B \in \{\mathbb{Z}, \mathbb{Q}, \mathbf{A} = \text{le corps des nombres algébriques}, \mathbb{R}\}$, $\underline{0} = (0, \dots, 0) \in A^p$, $a = (a_1, \dots, a_p) \in A(p)$, $b = (b_1, \dots, b_p) \in A(p)$, $a + b = (a_1 + b_1, \dots, a_p + b_p)$, $a \cdot b = (a_1 \cdot b_1, \dots, a_p \cdot b_p)$. Entre autres, sous quelles conditions, pour la solution F de $(*)$ (la fonction de choix), $F(\mathbf{xy})$ est-elle une fonction de $F(\mathbf{x})$ et $F(\mathbf{y})$, ou pour la solution f de $(**)$ (la fonction d'indice), $f(a+b)$ est-elle une fonction de $f(a)$ et $f(b)$, quand la solution de $(*)$ est en même temps une solution du conséquent de l'implication $(*)$ et est-ce que le prolongement de la solution de $(**)$ existe ? On considère aussi les modifications de l'équation $(**)$, entre autres l'équation suivante

$$f(a) \cdot f(b) = \underline{0} \iff f(a+b) = f(a) \cdot f(b) \qquad (***)$$

et les inclusions (les équations) pour les fonctions multi-valentes $Z(t)$, liées aux équations $(**)$ et $(***)$, de la forme suivante

$$\bigcap_{t \in T} Z(t)^{i(t)} + \bigcap_{t \in T} Z(t)^{j(t)} \subset \bigcap_{t \in T} Z(t)^{i(t)j(t)}$$
$$\Big(\bigcap_{t \in T} Z(t)^{i(t)} + \bigcap_{t \in T} Z(t)^{j(t)} = \bigcap_{t \in T} Z(t)^{i(t)j(t)} \ \Big) \qquad (****)$$

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et les inclusions (les équations) conditionnelles

$$\exists_{t\in T}:\ i(t)j(t)\neq 0 \iff (****) \quad \text{et} \quad \forall_{t\in T}:\ i(t)j(t)=0 \iff (****),$$

où T est un ensemble arbitraire, $(G,+)$ est un groupoïde, $Z(t)\colon T\longrightarrow 2^G$ est une fonction cherchée, $Z(t)^1=Z(t)$, $Z(t)^0=G\setminus Z(t)$, qui doivent avoir lieu pour chaques fonctions $i(t),j(t)\colon T\longrightarrow\{0,1\}$ non-idéntiques zéro. On pose aussi quelques problèmes ouverts.

1. Function de choix

F.S. Roberts a introduit dans [13] et [14] une fonction, qui peut être nommée la fonction de choix, de la manière suivante. Soit E un ensemble non-vide et définissons la fonction $F\colon E^*:=\bigcup_{n=1}^{\infty}E^n\longrightarrow 2^E\setminus\{\emptyset\}$ comme il suit

$$F(x_1,\dots,x_n) \text{ est l'ensemble des éléments de } E \text{ qui paraissent le plus souvent dans la suite } x_1,\dots,x_n\,. \tag{1}$$

On peut interpréter la valeur $F(x_1,\dots,x_n)$ comme le résultat de l'élection : l'électeur premier donne sa voix au candidat x_1, le deuxième au candidat x_2 et caetera, l'ensemble $F(x_1,\dots,x_n)$ est l'ensemble des candidats qui recevoient le plus des voix.

Cette fonction F remplit la condition suivante [13]

$$F(x_1,\dots,x_n)\cap(F(y_1,\dots,y_m)\neq\emptyset \implies F(x_1,\dots,x_n,y_1,\dots,y_m)=F(x_1,\dots,x_n)\cap F(y_1,\dots,y_m), \tag{2}$$

pour chaques n et m entiers et positifs, en abrégé

$$F(\mathbf{x})\cap F(\mathbf{y})\neq\emptyset \implies F(\mathbf{xy})=F(\mathbf{x})\cap F(\mathbf{y}) \tag{2'}$$

pour chaque $\mathbf{x}=(x_1,\dots,x_n)$, $\mathbf{y}=(y_1,\dots,y_m)$ de E^*, $\mathbf{xy}=(x_1,\dots,x_n,y_1,\dots,y_m)$, qui a l'interprétation suivante : si pour deux circonscriptions avec les élections $x_1,\dots,x_n$ et $y_1,\dots,y_m$ il existe au moins un candidat qui a gagné l'élection dans ces deux circonsciptions alors dans la circonscription commune de ces deux régions ces candidats et seulement ces candidats ont gagnés l'élection qui ont gagnés le choix dans chaque de ces deux régions. Nous nommons la solution F de (2) *la fonction de choix.*

L'équation

$$F(x_1,\dots,x_n,y_1,\dots,y_m)=F(x_1,\dots,x_n)\cap F(y_1,\dots,y_m) \quad (F(\mathbf{xy})=F(\mathbf{x})\cap F(\mathbf{y})) \tag{3}$$

dans le conséquent de l'implication (2) ou de l'implication (2′) est conditionnelle : nous pouvons exprimer $F(\mathbf{xy})$ comme une fonction de $F(\mathbf{x})$ et de $F(\mathbf{y})$

mais sous la restriction que $F(\mathbf{x}) \cap F(\mathbf{y}) \neq \emptyset$. Nous nommons pour cette raison la condition (2) ou (2′) l'équation fonctionnelle conditionnelle. Il se pose la question : est-ce qu'on peut exprimer $F(\mathbf{xy})$ comme une fonction de $F(\mathbf{x})$ et de $F(\mathbf{y})$ sans aucune restriction ? La réponse à cette question est négative. En effet, supposons qu'il existe une fonction ϕ telle que

$$F(x_1, \ldots, x_n, y_1, \ldots, y_m) = \phi\,[F(x_1, \ldots, x_n), F(y_1, \ldots, y_m)] \tag{4}$$

pour chaques n et m entiers et positifs ou en abrégé

$$F(\mathbf{xy}) = \phi\,[F(\mathbf{x}), F(\mathbf{y})] \tag{4′}$$

pour tous $\mathbf{x}, \mathbf{y} \in E^*$. Dans ce cas, pour $x, y \in E$, $x \neq y$

$$\{x\} = F(x, x, y) = \phi\,[F(x, x), F(y)] = \phi\,[F(x), F(y)] = F(x, y) = \{x, y\},$$

donc une contradiction.

Il existe des functions qui remplissent (2) et pour lesquelles il existe une fonction ϕ satisfaisante à (4). Telle est chaque fonction remplissante (3) (par ex. chaque fonction constante) et aussi la fonction $F(x_1, \ldots, x_n) = \{x_1\}$, qui remplit (2), ne satisfait pas à (3) et pour laquelle la fonction $\phi(\{x\}, \{y\}) = \{x\}$ satisfait à (4).

Remarquons que la fonction F vérifiant (2) satisfait à (3) si et seulement si $F(\mathbf{x}) \cap F(\mathbf{y}) \neq \emptyset$ pour chaque $\mathbf{x}, \mathbf{y} \in E^*$. Cela ne désigne pas que

$$\bigcap_{x \in E^*} F(\mathbf{x}) \neq \emptyset. \tag{5}$$

En effet, si E est infini, (3) est remplie pour la fonction

$$F(x_1, \ldots, x_n) = E \setminus \{x_1, \ldots, x_n\} \tag{6}$$

et la condition (5) n'a pas lieu. Pour E fini la fonction (6) n'est pas bonne puisque elle prend comme la valeur l'ensemble vide $\emptyset$.

La fonction F qui satisfait au conséquent de (2) doit être symétrique, puisque l'intersection des ensembles est commutative. On voit plus haut que cette symétrie ne suffit pas et elle n'est pas nécéssaire pour que la fonction F remplisse (2).

Il se pose donc la question : quelle est la condition suffisante et nécessaire au sujet d'une fonction F satisfaisante à (2) pour qu'il existe la fonction ϕ vérifiant (4) pour cette fonction F ? La réponse a cette question donne le

THÉORÈME 1
Il existe une fonction ϕ satisfaisante à (4) *pour la fonction F vérifiant* (2) *si et seulement si*

$$F(\mathbf{x}) = F(\mathbf{u}) \quad et \quad F(\mathbf{y}) = F(\mathbf{v}) \quad et \quad F(\mathbf{x}) \cap F(\mathbf{y}) = \emptyset \implies F(\mathbf{xy}) = F(\mathbf{uv})$$

pour chaques $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v} \in E^*$.

Démonstration. L'implication „seulement si" est évidente. Pour la démonstration de „si" remarquons que la fonction $\phi\colon \big(2^E \setminus \{\emptyset\}\big) \longrightarrow 2^E$ définie pour $B, C \in F(E^*)$ comme il suit

$$\phi(B, C) = F(\mathbf{xy}) \qquad \text{pour } F(\mathbf{x}) = B \text{ et } F(\mathbf{y}) = C$$

et arbitraire sinon, ne dépend pas du choix de $\mathbf{x}$ et $\mathbf{y}$ et elle remplit $(4')$.

On peut donner l'interprétation suivante de cette condition. Considérons la relation R définie comme il suit

$$\mathbf{x} R \mathbf{y} \iff F(\mathbf{x}) = F(\mathbf{y})$$

pour $\mathbf{x}, \mathbf{y} \in E^*$. Les classes d'équivalence de cette relation sont nommées les noyaux de la fonction F. La condition en question désigne pour la fonction F vérifiant (2) que la relation R est compatible avec l'opération $(\mathbf{x}, \mathbf{y}) \longrightarrow \mathbf{xy}$ (elle est une congruence).

Remarquons que la fonction ϕ vérifiant (4) doit avoir la propriété de l'associativité sur l'ensemble $F(E^*) \times F(E^*)$, c. à d.

$$\phi\,[\phi(B, C), D] = \phi\,[B, \phi(C, D)] \qquad \text{pour } B, C, D \in F(E^*).$$

Inversement, il existe pour chaque fonction $\phi\colon \big(2^E \setminus \{\emptyset\}\big) \times \big(2^E \setminus \{\emptyset\}\big) \longrightarrow \big(2^E \setminus \{\emptyset\}\big)$ associative une fonction $F\colon E^* \longrightarrow \big(2^E \setminus \{\emptyset\}\big)$ vérifiant (4). En effet, il suffit de prendre $F\colon E \longrightarrow \big(2^E \setminus \{\emptyset\}\big)$ arbitraire et définir F sur $E^* \setminus E$ récursivement par la formule

$$F(\mathbf{x}y) = \phi\,[F(\mathbf{x}), F(y)] \qquad \text{pour } \mathbf{y} \in E^* \text{ et } y \in E.$$

De plus, chaque fonction F qui remplit (4) avec ϕ donnée doit être obtenue par cette méthode. Cette fonction F naturellement ne doit pas satisfaire à (2). Elle remplit (2) si $\phi(B, C) = B \cap C$ pour $B \cap C \neq \emptyset$.

2. Fonction d'indice

Supposons à présent que l'ensemble E a p éléments $v_1, \ldots, v_p$ et soit $x_1, \ldots,\ x_n$ la suite des éléments de E telle que nous avons au début a_1 fois l'élément v_1, puis a_2 fois l'élément v_2, et caetera, a_p fois l'élément v_p, où $a_1, \ldots, a_p$ sont des nombres entiers non-négatifs et $a_1 + \cdots + a_p = n$. Soit $\mathbb{Z}(p)$ l'ensemble de toutes suites de p éléments des nombres non-négatifs et entiers sauf la suite de zéros, désignée dans la suite par $\underline{0}$. Nous allons définir la fonction nouvelle

$$f = (f_1, \ldots, f_p)\colon \mathbb{Z}(p) \longrightarrow O(p) := \{0,1\}^p \setminus \{\underline{0}\}$$

par l'équivalence suivante

$$f_k(a_1, \ldots, a_p) = 1 \iff v_k \in F(x_1, \ldots, x_n), \tag{7}$$

où la fonction F est définie dans (1), ou bien par l'équivalence

$$f_k(a_1, \ldots, a_p) = 1 \iff a_k \geq a_i \qquad \text{pour } i = 1, \ldots, p. \tag{8}$$

Cette fonction f est nommée la fonction d'indice puisque elle indique par ses valeurs 1 quels sont les candidats qui gagnent l'élection. Cette fonction f remplit la condition suivante [14]

$$f(a) \cdot f(b) \neq \underline{0} \implies f(a+b) = f(a) \cdot f(b), \tag{9}$$

où $a = (a_1, \ldots, a_p) \in \mathbb{Z}(p)$, $b = (b_1, \ldots, b_p) \in \mathbb{Z}(p)$, $a+b = (a_1+b_1, \ldots, a_p+b_p)$, $a \cdot b = (a_1 \cdot b_1, \ldots, a_p \cdot b_p)$. Nous nommons dans la suite par la fonction d'indice chaque solution de l'équation conditionnelle (9). La relation $f(a) \cdot f(b) \neq \underline{0}$ désigne que des vecteurs $f(a) = (f_1(a), \ldots, f_p(a))$ et $f(b) = (f_1(b), \ldots, f_p(b))$ ne sont pas orthogonaux puisque elle est équivalente à la condition $f_1(a)f_1(b) + \cdots + f_p(a)f_p(b) \neq 0$, car chaque des fonctions f_k $(k = 1, \ldots, p)$ a seulement les valeurs non-négatives.

Il existe des solutions de cette équation qui ne sont pas de la forme (8) et cette équation a déjà sa théorie (voir [1]-[6] et [9]-[17]). En particulier chaque solution $f = (f_1, \ldots, f_p)$ de (9) a la forme suivante

$$f_k(x) = \begin{cases} 1 & \text{pour } x \in Z_k\,, \\ 0 & \text{pour } x \in \mathbb{Z}(p) \setminus Z_k\,, \end{cases} \tag{10}$$

pour $k = 1, \ldots, p$, où les ensembles $Z_1, \ldots, Z_p$ remplissent les conditions

$$Z_1 \cup \ldots \cup Z_p = \mathbb{Z}(p), \tag{11}$$

$$\begin{aligned} &(i_1 j_1, \ldots, i_p j_p) \neq \underline{0} \\ &\qquad \implies Z_p^{i_1} \cap \ldots \cap Z_p^{i_p} + Z_1^{j_1} \cap \ldots \cap Z_p^{j_p} \subset Z_1^{i_1 j_1} \cap \ldots \cap Z_p^{i_p j_p}, \end{aligned} \tag{12}$$

pour tous $i_k, j_k \in \{0,1\}$, où $Z_i^1 = Z_i$, $Z_i^0 = \mathbb{Z}(p) \setminus Z_i$ pour $i = 1, \ldots, p$ et $E_1 + E_2 := \{a + b : \; a \in E_1,\; b \in E_2\}$ pour $E_1, E_2 \subset \mathbb{Z}(p)$ (voir [9], où la démonstration est donnée pour les nombres réels au lieu des nombres entiers, dans notre cas la démonstration est la même) et réciproquement.

Les ensembles Z_k vérifiant (11) et (12) sont fermés par rapport à l'addition. En effet pour chaque $a \in \mathbb{Z}(p)$ et pour chaque $j \in \{1,\dots,p\}$ il existe un $i_j(a)$ tel que $a \in Z_j^{i_j(a)}$. Supposons que $c \in Z_k$ et $d \in Z_k$, donc $i_k(c) = 1 = i_k(d)$. Donc de plus, d'après (12)

$$\begin{aligned} c + d &\in Z_1^{i_1(c)} \cap \ldots \cap Z_p^{i_p(c)} + Z_1^{i_1(d)} \cap \ldots \cap Z_p^{i_p(d)} \\ &\subset Z_1^{i_1(c)i_1(d)} \cap \ldots \cap Z_p^{i_p(c)i_p(d)} \subset Z_k^{i_k(c)i_k(d)} \\ &= Z_k \,. \end{aligned}$$

Il est intéressant que la condition (12) est équivalente à la suivante [3]

$$\begin{aligned} &\forall_{k,l\in\{1,\dots,p\}} \forall_{i_1,i_2,j_1,j_2\in\{0,1\}} : \ (i_1j_1, i_2j_2) \neq (0,0) \implies \\ &\qquad Z_k^{i_1} \cap Z_l^{i_2} + Z_k^{j_1} \cap Z_l^{j_2} \subset Z_k^{i_1j_1} \cap Z_l^{i_2j_2}, \end{aligned} \tag{12'}$$

c. à d. à la condition de la forme (12) mais demandée seulement pour chaque deux ensembles parmi les ensembles $Z_1,\dots,Z_p$. Remarquons que nous avons $2^{2p-1} + 2^{p-1} - 2^{-1}(3^p + 1)$ inclusions essentielles (c. à d. différentes et pour $(i_1j_1,\dots,i_pj_p) \neq \underline{0}$), qui nous trouvons dans le conséquent de (12) et il y a seulement $2^{-1}p(5p-3)$ inclusions de cette sorte (pour $(i_1j_1, i_2j_2) \neq (0,0)$) dans le conséquent de (12′) (alors moins si $p > 2$). Les autres conditions équivalentes à (12) sont données dans [3]. Remarquons enfin que p^2 est le nombre le plus petit des conditions équivalentes à (12) et ces sont les conditions suivantes [3]

$$\begin{aligned} &\forall_{k\in\{1,\dots,p\}} : \ Z_k + Z_k \subset Z_k \\ \text{et} \\ &\forall_{k,l\in\{1,\dots,p\},\, k\neq l} : \ Z_k + Z_k \cap Z_l^0 \subset Z_k \cap Z_l^0. \end{aligned} \tag{12''}$$

Ces conditions forment un système des relations *indépendantes* entre les ensembles $Z_1,\dots,Z_p$. En effet, considérons les deux exemples suivants.

Exemple 1

Si pour un $j \in \{1,\dots,p\}$ $Z_j = \{(x_1,\dots,x_p) \in \mathbb{Z}(p) : x_2 = \ldots = x_p = 0$ ou $x_1 = x_3 = \ldots = x_p = 0\}$, $Z_i = \mathbb{Z}(p)$ pour chaque $i = 1,\dots,p$, $i \neq j$, dans ce cas seulement l'inclusion $Z_j + Z_j \subset Z_j$ n'a pas lieu parmi les relations (12″).

Exemple 2

Si pour un $j \in \{1,\dots,p\}$ $Z_j = \{(x_1,\dots,x_p) \in \mathbb{Z}(p) : \ x_3 = \ldots = x_p = 0$ et $x_1 = x_2\}$, $Z_m = \mathbb{Z}(p)$ pour un $m \in \{1,\dots,p\}$, $m \neq j$ et $Z_i = \emptyset$ pour chaque $i \in \{1,\dots,p\}$, $j \neq i \neq m$, alors seulement l'inclusion $Z_m + Z_m \cap Z_j^0 \subset Z_m \cap Z_j^0$ n'a pas lieu parmi les relations (12″).

Ici, la condition nécessaire et suffisante pour que $f(a+b)$ soit la fonction de $f(a)$ et $f(b)$ pour une solution de (9), c. à d. pour qu'il existe une fonction ψ telle que

$$f(a+b) = \psi[f(a), f(b)] \qquad \text{pour } a, b \in \mathbb{Z}(p),$$

est suivante

$$f(a)\cdot f(b) = \underline{0}, \quad f(a) = f(c) \ \text{ et } \ f(b) = f(d) \implies f(a+b) = f(c+d) \quad (13)$$

pour tous $a, b, c, d \in \mathbb{Z}(p)$. Cela désigne que la relation $a\rho b \iff f(a) = f(b)$ est compatible avec l'addition.

Nous allons démontrer que la fonction f donnée par (7) pour la fonction $F(x_1, \ldots, x_n) = \{x_1\}$ remplit la condition (13). Remarquons d'abord que (7) nous donne pour chaque $k \in \{1, \ldots, p\}$, $(0, \ldots, 0, e_k, \ldots, e_p) \in \mathbb{Z}(p)$ et $e_k \neq 0$

$$f(0, \ldots, 0, e_k, \ldots, e_p) = (0, \ldots, 0, \underset{k}{1}, 0, \ldots, 0), \quad (14)$$

donc $f(a) = f(c)$ désigne qu'il existe un k tel que $a_\nu = 0 = c_\nu$ pour $\nu = 1, \ldots, k-1$ et $a_k \neq 0 \neq c_k$ et analogiquement $f(b) = f(d)$ désigne qu'il existe s tel que $b_\mu = 0 = d_\mu$ pour $\mu = 1, \ldots, s-1$ et $b_s \neq 0 \neq d_s$. Nous pouvons supposer que $k \leq s$ et dans ce cas $a_i + b_i = 0 = c_i + d_i$ pour $i = 1, \ldots, k-1$ et $a_k + b_k \neq 0 \neq c_k + d_k$, donc $f(a+b) = (0, \ldots, 0, \underset{k}{1}, 0, \ldots, 0) = f(c+d)$. On peut de même démontrer que notre fonction f remplit (9) et que pour cette fonction

$$Z_k = \{(c_1, \ldots, c_p) \in \mathbb{Z}(p) : \ c_1 = \ldots = c_{k-1} = 0 \text{ et } c_k \neq 0\}. \quad (15)$$

On peut faire au sujet de la fonction ψ les remarques analogues comme pour la fonction ϕ en fin de la partie 1.

Remarquons que la fonction f remplissante (9) satisfait au conséquent de (9)

$$f(a+b) = f(a)\cdot f(b) \quad (16)$$

si et seulement s'il existe au moins un $k \in \{1, \ldots, p\}$ tel que $f_k(a) = 1$ pour chaque $a \in \mathbb{Z}(p)$, c. à d. si et seulement s'il existe un $k \in \{1, \ldots, p\}$ tel que $Z_k = \mathbb{Z}(p)$ (la démonstration est la même que dans [10] pour les nombres réels). Pour l'ensemble E fini, la fonction F symétrique vérifiant (2) satisfait à (3) si et seulement si la fonction f bien définiée par (7) satisfait à (16), donc si et seulement s'il existe un k tel que $v_k \in F(\mathbf{x})$ pour chaque $\mathbf{x}$ de E^*, alors si et seulement si la condition (5) a lieu.

Nous avons donc le

Théorème 2
Pour l'ensemble E fini la fonction $F\colon E^* := \bigcup_{n=1}^{\infty} E^n \longrightarrow 2^E \setminus \{\emptyset\}$ satisfait à (3) *pour tous $x, y \in E^*$, si et seulement si elle remplit* (2) *et* (5).

Ce théorème n'est pas vrai pour l'ensemble infini, même si la fonction F est symétrique (voir la fonction (6)).

Puisque $f(x) = (i_1, \dots, i_p)$ pour $x \in Z_1^{i_1} \cap \dots \cap Z_p^{i_p}$, donc nous avons le théorème suivant :

Théorème 3

Pour une solution f de (8), *$f(a+b)$ est une fonction de $f(a)$ et $f(b)$ si et seulement si*

$$\begin{gathered}\forall_{(i_1,\dots,i_p),(j_1,\dots,j_p)\in O(p)}\ \exists_{(k_1,\dots,k_p)\in O(p)} : \\ Z_1^{i_1} \cap \dots \cap Z_p^{i_p} + Z_1^{j_1} \cap \dots \cap Z_p^{j_p} \subset Z_1^{k_1} \cap \dots \cap Z_p^{k_p}.\end{gathered} \tag{17}$$

D'après (11) ce théorème est intéressante seulement pour $(i_1 j_1, \dots, i_p j_p) = \underline{0}$.

Pour les ensembles $Z_1, \dots, Z_p$ disjoints la condition (11) désigne que ces ensembles sont fermés par rapport à l'addition et la condition (17) est équivalente à la suivante

$$\forall_{i,j\in\{1,\dots,p\}}\ \exists_{k\in\{1,\dots,p\}} :\ Z_i + Z_j \subset Z_k.$$

Ces deux conditions sont remplies par exemple par les ensembles (15), ici $k = \min(i,j)$.

La fonction f donnée par (8) ne remplit pas la condition (17). En effet, supposons que pour $(i_1, \dots, i_p) = (1, 0, \dots, 0)$ et $(j_1, \dots, j_p) = (0, 1, 0, \dots, 0)$ il existe $k_1, \dots, k_p \in O(p)$ tells que (17) a lieu. Puisque $f(n, 0, \dots, 0) = (1, 0, \dots, 0)$ et $f(0, m, 0, \dots, 0) = (0, 1, 0, \dots, 0)$ pour n et m entiers et positifs,

$$\begin{aligned}(n, m, 0, \dots, 0) &= (n, 0, \dots, 0) + (0, m, 0, \dots, 0) \\ &\in Z_1^{i_1} \cap \dots \cap Z_p^{i_p} + Z_1^{j_1} \cap \dots \cap Z_p^{i_p} \\ &\subset Z_1^{k_1} \cap \dots \cap Z_p^{k_p},\end{aligned}$$

donc $f(n, m, 0, \dots, 0) = (k_1, \dots, k_p)$ pour chaques n et m. Mais, d'après la définition de f on a $f(n, m, 0, \dots, 0) = (1, 0, \dots, 0)$ pour $n > m$ et $f(n, m, 0, \dots, 0) = (0, 1, 0, \dots, 0)$ pour $n < m$, donc une contradiction.

La fonction f définie par (7) est bien définie puisque elle a les valeurs dans l'ensemble $O(p)$ et de cette raison elle remplit l'équation (9). Elle peut être définie équivalentement aussi comme il suit

$$f_k(a_1, \dots, a_p) \neq 0 \iff v_k \in F(x_1, \dots, x_n), \tag{7'}$$

mais si nous considérons la fonction $f = (f_1, \dots, f_p) \colon \mathbb{Z}(p) \longrightarrow \mathbb{Z}(p)$, donc à valeurs dans $\mathbb{Z}(p)$, telle que seulement (7′) a lieu, cette fonction ne doit pas remplir (9), elle satisfait seulement à la condition

$$f(a) \cdot f(b) \neq \underline{0} \implies \forall_{k\in\{1,\dots,p\}} \left(f_k(a+b) \neq 0 \iff f_k(a) \cdot f_k(b) \neq 0\right) \tag{9'}$$

équivalente à la condition que la fonction $\operatorname{sgn} f$ satisfait à (9), où $\operatorname{sgn} x = 1$ pour $x > 0$ et $\operatorname{sgn} 0 = 0$ et $\operatorname{sgn}(a_1, \dots, a_p) = (\operatorname{sgn} a_1, \dots, \operatorname{sgn} a_p)$ pour $(a_1, \dots, a_p) \in \mathbb{Z}(p)$. La fonction f vérifiant (9′) doit avoir la forme analogue à (10)

$$f_k(x) \text{ arbitraire positive pour } x \in Z_k \text{ et } f_k(x) = 0 \text{ pour } x \in \mathbb{Z}(p) \backslash Z_k\,, \tag{10'}$$

pour $k = 1, \dots, p$ et pour les ensembles $Z_1, \dots, Z_p$ vérifiant les conditions (11) et (12) (la démonstration analogue à celle de [8]). Puisque (9) entraîne (9′), mais pas inversement, il est naturel, mais pas nécessaire, en considérant (9), supposer que f a les valeurs dans $O(p)$. Puisque il existe des solutions de (9) qui ont les valeurs en dehors de $O(p)$, p. ex. pour $p = 2$: $f(n, m) = (2^n, 0)$ pour $n \geq m$ et $f(n, m) = (0, 1)$ pour $n < m$, il est donc interéssant quelles solutions de (9) ont les valeurs seulement dans $O(p)$ (voir [2], [3], [8], [9]). Puisque la fonction $f = (f_1, \dots, f_p)\colon \mathbb{Z}(p) \longrightarrow \mathbb{Z}(p)$ vérifiant (9) doit avoir la forme

$$f_k(x) = \begin{cases} g_k(x) & \text{pour } x \in Z_k\,, \\ 0 & \text{pour } x \in \mathbb{Z}(p) \setminus Z_k\,, \end{cases}$$

où $g_k\colon \mathbb{Z}(p) \longrightarrow \mathbb{Z}$ est telle que

$$g_k(a + b) = g_k(a) \cdot g_k(b)$$

et $Z_1, \dots, Z_p$ remplissent (11) et (12), donc le théorème 3 est vrai pour cette fonction. Au contraire l'implication ,,si" dans le théorème 3 n'est pas évidemment vraie a cause de (10′) pour la fonction $f\colon \mathbb{Z}(p) \longrightarrow \mathbb{Z}(p)$ satisfaisante à (9′) (l'implication ,,seulement si" est vraie puisque $\operatorname{sgn} f$ satisfait à (9)), mais la condition (13) est nécessaire et suffisante pour que $f(a+b)$ soit une fonction de $f(a)$ et $f(b)$ dans ce cas.

3. Généralisations de la function d'indice

En généralisant ses considérations F.S. Roberts dans [14] remplace l'ensemble $\mathbb{Z}(p)$ des nombres entiers par l'ensemble $\mathbb{Q}(p)$ des suites de p-éléments des nombres rationnels non-négatifs sauf $\underline{0}$ ou par l'ensemble

$$\mathbb{R}(p) = [0, +\infty)^p \setminus \{\underline{0}\}$$

(cela désigne que nous considérons les choix avec les poids – la voix de chaque électeur a un poids rationnel ou réel, pas nécessairement égal à 1).

Les considérations plus haut sont valables aussi pour les ensembles $\mathbb{Q}(p)$ et $\mathbb{R}(p)$ au lieu de $\mathbb{Z}(p)$. Les ensembles Z_k forment dans ce cas des cônes sur le corps des nombres rationnels [9]. De plus, dans le cas de $\mathbb{R}(p)$ au lieu de $\mathbb{Z}(p)$ les ensembles $Z_1, \dots, Z_p$ dans les exemples 1 et 2 forment les cônes sur $\mathbb{R}$.

Il est intéressant remarquer qu'on ne peut pas remplacer l'inclusion ,,$\subset$" par l'égalité ,,=" dans les conditions (12) et (12′), même, si nous remplaçons

$\mathbb{Z}(p)$ dans ces conditions par $\mathbb{Q}(p)$ ou $\mathbb{R}(p)$. En effet, les ensembles $Z_1 = \mathbb{Z}(p)$ ($\mathbb{Q}(p)$ ou $\mathbb{R}(p)$) et $Z_2 = \ldots = Z_p = \emptyset$ remplissent (11) et (12) (alors aussi (12′)) pour $\mathbb{Z}(p)$ ou $\mathbb{Q}(p)$ ou $\mathbb{R}(p)$ respectivement mais nous avons

$$Z_1^{i_1} \cap \ldots \cap Z_p^{i_p} + Z_1^{j_1} \cap \ldots \cap Z_p^{j_p} = \emptyset + Z_1 = \emptyset \neq Z_1^{i_1 j_1} \cap \ldots \cap Z_p^{i_p j_p} = Z_1$$

pour $i_1 = i_2 = j_1 = 1$, $i_3 = \ldots = i_p = j_2 = \ldots = j_p = 0$, et analogiquement

$$Z_1^1 \cap Z_2^1 + Z_1^1 \cap Z_2^0 = \emptyset + Z_1 = \emptyset \neq Z_1^1 \cap Z_2^0 = Z_1 .$$

La même situation a lieu pour la condition (12″) dans le cas de $\mathbb{Z}(p)$, puisque $\mathbb{Z}(p) + \mathbb{Z}(p) \neq \mathbb{Z}(p)$ car $(1, 0, \ldots, 0) \in \mathbb{Z}(p)$ et $(1, 0, \ldots, 0) \in \mathbb{Z}(p) + \mathbb{Z}(p)$. Mais si nous considérons la condition (12″) pour $\mathbb{Q}(p)$ ou $\mathbb{R}(p)$, elle implique la condition

$$\begin{aligned} &\forall_{k\in\{1,\ldots,p\}}: \ Z_k + Z_k = Z_k \\ \text{et} \quad & \\ &\forall_{k,l\in\{1,\ldots,p\},\, k\neq l}: \ Z_k + Z_k \cap Z_l^0 = Z_k \cap Z_l^0. \end{aligned} \tag{12‴}$$

En effet, puisque $Z_1, \ldots, Z_p$ vérifiant (12″) (donc aussi (12)) forment les cônes sur $\mathbb{Q}$ ([7]), donc $\frac{1}{2}Z_k^i \subset Z_k^i$ pour chaque $k = 1, \ldots, p$ et $i = 0, 1$ et de là

a) si $x \in Z_k$, alors $x = \frac{1}{2}x + \frac{1}{2}x \in Z_k + Z_k$,

b) si $x \in Z_k \cap Z_l^0$, alors $x = \frac{1}{2}x + \frac{1}{2}x \in Z_k + Z_l^0$.

La fonction vérifiant (16) satisfait naturellement à (17). On peut aussi voir cela directement. Puisque pour la solution de (16) au moins un $Z_k = \mathbb{R}(p)$ (voir [9]), alors $Z_k^{i_k} = \emptyset$ ou $Z_k^{j_k} = \emptyset$ pour $i_k j_k = 0$, d'où la condition (17) pour $i \cdot j = \underline{0}$ est remplie car dans ce cas

$$Z_1^{i_1} \cap \ldots \cap Z_p^{i_p} + Z_1^{j_1} \cap \ldots \cap Z_p^{j_p} = \emptyset.$$

Cette condition n'est pas naturellement nécessaire pour que (17) ait lieu, les considérations plus haut au sujet de la fonction (14) cela montrent.

4. Prolongements

Les généralisations de l'ensemble $\mathbb{Z}(p)$ aux ensembles $\mathbb{Q}(p)$ ou $\mathbb{R}(p)$ impose la question : est-ce que chaque solution de (9) sur $\mathbb{Z}(p)$ possède un prolongement (unique ?) sur $\mathbb{Q}(p)$ (sur $\mathbb{R}(p)$) ?

Pour démontrer la réponse positive dans le cas de $\mathbb{Q}(p)$ nous allons montrer tout d'abord le

Lemme 1
La solution f de (9) sur $\mathbb{Z}(p)$ est stable sur chaque ensemble $\mathbb{Z}(a) = D(a) \cap \mathbb{Z}(p)$, où $D(a) = \{ta : \ t \in (0, +\infty)\}$ pour $a \in \mathbb{Z}(p)$.

Démonstration. Pour chaque $a \in \mathbb{Z}(p)$ il existe un $a^* \in \mathbb{Z}(a)$ tel que $\mathbb{Z}(a) = \{na^* \colon\ n \in \mathbb{N}\}$. Nous avons $f(na^*) = f(a^*)$ pour chaque $n \in \mathbb{N}$, d'où la thèse.

Théorème 4
Chaque solution de (9) *sur* $\mathbb{Z}(p)$ *possède un prolongement unique sur* $\mathbb{Q}(p)$.

Démonstration. Soit f une solution de (9) sur $\mathbb{Z}(p)$ donnée par (10), et soit $D(a) = \{ta :\ t \in (0, +\infty)\}$ pour $a \in \mathbb{Q}(p)$. Il existe pour chaque $a \in \mathbb{Q}(p)$ un $b \in \mathbb{Z}(p) \cap D(a)$. Nous posons $g(a) = f(b)$. La définition de la fonction g ne dépend pas du choix de b puisque si $b_1, b_2 \in \mathbb{Z}(p) \cap D(a)$, alors $D(a) = D(b_1) = D(b_2)$ et $f(b_1) = f(b_2)$, d'après le lemme 1. Pour $c \in \mathbb{Z}(p)$ nous avons $c \in \mathbb{Z}(p) \cap D(c)$, alors $g(c) = f(c)$, donc g est un prolongement de f de $\mathbb{Z}(p)$ sur $\mathbb{Q}(p)$. Nous allons montrer que g remplit (9). Il existe pour $x \in \mathbb{Q}(p)$ un $n \in \mathbb{N}$ tel que $nx \in \mathbb{Z}(p) \cap D(x)$ et dans ce cas $knx \in \mathbb{Z}(p) \cap D(p)$ pour chaque $k \in \mathbb{N}$. Analogiquement il existe pour $y \in \mathbb{Q}(p)$ un $m \in \mathbb{N}$ tel que $my \in \mathbb{Z}(p) \cap D(y)$ et donc $kmy \in \mathbb{Z}(p) \cap D(y)$, $k \in \mathbb{N}$. Il en résulte que $mnx \in \mathbb{Z}(p) \cap D(x)$ et $mny \in \mathbb{Z}(p) \cap D(y)$, alors $mn(x+y) \in \mathbb{Z}(p) \cap D(x+y)$ et de là $g(x) = f(mnx)$, $g(y) = f(nmy)$, $g(mn(x+y)) = f(mn(x+y))$.
Si $g(x) \cdot g(y) \neq \underline{0}$, donc $f(nx) \cdot f(my) \neq \underline{0}$, alors $f(mn(x+y)) = f(mnx) \cdot f(nmy)$, d'où $g(x+y) = g(x) \cdot g(y)$.

Soient à présent g_1, g_2 deux solutions de (9) sur $\mathbb{Q}(p)$, étant des prolongements de f. Remarquons que $g(nx) = g(x)$ pour chaque $n \in \mathbb{N}$ si g remplit (9). Il existe pour chaque $x \in \mathbb{Q}(p)$ un $n \in \mathbb{N}$ tel que $nx \in \mathbb{Z}(p)$. Nous avons

$$g_1(x) = g_1(nx) = f(nx) = g_2(nx) = g_2(x)$$

et alors le prolongement de f est unique.

Corollaire 1
Chaque solution de (8) *sur* $\mathbb{Q}(p)$ *est stable sur chaque ensemble*

$$\mathbb{Q}(p) \cap \{ta :\ t \in (0, +\infty)\} \qquad \textit{pour } a \in \mathbb{Q}(p).$$

Cela résulte d'après le lemme 1 de l'unicité du prolongement dans le théorème 4. Il résulte cela aussi de l'équation (9) puisque pour la solution f de cette équation on a $f(qx) = f(x)$ pour chaque $q \in \mathbb{Q}$ et $x \in \mathbb{Q}(p)$.

Nous entendrons dans la suite par *une demi-droite de la direction* ,,a" l'ensemble $\{(x, ax) :\ x \in \mathbb{R}(1)\}$ pour a réel, non-négatif et par *la demi-droite de la direction* $+\infty$ l'ensemble $\{(0, y) :\ y \in \mathbb{R}(1)\}$. Nous comprenons la direction $+\infty$ comme rationnelle et nous prenons par une demi-droite rationnelle l'intersection de chaque demi-droite de la direction rationnelle avec l'ensemble $\mathbb{Q}(2)$.

Pour donner la réponse partielle au sujet du prolongement de la solution de (9) de $\mathbb{Q}(p)$ à $\mathbb{R}(p)$ nous allons démontrer les deux lemmes.

LEMME 2
La fonction $f = (f_1, f_2)\colon \mathbb{Q}(2) \longrightarrow O(2)$ *(*$\mathbb{R}(2) \longrightarrow O(2)$*) remplit* (9) *si et seulement si les ensembles* Z_i *sur lesquels* $f_i \equiv 1$, $i = 1, 2$, *satisfont aux conditions*

$$\begin{aligned} &Z_i + Z_i \subset Z_i \qquad \text{pour } i = 1, 2, \\ &Z_1^0 + Z_2 \subset Z_1^0,\ Z_2^0 + Z_1 \subset Z_2^0 \text{ et } Z_1 \cup Z_2 = \mathbb{Q}(2) \text{ (ou } \mathbb{R}(2)). \end{aligned} \tag{18}$$

Rappelons que Z_i^0 désigne le complément de Z_i à $\mathbb{Q}(2)$ (ou à $\mathbb{R}(2)$).
La démonstration du lemme 2 dans le cas $\mathbb{R}(2)$ est donnée dans [9] et elle est analogue pour $\mathbb{Q}(2)$.

Remarquons que la condition première dans (18) désigne que l'ensemble Z_i est fermé par rapport à l'addition et que pour les ensembles Z_1 et Z_2 disjoints (18) est équivalente à la condition

$$Z_i + Z_i \subset Z_i \qquad \text{pour } i = 1, 2 \text{ et } Z_1 \cup Z_2 = \mathbb{Q}(2) \text{ (ou } \mathbb{R}(2)), \tag{19}$$

puisque $Z_1^0 = Z_2$ et $Z_2^0 = Z_1$ dans ce cas.

LEMME 3
Les ensembles différents $Z_1, Z_2 \subset \mathbb{Q}(2)$ *remplissent* (18) *si et seulement s'il existe une demi-droite* D *telle que* Z_1 *est la partie de* $\mathbb{Q}(2)$ *située d'un côté de* D *avec* D *ou non et* Z_2 *se compose des points de* $\mathbb{Q}(2)$ *qui se trouvent de l'autre côté de* D *avec* D *ou non et* $Z_1 \cup Z_2 = \mathbb{Q}(2)$.

Démonstration. La condition „si" est évidente. Si Z_1 et Z_2 remplissent (18) dans ce cas, d'après le corollaire, chaque demi-droite rationnelle est contenue dans Z_i ou elle est disjoint avec Z_i pour $i = 1, 2$. Puisque Z_i est fermé par rapport à l'addition donc l'ensemble des directions des demi-droites rationnelles contenues dans Z_i forme un intervalle I_i dans l'ensemble $[0, +\infty] \cap [\mathbb{Q} \cup \{+\infty\}]$. Si les ensembles Z_1 et Z_2 sont disjoints, les intervalles I_1 et I_2 sont aussi disjoints et $I_1 \cup I_2 = [0, +\infty] \cap [\mathbb{Q} \cup \{+\infty\}]$. Il existe alors un $\alpha \in [0, +\infty]$ tel que un de ces intervalles est de la forme $[0, \alpha| \cap [\mathbb{Q} \cup \{+\infty\}]$ et l'autre de la forme $|\alpha, +\infty] \cap [\mathbb{Q} \cup \{+\infty\}]$, où α appartient au plus à un de ces intervalles. La demi-droite D dans le lemme c'est la demi-droite de la direction α.
Si les ensembles Z_1 et Z_2 ne sont pas disjoints, il existe seulement une demi-droite rationnelle $D_1 \subset Z_1 \cap Z_2$ (cette demi-droite a naturellement la direction rationnelle). En effet, supposons au contraire qu'il existe deux demi-droites rationnelles différentes D_1 et D_2 dans $Z_1 \cap Z_2$. Il existe dans ce cas un $a \in \mathbb{Q}(2)$ et un $r > 0$ tels que $K(a, r) = K^*(a, r) \cap \mathbb{Q}(2) \subset Z_1 \cap Z_2$, où $K^*(a, r)$ est un disque dans $\mathbb{R}(2)$. Puisque $Z_1 \neq Z_2$, il existe soit un $x \in Z_1 \cup Z_2^0 \cap Z_2$, soit un $x \in Z_1 \cap Z_2^0$. Il existe aussi un $q \in \mathbb{Q}$ tel que $|qx| < r$, donc $a + qx \in K(a, r)$, alors $a + qx \in Z_2$, mais $qx \in Z_2^0$, d'où $a + qx \in Z_1 + Z_2^0 \subset Z_2^0$ d'après (18). Nous avons une contradiction, il existe donc seulement une demi-droite rationnelle

contenue dans $Z_1 \cap Z_2$ et cette demi-droite joue la rôle de la demi-droite D dans notre lemme.

THÉORÈME 5
Il existe toujours un prolongement de la solution de (9) *pour* $\mathbb{Q}(2)$ *sur* $\mathbb{R}(2)$ *stable sur chaque demi-droite, mais pas unique.*

Démonstration. Soient Z_1 et Z_2 les ensembles déterminés par une solution f de (9) sur $\mathbb{Q}(2)$. Si $Z_1 = Z_2$, on a $Z_1 = Z_2 = \mathbb{Q}(2)$, donc $f \equiv (1,1)$, alors $g \equiv (1,1)$ sur $\mathbb{R}(2)$ est un prolongement de f. Si $Z_1 \neq Z_2$ soit D la demi-droite donnée dans le lemme 3 et α sa direction. Si α est le nombre irrationnel, alors Z_1 et Z_2 sont disjoints et les ensembles $\mathbf{Z}_1 = \mathrm{cl}\, Z_1$ (la fermeture de Z_1 dans $\mathbb{R}(2)$) et $\mathbf{Z}_2 = \mathrm{cl}\, Z_2 \setminus D$ forment deux ensembles disjoints remplissants (19) pour $\mathbb{R}(2)$, donc satisfaisants à (18). La même situation a lieu pour les ensembles $\mathbf{Z}_1 = \mathrm{cl}\, Z_1 \setminus D$ et $\mathbf{Z}_2 = \mathrm{cl}\, Z_2$, donc l'unicité n'a pas lieu dans se cas.

Si α est rationnelle et

a) $D \subset Z_1 \cap Z_2$, posons $\mathbf{Z}_1 = \mathrm{cl}\, Z_1$ et $\mathbf{Z}_2 = \mathrm{cl}\, Z_2$,

b) D est contenue seulement dans Z_1, posons $\mathbf{Z}_1 = \mathrm{cl}\, Z_1$ et $\mathbf{Z}_2 = \mathrm{cl}\, Z_2 \setminus D$,

c) D est contenue seulement dans Z_2, posons $\mathbf{Z}_1 = \mathrm{cl}\, Z_1 \setminus D$ et $\mathbf{Z}_2 = \mathrm{cl}\, Z_2$.

On voit que les ensembles $\mathbf{Z}_1$ et $\mathbf{Z}_2$ remplissent la condition (18), donc la fonction $g = (g_1, g_2)\colon \mathbb{R}(2) \longrightarrow O(2)$, pour laquelle $g_i(x) = 1$ si et seulement si $x \in \mathbf{Z_i}$ pour $i = 1, 2$, est un prolongement de f sur $\mathbb{R}(2)$.

Ce prolongement ne doit pas être unique aussi de cette raison qu'il existe des solutions de (9) sur $\mathbb{R}(2)$ qui ne sont pas stables sur les demi-droites $D(\alpha)$. Telle est la fonction dans l'exemple suivant.

EXEMPLE 3
Soit g la fonction pour laquelle les ensembles $\mathbf{Z}_1$ et $\mathbf{Z}_2$ sont définis comme il suit. Soit B la base de Hamel des nombres réels sur le corps $\mathbb{Q}$ telle que $1 \in B$ et soit $\mathbf{Z}_1$ l'ensemble des paires $(x, 0) \in \mathbb{R}(2)$ pour lesquelles nous avons des coefficients positifs auprès de 1 dans le développement de x par rapport à cette base B et posons $\mathbf{Z}_2 = \mathbb{R}(2) \setminus \mathbf{Z}_1$. Ces ensembles $\mathbf{Z}_1$ et $\mathbf{Z}_2$ remplissent (19), donc aussi (18) puisque ils son disjoints. Cet exemple montre que l'unicité peut ne pas avoir lieu aussi quand D plus haut est rationnel (ici $D = \{(x, 0) :\ x \in \mathbb{R}(1)\}$).

Nous tirons le profit de l'axiome de Zermelo du choix dans cet exemple (l'existence de la base de Hamel B). Ce n'est pas par hazard puisque dans le cas de $\mathbb{R}(p)$ les solutions de (9), pour lesquelles toutes les intersections des ensembles $\mathbf{Z}_i$ avec chaque $D(\alpha)$ sont mesurables linéairement au sens de Lebesgue, doivent être stables sur $D(\alpha)$ (voir [9] théorème 3), donc au moins une des intersections en question ne peut pas être mesurable pour notre fonction g de l'exemple plus haut. La même situation n'a pas lieu dans le cas de l'ensemble

$\mathbf{A}(p)$ des suites de p-éléments des nombres algébriques non-négatifs sauf $\underline{0}$. Cela montre pour $p = 2$ la modification h de la fonction g de l'exemple plus haut qui consiste au remplacement de l'ensemble $\mathbb{R}(p)$ par $\mathbf{A}(p)$. Remarquons encore que la fonction g sera un prolongement de h de l'ensemble $\mathbf{A}(2)$ sur $\mathbb{R}(2)$ si la base dans $\mathbf{A}(2)$ sera un sous-ensemble de la base B dans $\mathbb{R}(2)$, considérée dans l'exemple.

On peut montrer d'une manière analogue à la demonstration du théorème 5 qu'il existe toujours un prolongement de la solution de (8) pour $\mathbb{Q}(2)$ sur $\mathbf{A}(2)$ stable sur l'intersection de chaque demi-droite avec $\mathbf{A}(2)$, mais pas unique.

Les **problèmes** du prolongement de la solution de (9) pour $\mathbb{Q}(p)$ à $\mathbf{A}(p)$ ou à $\mathbb{R}(p)$ dans le cas $p > 2$ restent ouverts, en particulier la question si le théorème 5 est vrai pour $p > 2$ est aussi ouverte. Il est aussi ouvert le **problème** du prolongement de la solution de (9) pour $\mathbf{A}(p)$ sur $\mathbb{R}(p)$.

5. Fonction d'indice du domaine restreint

On considère dans [6] l'équation (9) pour a et b tells que

$$a_1 + \cdots + a_p + b_1 + \cdots + b_p \leq \alpha, \tag{20}$$

pour un α donné réel et positif (la somme des voix dans l'élection est bornée dans la pratique), c. à d. l'équation conditionnelle de la forme

$$(20) \quad \text{et} \quad f(a) \cdot f(b) \neq \underline{0} \implies f(a+b) = f(a) \cdot f(b) \tag{21}$$

et on a démontré que dans le cas de $\mathbb{R}(p)$ chaque solution de cette équation dernière peut être uniquement prolongée à la solution de (9) (voir [9]) et dans le cas de $\mathbb{Z}(p)$ ce prolongement ne doit pas être unique (voir [10]). Il résulte du corollaire que chaque solution de (21) sur $\mathbb{Q}(p)$ peut être uniquement prolongée sur $\mathbb{R}(p)$. On peut formuler ces résutats comme il suit

Théorème 6
Si pour deux solutions de (9) *pour* $\mathbb{Q}(p)$ *(ou* $\mathbb{R}(p)$*) il existe un* $\alpha > 0$ *tel que ces solutions sont idéntiques sur l'ensemble*

$$\{(a_1, \ldots, a_p) \in \mathbb{Q}(p) : \ a_1 + \cdots + a_p \leq \alpha\}$$

(ou sur l'ensemble $\{(a_1, \ldots, a_p) \in \mathbb{R}(p) : \ a_1 + \cdots + a_p \leq \alpha\}$*),*

elles sont idéntiques sur $\mathbb{Q}(p)$ *(sur* $\mathbb{R}(p)$*).*

Cette situation n'a pas lieu dans le cas de $\mathbb{Z}(p)$, mais il résulte du lemme 1 que si deux solution de (9) pour $\mathbb{Z}(p)$ sont idéntiques sur un sélecteur des demi-droites $D(\alpha) \cap \mathbb{Z}(p)$ pour chaques α rationnels, dans ce cas elles sont égales. On peut prendre pour ce sélecteur par exemple l'ensemble $\{(k_1, \ldots, k_p) \in \mathbb{Z}(p) : k_1, \ldots, k_p$ sont premiers entre eux$\}$ et aucun de ces sélecteurs n'est pas borné. L'idéntité de deux solutions de (9) pour $\mathbb{Q}(p)$ sur un sélecteur des demi-droites $D(\alpha) \cap \mathbb{Q}(p)$ suffit naturellement pour qu'elles soient égales sur $\mathbb{Q}(p)$ (voir le corollaire 1), mais dans ce cas il existe des sélecteurs bornés. Au contraire il existe deux solutions de (9) pour $\mathbb{R}(p)$ qui sont idéntiques sur un sélecteur de la famille $D(\alpha)$ et qui ne sont pas égales sur $\mathbb{R}(p)$. Cela montrent pour $p = 2$ les deux fonctions : la fonction g de l'exemple 3 et la fonction h définie comme il suit : $h(a_1, a_2) = (0, 1)$ pour $(a_1, a_2) \in \mathbb{R}(2)$ et $a_2 > 0$ et $h(a_1, 0) = (1, 0)$ pour $a_1 > 0$, qui sont idéntiques sur le sélecteur $b_1 + b_2 = 1$ et $(b_1, b_2) \in \mathbb{R}(2)$ et ne sont pas idéntiques sur $\mathbb{R}(2)$ tout entier.

On considère dans [12] et dans les travaux postérieurs les solutions de (9) qui ont les valeurs dans l'ensemble $\mathbb{Z}(p)$ (ou $\mathbb{Q}(p)$ ou $\mathbb{R}(p)$) au lieu de $O(p)$. Il existe une solution $f\colon \mathbb{Z}(p) \longrightarrow \mathbb{Z}(p)$ de (8) pour laquelle le lemme 1 n'a pas lieu, c. à d. qui n'est pas stable sur les demi-droites en considération. Telle est la fonction $f(a_1, \ldots, a_p) = (2^{a_1}, \ldots, 2^{a_1})$. Le corollaire est vrai pour chaque solution $g = (g_1, \ldots, g_p)\colon \mathbb{Q}(p) \longrightarrow \mathbb{Q}(p)$ de (7), c. à d. chaque solution de (9) est stable sur les demi-droites en question, puisque dans ce cas nous avons $g(\mathbb{Q}(p) \subset O(p)$. En effet, nous avons $g_k(qa) = g_k(a)^q \in \mathbb{Q}$ pour chaque $a \in \mathbb{Q}(p)$ et pour chaque q rationnel et positif, d'où si $g_k(a) \neq 0$, alors $g_k(a) = 1$. Il résulte de nos considérations qu'il ne doit pas exister toujours un prolongement de la solution $f\colon \mathbb{Z}(p) \longrightarrow \mathbb{Z}(p)$ de (9) sur $\mathbb{Q}(p)$. Au contraire, chaque solution $g\colon \mathbb{Q}(2) \longrightarrow \mathbb{Q}(2)$ a un prolongement sur $\mathbb{R}(2)$ puisque la démonstration du théorème 5 est valable aussi dans ce cas.

6. Modification

D. Gronau a posé pendant 39-ème International Symposium on Functional Equations [12] la question : quelle est la solution générale de l'équation fonctionnelle conditionnelle

$$f(a) \cdot f(b) = \underline{0} \implies f(a+b) = f(a) \cdot f(b), \tag{22}$$

qui est une modification de l'équation (9) ? La relation $f(a) \cdot f(b) = \underline{0}$ peut être interprétée comme l'orthogonalité des vecteurs $f(a)$ et $f(b)$. Soit $A, B \in \{\mathbb{Z}, \mathbb{Q}, \mathbf{A}, \mathbb{R}\}$. La réponse à la question de Gronau donne le

Théorème 7
La fonction $f\colon A(p) \longrightarrow B(p)$ est une solution de (22) si et seulement s'il n'y a pas de deux éléments c et d dans l'ensemble des valeurs de f pour lesquels $c \cdot d = \underline{0}$, c. à d. si et seulement si

$$f(a) \cdot f(b) \neq \underline{0} \qquad \textit{pour chaque } a, b \in A(p), \tag{23}$$

intéressant pour $a \neq b$, alors si et seulement si

$$A(p) \times A(p) \subset (Z_1 \times Z_1) \cup \ldots \cup (Z_p \times Z_p). \tag{23'}$$

Démonstration. S'il existe deux éléments c et d dans l'ensemble des valeurs de f pour lesquels $c \cdot d = \underline{0}$, alors il existe a et b tells que $f(a) = c$ et $f(b) = d$. La fonction f ne peut pas remplir (22) dans ce cas puisque la première partie dans (22) est vraie et $f(a+b) = f(a) + f(b)$ n'a pas lieu (la fonction f ne peut pas prendre de la valeur $\underline{0}$).

La fonction pour laquelle il n'existe pas deux éléments c et d dans l'ensemble de ses valeurs pour lesquels $c \cdot d = \underline{0}$ satisfait a (22) puisque la première partie dans (22) toujours n'a pas lieu dans ce cas.

Il résulte du théorème démontré que chaque fonction $f\colon A(1) \longrightarrow B(1)$ est une solution de (22) pour $p = 1$.

La relation (23) est une condition nécessaire et suffisante pour qu'une solution de (9) soit en même temps une solution de l'équation (16) (pour $A(p) = \mathbb{R}(p)$ voir [10], la démonstration est analogue pour les autres A). La situation est différente pour une solution de (22). En effet, $f(a) = (2, \ldots, 2)$ pour chaque $a \in A(p)$ est une solution de (22) ne vérifiant pas (23).

On a démontré dans [9] aussi que pour une solution de (9) la relation (23) pour $A(p) = \mathbb{R}(p)$ entraîne que au moins un $Z_k = \mathbb{R}(p)$. Cette situation n'a pas lieu pour la solution de (22) pour $p > 2$. En effet, pour $A(p) = \mathbb{Z}(p)$ soit

$$Z_1 = \{(a_1, \ldots, a_p) \in \mathbb{Z}(p) : \ a_1 = 3n \text{ ou } a_1 = 3n+1 \text{ pour } n \in \mathbb{N}\},$$
$$Z_2 = \{(a_1, \ldots, a_p) \in \mathbb{Z}(p) : \ a_1 = 3n \text{ ou } a_1 = 3n+2 \text{ pour } n \in \mathbb{N}\},$$
$$Z_3 = \{(a_1, \ldots, a_p) \in \mathbb{Z}(p) : \ a_1 = 3n+1 \text{ ou } a_1 = 3n+2 \text{ pour } n \in \mathbb{N}\}$$

et

$$Z_4 = \cdots = Z_p = \emptyset.$$

On peut vérifier que

$$\mathbb{Z}(p) \times \mathbb{Z}(p) \subset (Z_1 \times Z_1) \cup (Z_2 \times Z_2) \cup (Z_3 \times Z_3) \cup \ldots \cup (Z_p \times Z_p),$$

donc la fonction $f\colon \mathbb{Z}(p) \longrightarrow \mathbb{Z}(p)$ pour laquelle $Z_k = \{a \in \mathbb{Z}(p) : \ f_k(a) \neq 0\}$ pour $k = 1, \ldots, p$, remplit (22) d'après le théorème 7, mais aucun Z_k n'est égal à $\mathbb{Z}(p)$.

Remarquons encore que (23) pour $p = 2$ entraîne que au moins un $Z_k = A(p)$. Supposons contrairement qu'il existe $a, b \in A(p)$ tels que $a \notin Z_1$ et $b \notin Z_2$. Dans ce cas $(a, b) \notin Z_1 \times Z_1$ et $(a, b) \notin Z_2 \times Z_2$ et $(a, b) \in A(p) \times A(p)$, donc une contradiction avec (23). S'il existe un élément a appartenant seulement à un ensemble Z_k $(k \in \{1, \ldots, p\})$, dans ce cas (22) donne $(a, x) \in Z_k \times Z_k$ pour chaque x de $A(p)$, d'où $Z_k = A(p)$. Pour $p = 2$ on a $Z_1 = Z_2$ ou il existe un élément a appartenant seulement à un Z_k $(k = 1, 2)$.

Une fonction $f\colon A(p) \longrightarrow B(p)$ est une solution de (9) et de (22) en même temps si et seulement si elle est une solution de (16). Il existe naturellement des solutions de (9) qui ne sont pas des solutions de (22) et inversement.

On a vu que la fonction $f\colon A(p) \longrightarrow B(p)$ remplit (9) si et seulement si la condition (12) a lieu. Il est donc intéressant remarquer que nous avons le

THÉORÈME 8
La fonction $f = (f_1, \ldots, f_p)\colon A(p) \longrightarrow B(p)$ *remplit* (22) *si et seulement si la modification simple de* (12) *:*

$$\begin{aligned}&(i_1 j_1, \ldots, i_p j_p) = \underline{0}\\ &\qquad \Longrightarrow\ Z_1^{i_1} \cap \ldots \cap Z_p^{i_p} + Z_1^{j_1} \cap \ldots \cap Z_p^{j_p} \subset Z_1^{i_1 j_1} \cap \ldots \cap Z_p^{i_p j_p}\end{aligned} \tag{24}$$

a lieu pour tous $(i_1, \ldots, i_p), (j_1, \ldots, j_p) \in O(p)$*, où*

$$Z_k = \{a \in A(p) :\ f_k(a) \neq 0\} \qquad \textit{pour } k = 1, \ldots, p.$$

Cette modification (24) *désigne d'après la relation* $Z_1 \cup \ldots \cup Z_p = A(p)$ *que*

$$\begin{aligned}&(i_1 j_1, \ldots, i_p j_p) = \underline{0}\\ &\qquad \Longrightarrow\ Z_1^{i_1} \cap \ldots \cap Z_p^{i_p} + Z_1^{j_1} \cap \ldots \cap Z_p^{j_p} \subset Z_1^{0} \cap \ldots \cap Z_p^{0}\\ &\qquad\qquad = A \setminus (Z_1 \cup \ldots \cup Z_p) = \emptyset,\end{aligned}$$

donc elle est équivalente à la condition

$$\begin{aligned}&(i_1 j_1, \ldots, i_p j_p) = \underline{0}\\ &\qquad \Longrightarrow\ Z_1^{i_1} \cap \ldots \cap Z_p^{i_p} = \emptyset \ \textit{ou}\ Z_1^{j_1} \cap \ldots \cap Z_p^{j_p} = \emptyset\end{aligned} \tag{24'}$$

pour tous $(i_1, \ldots, i_p), (j_1, \ldots, j_p) \in O(p)$.

Démonstration. Il suffit de montrer d'après le théorème 7 que la condition

$$\exists_{a,b \in A(p)} :\ f(a) \cdot f(b) = \underline{0} \tag{25}$$

est équivalente à la condition

$$\begin{aligned}&\exists_{(i_1,\ldots,i_p),(j_1,\ldots,j_p) \in O(p)} :\ (i_1 j_1, \ldots, i_p j_p) = \underline{0}\\ &\qquad \text{et}\quad Z_1^{i_1} \cap \ldots \cap Z_p^{i_p} \neq \emptyset \ \text{ et }\ Z_1^{j_1} \cap \ldots \cap Z_p^{j_p} \neq \emptyset.\end{aligned}$$

Pour a et b dans (25) il existe $(i_1, \ldots, i_p), (j_1, \ldots, j_p) \in \{0, 1\}^p \setminus \{\underline{0}\}$ tels que

$$a \in Z_1^{i_1} \cap \ldots \cap Z_p^{i_p} \quad \text{et} \quad b \in Z_1^{j_1} \cap \ldots \cap Z_p^{j_p}. \tag{26}$$

La condition (25) désigne que pour chaque $k \in \{1, \ldots, p\}$: $f_k(a) = 0$ ou $f_k(b) = 0$ et cela désigne que pour chaque $k \in \{1, \ldots, p\}$: $i_k = 0$ ou $j_k = 0$. Cette alternative est équivalente à la condition $(i_1 j_1, \ldots, i_p j_p) = \underline{0}$ et la condition (26) désigne que $Z_1^{i_1} \cap \ldots \cap Z_p^{i_p} \neq \emptyset$ et $Z_1^{j_1} \cap \ldots \cap Z_p^{j_p} \neq \emptyset$.

Remarquons qu'il suffit vérifier (24′) seulement pour les $2^{-1}(3^p - 2^{p+1} + 1)$ paires $(i_1, \dots, i_p), (j_1, \dots, j_p) \in O(p)$ à cause de la condition $(i_1 j_1, \dots, i_p j_p) = 0$ et par raison de la symétrie.

Pour la fonction f vérifiant (22), $f(a+b)$ soit une fonction de $f(a)$ et $f(b)$ si et seulement si

$$f(a) = f(c) \quad \text{et} \quad f(b) = f(d) \implies f(a+b) = f(c+d)$$

pour chaques $a, b, c, d \in A(p)$ (l'analogie à la condition (13) ; $f(a) \cdot f(b) \neq \underline{0}$ toujours pour la fonction remplissante (22)).

Constatons que la fonction $F\colon E^* := \bigcup_{n=1}^{\infty} E^n \longrightarrow 2^E \setminus \{\emptyset\}$ satisfait à l'équation fonctionnelle conditionnelle

$$F(\mathbf{x}) \cap F(\mathbf{y}) = \emptyset \implies F(\mathbf{xy}) = F(\mathbf{x}) \cap F(\mathbf{y}), \tag{27}$$

étante une analogie de la condition (2′), si et seulement si chaque paire des valeurs de cette fonction a au moins un element commun ($F(\mathbf{x}) \cap F(\mathbf{y}) \neq \emptyset$ pour chaque $\mathbf{x}$ et $\mathbf{y}$ de E^*). Cela ne signifie pas que (5) a lieu (voir la fonction (6)). Si nous remplaçons dans le théorème 2, l'équation (2) par l'équation (22) (en vérité nous rejetons (2) puisque (5) entraine (22)) nous constatons que l'implication „seulement si" est vraie dans ce théorème modifié, mais l'implication „si" n'a pas lieu. La fonction $F(x_1, \dots, x_n) = \{x_1, \dots, x_n\} \cup \{a\}$ cela montre, où a est un element fixé de E. Le théorème analogue au théorème 1 a la forme

Théorème 1′
Il existe pour la fonction F vérifiant (27) *une fonction ϕ satisfaisante à* (4′) *si et seulement si*

$$F(\mathbf{x}) = F(\mathbf{u}) \quad \text{et} \quad F(\mathbf{y}) = F(\mathbf{v}) \implies F(\mathbf{xy}) = F(\mathbf{uv})$$

pour chaques $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v} \in E^*$,
puisque pour $\mathbf{x}$ et $\mathbf{y}$ de E^* on a $F(\mathbf{x}) \cap F(\mathbf{y}) \neq \emptyset$ pour la fonction F vérifiant (27).

S'il s'agit du problème du prolongement des solutions de l'équation (22), il existe toujours ce prolongement pour chaque des ensembles $\mathbb{Z}(p)$, $\mathbb{Q}(p)$, $\mathbf{A}(p)$ sur les ensembles plus vaste, mais pas unique.

Nous savons ([10]) que la fonction $f = (f_1, \dots, f_p)\colon A(p) \longrightarrow B(p)$ vérifiant (9) satisfait à (23) si et seulement s'il existe un $k \in \{1, \dots, p\}$ tel que

$$Z_k = \{a \in A(p) : \; f_k(a) \neq 0\} = A(p). \tag{28}$$

Cette condition (28) est seulement nécessaire pour que la fonction $f\colon A(p) \longrightarrow B(p)$ vérifiant (22) satisferait en même temps à (23), une condition nécessaire et suffisante (en vérité pas intéressante) est dans ce cas le remplissage par f de l'équation (9).

La fonction $f = (f_1, \ldots, f_p)\colon \mathbb{R}(p) \longrightarrow \mathbb{R}(p)$ vérifiant (9) et (22) doit satisfaire aussi à (23), donc elle doit avoir la forme ([10])

$$f_k(x) = \begin{cases} \exp a_k(x) & \text{pour} x \in Z_k = \{(x_1, \ldots, x_p) \in \mathbb{R}(p) : \\ & \qquad\qquad \forall_{j \in M_k} : \ x_j = 0\}, \\ 0 & \text{pour } x \in \mathbb{R}(p) \setminus Z_k, \end{cases} \tag{29}$$

où M_k est, pour $k = 1, \ldots, p$, un sous-ensemble de l'ensemble $\{1, \ldots, p\}$ tel que au moins un M_k, soit M_m, est vide (alors $Z_m = \mathbb{R}(p)$) et $a_k\colon \mathbb{R}^p \longrightarrow \mathbb{R}$ est une fonction additive. Puisque f satisfait à (22) elle remplit (23) et (24′). On peut voir cela aussi directement puisque $f_m(a) \neq 0 \neq f_m(b)$ pour chaque $a, b \in \mathbb{R}(p) = Z_m$ d'après (29), donc (23) a lieu. Si $i_m j_m = 0$, donc $i_m = 0$ ou $j_m = 0$, alors $Z_m = \mathbb{R}(p)$ implique $Z_m^{i_m} = Z_m^0 = \emptyset$ ou $Z_m^{j_m} = Z_m^0 = \emptyset$, donc (24′) a lieu.

Il résulte de nos considérations que seulement les ensembles $Z_1, \ldots, Z_p$ définis dans (29) remplissent les conditions (12), (24′) et

$$Z_1 \cup \ldots \cup Z_p = \mathbb{R}(p), \tag{30}$$

c. à d. les conditions (30) et

$$\forall_{(i_1,\ldots,i_p),(j_1,\ldots,j_p) \in O(p)} : \ Z_1^{i_1} \cap \ldots \cap Z_p^{i_p} + Z_1^{j_1} \cap \ldots \cap Z_p^{j_p} \subset Z_1^{i_1 j_1} \cap \ldots \cap Z_p^{i_p j_p}. \tag{31}$$

Ces ensembles $Z_1, \ldots, Z_p$ forment les cônes sur $\mathbb{R}$ et il existe parmi eux tels qui remplissent dans (31) l'égalité au lieu de l'inclusion (p. ex. $Z_k = \mathbb{R}(p)$ (donc $M_k = \emptyset$) pour $k = 1, \ldots, p$) et tels qui satisfont seulement à l'inclusion [p. ex. $Z_1 = \mathbb{R}(p)$ (donc $M_1 = \emptyset$), $Z_2 = \ldots = Z_p = \emptyset$ (donc $M_2 = \ldots = M_p = \{1, \ldots, p\}$)].

7. Les systèmes des inclusions et des équations des fonctions multivalentes

Il serait intéressant considérer les géneralisations suivantes de (31). Soit T un ensemble arbitraire, $(G, +)$ un groupoïde (un espace vectoriel) et considérons l'inclusion suivante

$$\bigcap_{t \in T} Z(t)^{i(t)} + \bigcap_{t \in T} Z(t)^{j(t)} \subset \bigcap_{t \in T} Z(t)^{i(t)j(t)}, \tag{32}$$

où $Z(t)\colon T \longrightarrow 2^G$ est une fonction cherchée, $Z(t)^1 = Z(t)$, $Z(t)^0 = G \setminus Z(t)$ et (32) doit avoir lieu pour chaques fonctions $i(t), j(t)\colon T \longrightarrow \{0, 1\}$ non-idéntiques zéro et les deux généralisations de (12) et de (24)

$$\exists_{t \in T} : \ i(t)j(t) \neq 0 \implies (32) \tag{33}$$

et

$$\forall_{t\in T}: \; i(t)j(t)=0 \implies (32), \tag{34}$$

qui ont lieu pour chaques (pour deux) fonctions $i(t), j(t)\colon T \longrightarrow \{0,1\}$ non-idéntiques zéro. On peut nommée (32) comme un système des inclusions (un système des inégalités) pour la fonction multivalente $Z(t)$ et (33) et (34) comme un système des inclusions conditionnelles de cette sorte. On peut aussi ajouter ici l'analogue de (30)

$$\bigcup_{t\in T} Z(t) = G. \tag{35}$$

Il se pose le **problème** : quels résultats au sujet de (12), (24) ou (31) (sans ou avec (30)) donnés dans cette note ou dans [3] et [10] sont valables aussi pour (32), (33) et (34) (sans ou avec (35)) ? On peut aussi remplacer dans (32) "$\subset$" par "$=$" dans ce problème (donc considérer l'équation au lieu de l'inégalité).

Par exemple il est vraie la généralisation suivante de l'équivalence (i) et (v) dans le théorème 1 dans [3].

Théorème 9
Soit $(G,+)$ *un groupoïde et* T *un ensemble arbitraire. La fonction* $Z(t)\colon T \longrightarrow 2^G$ *vérifiant* (35) *satisfait à* (33) *si et seulement si*

$$Z(t_1)^{k_1} \cap Z(t_2)^{k_2} + Z(t_1)^{l_1} \cap Z(t_2)^{l_2} \subset Z(t_1)^{k_1 l_1} \cap Z(t_2)^{k_2 l_2} \tag{36}$$

pour chaques $t_1, t_2 \in T$ *et pour chaques* $k_1, k_2, l_1, l_2 \in \{0,1\}$ *et telles que* $k_1 l_1 + k_2 l_2 \neq 0$.

Remarquons qu'il suffit vérifier (33) seulement pour chaques deux parmi les ensembles $Z(t)$.

Démonstration ,,si". Pour $z \in \bigcap_{t\in T} Z(t)^{i(t)} + \bigcap_{t\in T} Z(t)^{j(t)}$ il existe $x \in \bigcap_{t\in T} Z(t)^{i(t)}$ et $y \in \bigcap_{t\in T} Z(t)^{j(t)}$ tels que $z = x+y$. Puisque $i(t)j(t)$ n'est pas idéntiquement zéro, il existe un t_0 tel que $i(t_0)j(t_0) \neq 0$. Soit t quelconque de T et posons $k_1 = i(t_0)$, $l_1 = j(t_0)$, $k_2 = i(t)$, $l_2 = j(t)$. Nous avons $k_1 l_1 + k_2 l_2 \neq 0$, donc d'après (36)

$$\begin{aligned} z = x + y &\in Z(t_0)^{k_1} \cap Z(t)^{k_2} + Z(t_0)^{l_1} \cap Z(t)^{l_2} \\ &\subset Z(t_0)^{k_1 l_1} \cap Z(t)^{k_2 l_2} \subset Z(t)^{k_2 l_2} \\ &= Z(t)^{i(t)j(t)}, \end{aligned}$$

alors $z \in \bigcap_{t\in T} Z(t)^{i(t)j(t)}$.

Pour la *démonstration* de ,,seulement si" remarquons que pour chaque a de G et pour chaque t de T il existe exactement un $j_t \in \{0,1\}$ tel que $a \in Z(t)^{j_t}$ (nous allons désigner dans la suite ce j_t par $j_t(a)$) et il existe au moins un u

de T tel que $j_u(a) = 1$. Soit $z \in Z(t_1)^{k_1} \cap Z(t_2)^{k_2} + Z(t_1)^{l_1} \cap Z(t_2)^{l_2}$, donc il existe $x \in Z(t_1)^{k_1} \cap Z(t_2)^{k_2}$ et $y \in Z(t_1)^{l_1} \cap Z(t_2)^{l_2}$ tels que $z = x + y$. Puisque $x \in \bigcap_{t\in T} Z(t)^{j_t(x)}$ et $y \in \bigcap_{t\in T} Z(t)^{j_t(y)}$, alors $j_{t_1}(x) = k_1$, $j_{t_2}(x) = k_2$, $j_{t_1}(y) = l_1$, $j_{t_2}(y) = l_2$. Il en résulte que

$$\begin{aligned} z = x + y &\in \bigcap_{t\in T} Z(t)^{j_t(x)} + \bigcap_{t\in T} Z(t)^{j_t(y)} \\ &\subset \bigcap_{t\in T} Z(t)^{j_t(x) j_t(y)} \\ &\subset Z(t_1)^{k_1 l_1} \cap Z(t_2)^{k_2 l_2}. \end{aligned}$$

On peut montrer analogiquement comme plus haut le

COROLLAIRE 2
Si la fonction $Z(t)\colon T \longrightarrow 2^G$ remplit (33) *et* (35), *elle satisfait à* (35) *avec T_1 au lieu de T pour chaque $T_1 \subset T$.*

Remarquons qu'on peut toujours prolonger banalement la fonction $Z(t)\colon T \longrightarrow 2^G$ remplissante (33) et (35) à la fonction $Z^*(t)\colon T_2 \longrightarrow 2^G$, où $T \subset T_2$, vérifiant (33) et (35) avec T_2 au lieu de T, en posant $Z^*(t) = G$ pour $t \in T_2 \setminus T$.

Il est vraie aussi la généralisation suivante du lemme 1 de [3]

THÉORÈME 10
Soit $(G, +)$ un groupoïde uniquement divisible. Si la fonction $Z(t)\colon T \longrightarrow 2^G$ remplit (33) *et* (35), *alors pour chaque sous-ensemble T_1 non-vide de l'ensemble T et pour chaque fonction $l(t)\colon T_1 \longrightarrow \{0, 1\}$ non-idéntique zéro, l'ensemble $Z = \bigcap_{t\in T_1} Z(t)^{l(t)}$ forme un cône sur $\mathbb{Q}$, c. à d. $x + y \in Z$ et $\frac{n}{k}x \in Z$ pour $x, y \in Z$ et $n, k \in \mathbb{N}$.*

Démonstration. Soit $x, y \in Z$. Puisque la fonction $l : T_1 \longrightarrow \{0, 1\}$ n'est pas idéntiquement zéro il existe $v \in T_1$ tel que $l(v) = 1$. En prenant les notations de la deuxième partie de la démonstration du théorème 9 nous avons $x \in \bigcap_{t\in T} Z(t)^{j_t(x)}$ et $y \in \bigcap_{t\in T} Z(t)^{j_t(y)}$ et $j_t(x) = j_t(y) = l(t)$ pour $t \in T_1$. De là $j_v(x) = j_v(y) = 1$, donc d'après (33) nous recevons

$$\begin{aligned} x + y &\in \bigcap_{t\in T} Z(t)^{j_t(x)} + \bigcap_{t\in T} Z(t)^{j_t(y)} \\ &\subset \bigcap_{t\in T} Z(t)^{j_t(x) j_t(y)} \subset \bigcap_{t\in T_1} Z(t)^{l(t)^2} \subset \bigcap_{t\in T_1} Z(t)^{l(t)} \\ &= Z. \end{aligned}$$

Soit à présent $x \in Z$ et $k \in \mathbb{N}$. Puisque $\frac{1}{k}x$ appartient à $\bigcap_{t\in T} Z(t)^{j_t[\frac{1}{k}x]}$, qui est fermé par rapport à ,,+" d'après la première partie de cette démonstration, donc $x = k(\frac{1}{k}x) \in \bigcap_{t\in T} Z(t)^{j_t[\frac{1}{k}x]}$. Il en résulte que $j_t[\frac{1}{k}x] = l(t)$ pour chaque t de T_1, donc

$$\frac{1}{k}\,x \in \bigcap_{t\in T} Z(t)^{j_t[\frac{1}{k}x]} \subset \bigcap_{t\in T_1} Z(t)^{l(t)} = Z.$$

De là $\frac{n}{k}x = n(\frac{1}{k}x) \in Z$ pour $n \in \mathbb{N}$, puisque Z est fermé par rapport à „+".

On peut aussi généraliser le théorème 3 (c) de [11].

Théorème 11
Soit $(G,+) = (\mathbb{R}(p),+)$ *et prenons les notations et les suppositions du théorème 10. De plus, supposons que* T *est au plus dénombrable et que l'ensemble* $Z_c(t) = \{\alpha c \in Z(t)\colon \alpha \in \mathbb{R}\}$ *est mesurable linéairement au sens de Lebesgue pour chaque* $c \in \mathbb{R}(p)$ *et* $t \in T$*. Dans ce cas, l'ensemble* $\bigcap_{t\in T_1} Z(t)^{l(t)}$ *forme un cône sur* $\mathbb{R}$*.*

Démonstration. D'après (35) la demi-droite

$$D(c) = \{\alpha c:\ \alpha \in \mathbb{R} \text{ et } \alpha > 0\} = \bigcup_{t\in T} Z_c(t),$$

donc il existe au moins un $t_0 \in T$ tel que $Z_c(t_0)$ a la mesure positive. Soit t de T fixé arbitrairement. Puisque

$$Z_c(t_0) = [Z_c(t_0)^1 \cap Z_c(t)^1] \cup [Z_c(t_0)^1 \cap Z_c(t)^0]$$

nous avons deux possibilités

i) $Z_c(t_0) \cap Z_c(t)$ a la mesure positive

ou

ii) $Z_c(t_0) \cap Z_c(t)^0$ a la mesure positive.

Dans le cas i), d'après le théorème de Steinhaus ([17]), il existe un sous-intervalle de $Z_c(t_0) \cap Z_c(t)$ de la longueur positive et de là $Z_c(t_0) \cap Z_c(t) = D(c)$, puisque l'ensemble $Z_c(t_0) \cap Z_c(t)$ forme un cône sur $\mathbb{Q}$ d'après le théorème 9. Il en résulte que $Z_c(t) = D(c)$, donc $Z_c(t)$ forme un cône sur $\mathbb{R}$. Dans le cas ii), $Z_c(t_0) \cap Z_c(t)^0 = D(c)$, d'où $Z_c(t)^0 = D(c)$, alors $Z_c(t) = \emptyset$ forme un cône sur $\mathbb{R}$. Puisque $\bigcap_{t\in T_1} Z(t)^{l(t)}$ forme un cône sur $\mathbb{Q}$, donc il est un cône sur $\mathbb{R}$.

Remarquons que la supposition que l'ensemble T est dénombrable est essentielle dans le théorème 11. En effet, considérons dans $\mathbb{R}(1)$ la relation ρ suivante : $x\rho y \iff \exists_{w\in\mathbb{Q}}:\ x = wy$. Les classes d'équivalence $Z(t)$ de cette relation pour t dans T (indénombrable) sont dénombrables, donc mesurables et elles forment des cônes sur $\mathbb{Q}$. Puisque elles sont disjointes, elles remplissent (36), donc aussi (33) d'après le théorème 9. Mais elles ne forment pas évidemment des cônes sur $\mathbb{R}$.

Le théorème 11 sera vrai pour T arbitraire (aussi indénombrable) si nous remplaçons la supposition de la mesurabilité des ensembles $Z_c(t)$ par la condition (incommode pour la vérification) : pour chaque $c \in \mathbb{R}(p)$ il existe une fonction $i: T \longrightarrow \{0,1\}$, non-idéntique zéro, telle que l'ensemble $\bigcap_{t\in T} Z_c(t)^{i(t)}$ a la mesure intérieure de Lebesgue positive ou jouit de la propriété de Baire et il est de la deuxième catégorie, puisque dans ce cas $\bigcap_{t\in T} Z_c(t)^{i(t)} = D(c)$ d'après le théorème de Steinhaus ou de Piccard ([7] p. 48), d'où $\bigcap_{t\in T} Z(t)^{i(t)} \cap D(c) = D(c)$. Si $c \in Z(t_0)$, alors $i(t_0) \neq 0$, d'où $\bigcap_{t\in T} Z(t)^{i(t)} \cap D(c) \subset Z(t_0) \cap D(c)$, donc $Z(t_0) \cap D(c) = D(c)$, alors $\alpha c \in Z(t_0)$ pour chaque $\alpha > 0$. Si $c \notin Z(t_0)$, alors $i(t_0) = 0$ et $Z(t_0) \cap D(c) = \emptyset$. Puisque $Z(t_0)$ forme un cône sur $\mathbb{Q}$, il forme un cône sur $\mathbb{R}$, d'où $\bigcap_{t\in T_1} Z(t)^{l(t)}$ forme un cône sur $\mathbb{R}$ pour chaque $\emptyset \neq T_1 \subset T$ et pour chaque $l(t): T_1 \longrightarrow \{0,1\}$ non-idéntique zéro.

Nous allons démontrer le

Lemme 4
Si $Z \subset \mathbb{R}(p)$, $Z + Z \subset Z$, $Z + Z^0 \subset Z^0$ et $Z^0 + Z^0 \subset Z^0$, alors il existe un ensemble $M \subset \{1, \ldots, p\}$ tel que

$$Z = \{(x_1, \ldots, x_p) \in \mathbb{R}(p) : \ \forall_{i\in M} : \ x_i = 0\}.$$

Démonstration. Si $Z = \mathbb{R}(p)$, alors $M = \emptyset$. Si $Z \neq \mathbb{R}(p)$, alors il existe $x_0 \in Z^0$ et de là $x_0 + x \in Z^0$ pour chaque $x \in \mathbb{R}(p)$, d'où

$$\operatorname{Int}\mathbb{R}(p) = \{(x_1, \ldots, x_p) \in \mathbb{R}(p) : \ x_i > 0 \text{ pour } i = 1, \ldots, p\} \subset Z^0, \tag{37}$$

puisque, en supposant au contraire, qu'il existe un $y \in \operatorname{Int}\mathbb{R}(p) \cap Z$, nous constatons qu'il existe un $n \in \mathbb{N}$ tel que $ny - x_0 \in \mathbb{R}(p)$, d'où $ny = x_0 + (ny - x_0) \in Z^0$, donc une contradiction avec $ny \in Z$.

Il doit exister un $k \in \{1, \ldots, p\}$ tel que

$$Z \subset \{(x_1, \ldots, x_p) \in \mathbb{R}(p) : \ x_k = 0\},$$

puisque dans le cas contraire, il existe dans Z des points $(x_1^i, \ldots, x_p^i)$ pour $i = 1, \ldots, p$ tels que $x_i^i \neq 0$ et de là leurs somme appartient à Z et à $\operatorname{Int}\mathbb{R}(p)$, donc une contradiction avec (37). Nous pouvons supposer que $k = 1$. L'ensemble

$$Z_1 = \{(x_2, \ldots, x_p) \in \mathbb{R}(p-1) : \ (x_1, \ldots, x_p) \in Z\}$$

remplit les conditions $Z_1 \subset \mathbb{R}(p-1)$, $Z_1 + Z_1 \subset Z_1$, $Z_1 + Z_1^0 \subset Z_1^0$ et $Z_1^0 + Z_1^0 \subset Z_1^0$. Si $Z_1 = \mathbb{R}(p-1)$, donc $Z = \{(x_1, \ldots, x_p) \in \mathbb{R}(p-1) : \ x_1 = 0\}$ et la démonstration est finie ($M = \{1\}$). Si $Z_1 \neq \mathbb{R}(p-1)$, donc il existe un $l \in \{2, \ldots, p\}$ tel que

$$Z_1 \subset \{(x_2, \ldots, x_p) \in \mathbb{R}(p-1) : \ x_l = 0\}$$

d'après le raisonnement plus haut (avec $p-1$ au lieu de p). Nous pouvons prendre $l = 2$. En raisonnant comme plus haut nous constatons que $Z \subset \{(x_1, \dots, x_p) \in \mathbb{R}(p) :\ x_1 = x_2 = 0\}$, donc $Z = \{(x_1, \dots, x_p) \in \mathbb{R}(p) :\ x_1 = x_2 = 0\}$ et nous avons la thèse ($M = \{1, 2\}$) ou $Z \neq \{(x_1, \dots, x_p) \in \mathbb{R}(p) :\ x_1 = x_2 = 0\}$ et en repetant le raisonnement nous recevons la thèse après le nombre fini des pas.

Nous pouvons à présent démontrer le

Théorème 12
La fonction $Z(t)\colon T \longrightarrow 2^{\mathbb{R}(p)}$ *est une solution de* (32) *et* (35) *si et seulement si*

$$Z(t) = \left\{(x_1, \dots, x_p) \in \mathbb{R}(p) :\ \forall_{j \in M(t)} :\ x_j = 0\right\}, \tag{38}$$

où $M(t)$*, pour chaque* $t \in T$*, est un sous-ensemble de l'ensemble* $\{1, \dots, p\}$ *tel que au moins un* $M(t) = \emptyset$*.*

Démonstration. La partie „si" de la démonstration est evidente. Pour la démonstration de „seulement si" nous allons montrer que $Z(t_0)$ remplit les suppositions du lemme 4 pour chaque $t_0 \in T$. Nous avons $Z(t_0) + Z(t_0) \subset Z(t_0)$ puisque $Z(t_0)$ forme un cône sur $\mathbb{Q}$. Soit $x \in Z(t_0) + Z(t_0)^0$, donc $x = y + z$, où $y \in Z(t_0)$ et $z \in Z(t_0)^0$. Il existe $i(t)$ et $j(t)$ telles que $y \in \bigcap_{t \in T} Z(t)^{i(t)}$ et $z \in \bigcap_{t \in T} Z(t)^{j(t)}$, d'où $i(t_0) = 1$ et $j(t_0) = 0$, alors $i(t_0)j(t_0) = 0$. Nous recevons $x \in \bigcap_{t \in T} Z(t)^{i(t)j(t)}$ d'après (32), donc $x \in Z(t_0)^0$, c. q. f. d. Nous montrons analogiquement que $Z(t_0)^0 + Z(t_0)^0 \subset Z(t_0)^0$. Puisque $(1, \dots, 1) \in \mathbb{R}(p) = \bigcup_{t \in T} Z(t)$, il existe alors un t_0 tel que $(1, \dots, 1) \in Z(t_0)$, donc $M(t_0) = \emptyset$.

Corollaire 3
Si $Z(t)\colon T \longrightarrow 2^{\mathbb{R}(p)}$ *remplit* (32) *et* (35)*, alors* $Z(t)$ *est un cône sur* $\mathbb{R}$ *pour chaque* $t \in T$*.*

Remarquons que la fonction $f = P_{t \in T} f_t$ (le produit cartésien des fonctions f_t), où

$$f_t(x) \begin{cases} 1 & \text{pour } x \in Z(t), \\ 0 & \text{pour } x \in G \setminus Z(t), \end{cases} \tag{39}$$

pour $Z\colon T \longrightarrow 2^G$, remplit (16) ou (9), ou (22), où $\underline{0} = P_{t \in T} 0$ et $f(b)f(b) = P_{t \in T} f_t(a) f_t(b)$, si et seulement si $Z(t)$ satisfont aux (32) ou (33), ou (34).

Nous avons le

Corollaire 4
La fonction $f = P_{t \in T} f_t\colon \mathbb{R}(p) \longrightarrow \{0, 1\}^T$ *remplit* (16) *si et seulement si les fonctions* f_t *ont la forme* (39) *avec* $Z(t)$ *donnés par* (38)*.*

Remarquons que le corollaire est une généralisation pour T infini du résutat de [11].

Nous avons aussi le

THÉORÈME 13
$Z(t) = \mathbb{R}(p)$ est la seule solution du systeme des équations

$$\bigcap_{t\in T} Z(t)^{i(t)} + \bigcap_{t\in T} Z(t)^{j(t)} = \bigcap_{t\in T} Z(t)^{i(t)j(t)} \tag{40}$$

et

$$\bigcup_{t\in T} Z(t) = G. \tag{35}$$

Démonstration. On peut facilement montrer que la fonction $Z(t) = \mathbb{R}(p)$ satisfait aux conditions (35) et (40). Si la fonction $Z(t)$ remplit (35) et (40) les ensembles $Z(t)$ forment les cônes sur $\mathbb{R}$ d'après le corollaire 2. Supposons qu'il existe $t_0 \in T$ tel que $Z(t_0) \neq \mathbb{R}(p)$. Il existe dans ce cas un demi-axe du système des coordonnes qui n'est pas contenu dans $Z(t_0)$ ($Z(t_0)$ forme un cône sur $\mathbb{R}$). Nous pouvons prendre que ce demi-axe c'est $X_1 = \{(x_1, \ldots, x_p) \in \mathbb{R}(p) :\ x_2 = \ldots = x_p = 0\}$. Nous avons donc

$$x_2 + \cdots + x_p \neq 0 \qquad \text{pour } (x_1, \ldots, x_p) \in Z(t_0). \tag{41}$$

Soit pour $t \in T_1 : X_1 \subset Z(t)$ (T_1 peut être vide) et pour $t \in T_2 : X_1$ n'est pas contenu dans $Z(t)$, donc $X_1 \subset Z(t)^0$. Posons dans (40), $i(t) = 1$ pour $t \in T_1$ et $i(t) = 0$ pour $t \in T_2$ et $j(t) = 1$ pour $t \in T$. Dans ce cas

$$X_1 \subset \bigcap_{t\in T} Z(t)^{i(t)} = \bigcap_{t\in T} Z(t)^{i(t)j(t)}.$$

Il en résulte que $(1, 0, \ldots, 0) \in \bigcap_{t\in T} Z(t)^{i(t)j(t)}$, alors $(1, 0, \ldots, 0) = (a_1, \ldots, a_p) + (b_1, \ldots, b_p)$, où $(a_1, \ldots, a_p) \in \bigcap_{t\in T} Z(t)^{i(t)}$ et $(b_1, \ldots, b_p) \in \bigcap_{t\in T} Z(t)^{j(t)}$, alors $a_2 + b_2 = \ldots = a_p + b_p = 0$, d'où $b_2 = \ldots = b_p = 0$. Nous avons une contradiction avec (41) puisque $(b_1, \ldots, b_p) \in Z(t_0)$.

On nomme chaque ensemble $\bigcap_{t\in T} Z(t)^{i(t)}$ comme une composante de la famille $Z(T)$ (voir [8] pour T fini). La condition (32) nous suggère qu'on peut définir dans la famille $Z(T)$ des composantes d'une fonction $Z(t)$ vérifiant (32), l'opération $\oplus$ de la manière suivante

$$C_1 \oplus C_2 = \bigcap_{t\in T} Z(t)^{t)j(t)} \tag{42}$$

pour $C_1 = \bigcap_{t\in T} Z(t)^{i(t)}$ et $C_2 = \bigcap_{t\in T} Z(t)^{j(t)}$. Cette définition n'est pas correcte puisque le résultat de l'opération $C_1 \oplus C_2$ peut dépendre de la représentation des composantes C_1 et C_2. En effet, pour $(G, +) = (\mathbb{R}(2), +)$, $T = \{1, 2, 3\}$,

$Z(k) = \{(x_1, x_2) \in \mathbb{R}(2) : \ x_k = 0\}$ pour $k = 1, 2$ et $Z(3) = \mathbb{R}(2)$, la fonction $Z(t)$, d'après ce qui prècede, remplit (32), la compopsante $C_1 = \emptyset$ a les deux représentations $Z(1)^1 \cap Z(2)^1 \cap Z(3)^1$ et $Z(1)^1 \cap Z(2)^1 \cap Z(3)^0$ et nous avons pour $C_2 = Z(1)^0 \cap Z(2)^0 \cap Z(3)^1$ d'un côté

$$\begin{aligned} C_1 \oplus C_2 &= Z(1)^1 \cap Z(2)^1 \cap Z(3)^1 \oplus Z(1)^0 \cap Z(2)^0 \cap Z(3)^1 \\ &= Z(1)^0 \cap Z(2)^0 \cap Z(3)^1 \neq \emptyset \end{aligned}$$

et d'autre côté

$$\begin{aligned} C_1 \oplus C_2 &= Z(1)^1 \cap Z(2)^1 \cap Z(3)^1 \oplus Z(1)^0 \cap Z(2)^0 \cap Z(3)^1 \\ &= Z(1)^0 \cap Z(2)^0 \cap Z(3)^1 = \emptyset. \end{aligned}$$

Mais, si nous bornons dans le cas arbitraire de $(G, +)$ seulement aux composantes non-vides nous constatons que la définition (42) est déjà correcte puisque dans ce cas les composantes $\bigcap_{t\in T} Z(t)^{i(t)}$ et $\bigcap_{t\in T} Z(t)^{j(t)}$ sont disjoints, donc différentes, pour $i(t)$ et $j(t)$ diverses.

En considérant dans l'ensemble des fonctions $T \longrightarrow \{0, 1\}$, pour lesquelles $\bigcap_{t\in T} Z(t)^{i(t)} \neq \emptyset$, l'opération $(i(t), j(t)) \longrightarrow i(t)j(t)$, nous constatons que la fonction $\phi[i] = \bigcap_{t\in T} Z(t)^{i(t)}$ est un isomorphisme entre cette opération et l'opération $\oplus$. De même, si nous remplaçons dans (32) l'inclusion ,,$\subset$" par l'égalité ,,$=$" nous constatons que la fonction analogue à la fonction ϕ est un homomorphisme de la famille de toutes les fonctions de T à $\{0, 1\}$ dans la famille de toutes composantes de $Z(T)$, puisque dans ce cas la somme ,,$\oplus$" des composantes, définie par (42), ne dépend pas de ces représentations.

Remarquons encore que la somme $\bigcap_{t\in T} Z(t)^{i(t)} + \bigcap_{t\in T} Z(t)^{j(t)}$ de deux composantes ne doit pas être une composante, même si $Z(t)$ remplit (32) et (35). En effet, cette situation a lieu p. ex. pour $(G, +) = (\mathbb{R}(2), +)$, $T = \{1, 2\}$, $Z(1) = \mathbb{R}(2)$ et $Z(2) = \{(x_1, x_2) \in \mathbb{R}(2) : \ x_2 = 0\}$, $i(1) = i(2) = 1$, $j(1) = 1$ et $j(2) = 0$. La condition (40) n'est pas naturellement remplie dans ce cas. Si, pour une fonction $Z(t) \colon T \longrightarrow 2^G$ vérifiant (32), la somme $\bigcap_{t\in T} Z(t)^{i(t)} + \bigcap_{t\in T} Z(t)^{j(t)}$ forme une composante non-vide, dans ce cas (40) a lieu puisque deux composantes sont idéntiques ou disjoints. Mais au contraire si cette some est une composante vide, elle ne doit pas remplir (38), p. ex. pour $T = \{1, 2\}$, $(G, +) = (\mathbb{R}(2), +)$, $Z(1) = \mathbb{R}(2)$, $Z(2) = \emptyset$, $i(1) = i(2) = j(1) = 1$ et $j(2) = 0$.

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On nondegenerate umbilical affine hypersurfaces in recurrent affine manifolds

Dedicated to Professor Andrzej Zajtz
on the occasion of his 70th birthday

Abstract. Let $\widetilde{M}$ be a differentiable manifold of dimension $\geqslant 5$, which is endowed with a (torsion-free) affine connection $\widetilde{\nabla}$ of recurrent curvature. Let M be a nondegenerate umbilical affine hypersurface in $\widetilde{M}$, whose shape operator does not vanish at every point of M. Denote by ∇ and h, respectively, the affine connection and the affine metric induced on M from the ambient manifold. Under the additional assumption that the induced connection ∇ is related to the Levi-Civita connection ∇^* of h by the formula

$$\nabla_X Y = \nabla^*_X Y + \varphi(X)Y + \varphi(Y)X + h(X,Y)E,$$

φ being a 1-form and E a vector field on M, it is proved that the affine metric h is conformally flat. Relations to totally umbilical pseudo-Riemannian hypersurfaces are also discussed.

In this paper, certain ideas from my unpublished report [14] (cf. also [15]) are generalized.

1. Preliminaries ([11, 10])

Let $\widetilde{M}$ be an $(n+1)$-dimensional affine manifold, that is, a connected differentiable manifold endowed with an affine connection $\widetilde{\nabla}$ (only torsion-free affine connections will be considered).

Let M be an n-dimensional connected differentiable manifold immersed into $\widetilde{M}$ and assume that there exists a transversal vector field ξ along the submanifold M. If $\widetilde{X}$ is a vector field defined along the submanifold M (which is not tangent to M in general), by $\widetilde{X}^\top$ and $\widetilde{X}^\perp$ we indicate its tangential and transversal parts, respectively.

Denote by ∇ the affine connection induced on M by assuming $\nabla_X Y = (\widetilde{\nabla}_X Y)^\top$ for all vector fields X, Y tangent to M. In the sequel, M will be

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called an affine hypersurface of the affine manifold $\widetilde{M}$. Thus, we have the Gauss equation for M

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi \tag{1}$$

for all vector fields X, Y tangent to M, where h is a symmetric $(0,2)$-tensor field, which is called the affine fundamental form of M or the affine metric corresponding to ξ.

The affine hypersurface M is said to be nondegenerate if the affine metric h is nondegenerate. In this case, h is a Riemannian or pseudo-Riemannian metric on M. It should be mentioned that there is no relation between the affine metric h and the induced connection ∇ in general.

For the affine hypersurface M, we also have the so-called Weingarten equation

$$\widetilde{\nabla}_X \xi = -AX + \tau(X)\xi, \tag{2}$$

where A is a (1,1)-tensor field and τ is a 1-form on M. A and τ are called, respectively, the shape operator and the transversal connection form of M.

Let $\widetilde{R}$ and R be the curvature tensor fields of the connection $\widetilde{\nabla}$ and the induced connection ∇. Thus,

$$\widetilde{R}\big(\widetilde{X},\widetilde{Y}\,\big) = [\widetilde{\nabla}_{\widetilde{X}}, \widetilde{\nabla}_{\widetilde{Y}}] - \widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]} \qquad \text{for any vector fields } \widetilde{X},\ \widetilde{Y} \text{ on } \widetilde{M}$$

and

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \qquad \text{for any vector fileds } X,\ Y \text{ on } M.$$

As the integrability conditions of (1) and (2), we have the so-called Gauss and Codazzi equations

$$\begin{aligned} \widetilde{R}(X,Y)Z = R(X,Y)Z &- h(Y,Z)AX + h(X,Z)AY \\ &+ \big((\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) \\ &- (\nabla_Y h)(X,Z) - \tau(Y)h(X,Z)\big)\xi, \end{aligned} \tag{3}$$

$$\begin{aligned} \widetilde{R}(X,Y)\xi = -(\nabla_X A)Y &+ \tau(X)AY + (\nabla_Y A)X - \tau(Y)AX \\ &+ \big(-h(X,AY) + h(Y,AX) + 2d\tau(X,Y)\big)\xi. \end{aligned} \tag{4}$$

In the above formulas and in the sequel, symbols X, Y, $Z, \ldots$ denote arbitrary vector fields tangent to M if it is not otherwise stated.

Remark

Note that for an immersion of a differentiable manifold M into an affine manifold $\widetilde{M}$, a choice of a transversal vector field ξ provides the induced connection ∇ on M in such a way that this immersion becomes an affine immersion of (M,∇) into $(\widetilde{M},\widetilde{\nabla})$ in the sense of [9].

2. Umbilical affine hypersurfaces

An affine hypersurface M is said to be umbilical ([5, 8, 10]) if its shape operator A is proportional to the identity tensor at every point of the hypersurface, that is, we have $A = \rho\,\mathrm{Id}$, where Id is the identity tensor field and ρ is a certain function on M. Consequently, for such a hypersurface, we also have $\nabla A = d\rho \otimes \mathrm{Id}$, where d indicates the exterior derivative.

For an umbilical affine hypersurface, the Gauss and Codazzi equations (3) and (4) take the forms

$$\begin{aligned}\widetilde{R}(X,Y)Z = R(X,Y)Z &- \rho h(Y,Z)X + \rho h(X,Z)Y \\ &+ \big((\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) \\ &- (\nabla_Y h)(X,Z) - \tau(Y)h(X,Z)\big)\xi,\end{aligned} \tag{5}$$

$$\widetilde{R}(X,Y)\xi = (\rho\tau - d\rho)(X)Y - (\rho\tau - d\rho)(Y)X + 2d\tau(X,Y)\xi. \tag{6}$$

The following proposition can be found in my unpublished report [14], and we include its proof to the presented paper for completness only.

Proposition 1
For an umbilical affine hypersurface M in an affine manifold $\widetilde{M}$, we have

$$\begin{aligned}((\widetilde{\nabla}_Z \widetilde{R})(X,Y)\xi)^\top = \rho R(X,Y)Z& \\ &- 2\rho d\tau(X,Y)Z - \rho^2(h(Y,Z)X - h(X,Z)Y) \\ &- \big((\nabla_Z(\rho\tau - d\rho))(Y) - \tau(Z)(\rho\tau - d\rho)(Y)\big)X \\ &+ \big((\nabla_Z(\rho\tau - d\rho))(X) - \tau(Z)(\rho\tau - d\rho)(X)\big)Y \\ &+ h(Y,Z)(\widetilde{R}(\xi,X)\xi)^\top - h(X,Z)(\widetilde{R}(\xi,Y)\xi)^\top.\end{aligned} \tag{7}$$

Proof. Applying the equalities (1), (2) and $A = \rho\,\mathrm{Id}$ into the general formula

$$\begin{aligned}(\widetilde{\nabla}_Z \widetilde{R})(X,Y)\xi = \widetilde{\nabla}_Z \widetilde{R}(X,Y)\xi &- \widetilde{R}(\widetilde{\nabla}_Z X, Y)\xi \\ &- \widetilde{R}(X, \widetilde{\nabla}_Z Y)\xi - \widetilde{R}(X,Y)\widetilde{\nabla}_Z \xi,\end{aligned}$$

we find

$$\begin{aligned}(\widetilde{\nabla}_Z \widetilde{R})(X,Y)\xi = \widetilde{\nabla}_Z \widetilde{R}(X,Y)\xi &- \widetilde{R}(\nabla_Z X, Y)\xi - \widetilde{R}(X, \nabla_Z Y)\xi \\ &- h(Z,X)\widetilde{R}(\xi,Y)\xi + h(Z,Y)\widetilde{R}(\xi,X)\xi \\ &+ \rho\widetilde{R}(X,Y)Z - \tau(Z)\widetilde{R}(X,Y)\xi.\end{aligned} \tag{8}$$

On the other hand, with the help of (6), (1) and (2), we find

$$
\begin{aligned}
(\widetilde{\nabla}_Z \widetilde{R}(X,Y)\xi - \widetilde{R}(\nabla_Z X,Y)\xi - \widetilde{R}(X,\nabla_Z Y)\xi)^{\top} \\
= (\nabla_Z(\rho\tau - d\rho))(X)Y - (\nabla_Z(\rho\tau - d\rho))(Y)X \\
- 2\rho d\tau(X,Y)Z.
\end{aligned} \tag{9}
$$

Moreover, (5) and (6) imply

$$
(\widetilde{R}(X,Y)Z)^{\top} = R(X,Y)Z - \rho h(Y,Z)X + \rho h(X,Z)Y, \tag{10}
$$

$$
(\widetilde{R}(X,Y)\xi)^{\top} = (\rho\tau - d\rho)(X)Y - (\rho\tau - d\rho)(Y)X. \tag{11}
$$

Now, to obtain (7) it is sufficient to take the tangential parts of the both sides of (8) and use identities (9)-(11).

In the final section, we will study the case when the ambient affine manifold $\widetilde{M}$ is a recurrent affine manifold, that is, the curvature tensor field $\widetilde{R}$ of $\widetilde{M}$ is non-zero and its covariant derivative $\widetilde{\nabla}\widetilde{R}$ satisfies the conditon ([19, 20, 6])

$$
\widetilde{\nabla}\widetilde{R} = \psi \otimes \widetilde{R} \tag{12}
$$

for a certain 1-form ψ.

We will need the following result:

Proposition 2

Let M be an umbilical affine hypersurface in a recurrent affine manifold $\widetilde{M}$. Then the curvature tensor R of the induced connection ∇ is given by

$$
\begin{aligned}
\rho R(X,Y)Z \\
= 2\rho d\tau(X,Y)Z + \rho^2(h(Y,Z)X - h(X,Z)Y) \\
+ \big((\nabla_Z(\rho\tau - d\rho))(Y) - (\tau+\psi)(Z)(\rho\tau - d\rho)(Y)\big)X \\
- \big((\nabla_Z(\rho\tau - d\rho))(X) - (\tau+\psi)(Z)(\rho\tau - d\rho)(X)\big)Y \\
- h(Y,Z)(\widetilde{R}(\xi,X)\xi)^{\top} + h(X,Z)(\widetilde{R}(\xi,Y)\xi)^{\top}
\end{aligned} \tag{13}
$$

Proof. At first, note that (12) and (6) enable us to find

$$
(\widetilde{\nabla}_Z \widetilde{R})(X,Y)\xi = \psi(Z)\big((\rho\tau - d\rho)(X)Y - (\rho\tau - d\rho)(Y)X + 2d\tau(X,Y)\xi\big).
$$

Then, applying the above into (7), we obtain (13).

3. A special class of affine connections

In the next section, a geometric situation occurs in which a pseudo-Riemannian manifold (M,g) admits an affine connection ∇ which is related to the Levi-Civita connection ∇^* of the metric g by the formula

$$
\nabla_X Y = \nabla^*_X Y + \varphi(X)Y + \varphi(Y)X + g(X,Y)E, \tag{14}
$$

where φ is a 1-form and E a vector field on a M.

The following proposition is of our special interest in the next section.

Proposition 3
Let ∇ be an affine connection on a pseudo-Riemannian manifold (M,g), which is related to the Levi-Civita connection ∇^* of g by the formula (14)*. Then for the curvature tensor fields R and R^* of ∇ and ∇^*, respectively, it holds*

$$\begin{aligned} R^*(X,Y)Z \\ &= R(X,Y)Z - 2d\varphi(X,Y)Z - \varphi(E)(g(Y,Z)X - g(X,Z)Y) \\ &\quad + ((\nabla^*_Y\varphi)(Z) - \varphi(Y)\varphi(Z))X - ((\nabla^*_X\varphi)(Z) - \varphi(X)\varphi(Z))Y \\ &\quad - g(Y,Z)(\nabla^*_X E + g(X,E)E) + g(X,Z)(\nabla^*_Y E + g(Y,E)E). \end{aligned} \tag{15}$$

Proof. Let ∇^2 and ∇^{*2} denote the second covariant derivatives with respect to ∇ and ∇^*, respectively,

$$\nabla^2_{XY}Z = \nabla_X\nabla_Y Z - \nabla_{\nabla_X Y}Z, \qquad \nabla^{*2}_{XY}Z = \nabla^*_X\nabla^*_Y Z - \nabla^*_{\nabla^*_X Y}Z.$$

Then obviously

$$R(X,Y) = \nabla^2_{XY} - \nabla^2_{YX}, \qquad R^*(X,Y) = \nabla^{*2}_{XY} - \nabla^{*2}_{YX}\,. \tag{16}$$

At first, using (14), we find the following relation for the second covariant derivatives

$$\begin{aligned} \nabla^{*2}_{XY}Z = \nabla^2_{XY}Z &- (\nabla^*_X\varphi)(Y)Z - \varphi(E)g(Y,Z)E - (\nabla^*_X\varphi)(Z)Y \\ &- \varphi(Y)\varphi(Z)X - g(Y,Z)(\nabla^*_X E + g(X,E)E) \\ &+ \mathrm{SP}(X,Y)Z, \end{aligned} \tag{17}$$

where $\mathrm{SP}(X,Y)Z$ indicates an expression which is symmetric with respect to X and Y. Next, we find (15), by applying (17), (16) and the following expression for the exterior derivative

$$d\varphi(X,Y) = \frac{1}{2}((\nabla^*_X\varphi)(Y) - (\nabla^*_Y\varphi)(X)).$$

Below, we discuss two typical geometric circumstances leading to (14).

A. Weyl connections ([2, 4, 11]). A Weyl structure on a differentiable manifold M is a conformal class of pseudo-Riemannian metrics $\mathfrak{C}$ together with a mapping $F\colon \mathfrak{C} \longrightarrow \Lambda^1(M)$ such that

$$F(e^\lambda g) = F(g) - d\lambda$$

for any $\lambda\colon M \longrightarrow \mathbb{R}$ and $g \in \mathfrak{C}$, $\Lambda^1(M)$ being the space of 1-forms on M. We say that an affine connection ∇ is compatible with the given Weyl structure $\mathfrak{C}$ on M if

$$\nabla g + F(g) \otimes g = 0 \qquad \text{for all } g \in \mathfrak{C}.$$

Given a Weyl structure $\mathfrak{C}$ on M, there exists a unique connection compatible with this structure, and this connection can be described in the following way

$$\nabla = \nabla^* + \varphi \otimes \mathrm{Id} + \mathrm{Id} \otimes \varphi - g \otimes \varphi^\sharp,$$

where g is a (pseudo-)Riemannian metric belonging to the conformal class, ∇^* is the Levi-Civita connection of g, $\varphi = F(g)/2$ and $\varphi^\sharp$ is the vector field related to the 1-form φ by $g(\cdot, \varphi^\sharp) = \varphi(\cdot)$.

Given a pseudo-Riemannian metric g, an affine connection ∇ and a 1-form φ satisfying the condition

$$\nabla g + 2\varphi \otimes g = 0 \tag{18}$$

on a manifold M, there is a Weyl structure on M for which ∇ is compatible. Namely it is sufficient to suppose $\mathfrak{C} = [g]$ ($\mathfrak{C}$ is the equivalence class of pseudo-Riemannian metrics conformal to g) and define $F\colon \mathfrak{C} \longrightarrow \Lambda^1(M)$ by $F(e^\lambda g) = 2\varphi - d\lambda$.

To be consistent with a certain geometrical tradition, an affine connection ∇ is called a Weyl connection for a pseudo-Riemannian metric g if there exists a 1-form φ such that the relation (18) is fulfilled. Of course, then ∇ is related to the Levi-Civita connection ∇^* of g by

$$\nabla_X Y = \nabla^*_X Y + \varphi(X)Y + \varphi(Y)X - g(X,Y)\varphi^\sharp,$$

so that we have (14) with $E = -\varphi^\sharp$.

B. Projectively related connections ([2, 10, 18], cf. also [16]). Let M be a differentiable manifold endowed with an affine connection ∇. A curve γ in M is called a ∇-pregeodesic (or a path with respect to ∇) if $\nabla_t\dot\gamma(t) = \sigma(t)\dot\gamma(t)$ for a function σ of the parameter t. Geometrically, this condition means that the tangent line field is parallel along γ. A ∇-pregeodesic γ can always be reparametrized so that $\nabla_s\dot\gamma(s) = 0$ with respect to the new parameter s. Two affine connections ∇ and ∇^* on M have the same paths if and only if there is a 1-form φ such that

$$\nabla_X Y = \nabla^*_X Y + \varphi(X)Y + \varphi(Y)X.$$

Clearly, if ∇^* is taken to be the Levi-Civita connection of a pseudo-Riemannian metric g on M, then we get (14) with $E = 0$.

4. Main result

THEOREM 4
Let $\widetilde{M}$ be a recurrent affine manifold with $\dim \widetilde{M} \geqslant 5$. *Let M be a nondegenerate umbilical affine hypersurface in $\widetilde{M}$, whose shape operator A does not vanish at every point of M. Moreover, assume that the induced connection ∇ is related to the Levi-Civita connection ∇^* of h by the formula*

$$\nabla_X Y = \nabla^*_X Y + \varphi(X)Y + \varphi(Y)X + h(X,Y)E, \tag{19}$$

where φ is a 1-form and E a vector field on M. Then the induced affine metric h is conformally flat.

Proof. Note that (19) is just of the form (14) with $g = h$, so we can apply Proposition 3. Using (13) and (15) with $g = h$, we conclude the following

$$\begin{aligned}\rho h(R^*(X,Y)Z,W) = \omega_0(X,Y)h(Z,W) & \\ + \alpha(h(Y,Z)h(X,W) - h(X,Z)h(Y,W)) & \\ + h(Y,Z)\omega_1(X,W) - h(X,Z)\omega_1(Y,W) & \\ + \omega_2(Y,Z)h(X,W) - \omega_2(X,Z)h(Y,W), &\end{aligned} \tag{20}$$

where α is the scalar function and ω_i's are the (0,2)-tensor fields defined by

$$\begin{aligned}\alpha &= \rho^2 - \rho\varphi(E),\\ \omega_0(X,Y) &= 2\rho(d\tau - d\varphi)(X,Y),\\ \omega_1(X,Y) &= -h\big(\rho h(X,E)E + \rho\nabla^*_X E + (\widetilde{R}(\xi,X)\xi)^\top, Y\big),\\ \omega_2(X,Y) &= \rho(\nabla^*_X\varphi)(Y) - \rho\varphi(X)\varphi(Y) + (\nabla_Y(\rho\tau - d\rho))(X)\\ &\quad - (\tau+\psi)(Y)(\rho\tau - d\rho)(X).\end{aligned}$$

The antisymmetrization of (20) with respect to Z and W gives

$$\begin{aligned}\rho h(R^*(X,Y)Z,W) = \alpha(h(Y,Z)h(X,W) - h(X,Z)h(Y,W)) & \\ + h(Y,Z)\omega(X,W) - h(X,Z)\omega(Y,W) & \\ + \omega(Y,Z)h(X,W) - \omega(X,Z)h(Y,W), &\end{aligned} \tag{21}$$

where

$$\omega = \frac{1}{2}(\omega_1 + \omega_2).$$

From (21), for the Ricci tensor S^* and the scalar curvature r^* of ∇^*, we find

$$\begin{aligned}\rho S^*(Y,Z) &= (n-2)\omega(Y,Z) + ((n-1)\alpha + \mathrm{Tr}_h(\omega))h(Y,Z),\\ \rho r^* &= 2(n-1)\,\mathrm{Tr}_h(\omega) + n(n-1)\alpha,\end{aligned}$$

where $\mathrm{Tr}_h(\omega)$ indicates the trace of the tensor ω with respect to the metric h. Next, from the last two equalities, one gets

$$\omega(Y,Z) = \frac{1}{n-2}\rho S^*(Y,Z) - \frac{1}{2}\left(\frac{1}{(n-1)(n-2)}\rho r^* + \alpha\right)h(Y,Z).$$

This applied to (21), gives

$$\rho\Big(h(R^*(X,Y)Z,W) - \frac{1}{n-2}\big(S^*(Y,Z)h(X,W) \\ -S^*(X,Z)h(Y,W) + h(Y,Z)S^*(X,W) - h(X,Z)S^*(Y,W)\big) \\ +\frac{r^*}{(n-1)(n-2)}\big(h(Y,Z)h(X,W) - h(X,Z)h(Y,W)\big)\Big) = 0,$$

that is, $\rho C^* = 0$, where C^* is the Weyl conformal curvature tensor of the metric h. This implies the assertion since $n = \dim M \geqslant 4$ and ρ is non-zero everywhere on M.

5. The case of pseudo-Riemannian hypersurfaces

Let $\widetilde{M}$ be a connected differentiable manifold, which is endowed with a pseudo-Riemannian metric $\widetilde{g}$. Denote by $\widetilde{\nabla}$ the Levi-Civita connection of the metric $\widetilde{g}$. Let us assume that M is a pseudo-Riemannian hypersurface of $\widetilde{M}$, that is, M is a submanifold of codimension 1 in $\widetilde{M}$, on which a pseudo-Riemannian metric g is induced by $g(X,Y) = \widetilde{g}(X,Y)$ for any vector fields X, Y on M. Then the induced connection ∇ on M is just the Levi-Civita connection of g.

As it follows from [12, Theorem and Corollary 3], if $\dim\widetilde{M} \geqslant 5$, $(\widetilde{M},\widetilde{g})$ is of recurrent curvature (more generally, of recurrent Weyl conformal curvature) and M is totally umbilical and not-totally geodesic ($g = \rho h$, $\rho \neq 0$, h being the second fundamental form), then (M,g) must be conformally flat. It is obvious that in this case, the second fundamental form h must be conformally flat too (h becomes the affine metric when we treat the pseudo-Riemannian submanifold as the affine hypersurface).

Thus, we claim that our Theorem 4 is an extension of the above result to the case of umbilical affine hypersurfaces.

Another theorems about totally umbilical hypersurfaces in pseudo-Riemannian manifolds of recurrent curvature are presented in [3, 7, 17], and of Riemannian or pseudo-Riemannian (locally) symmetric spaces in [1, 13] and in many others papers.

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Sur la monotonie en moyenne des suites

A Monsieur le Professeur Andrzej Zajtz
à l'oecasion de son 70$^{\text{ème}}$ anniversaire

Résumé. On donne la généralisation de la notion de suites monotones en moyenne considerée par F. Leja dans [3]. On démontre que les suites monotones en moyenne généralisées sont convergentes.

1. Introduction

F. Leja dans sa note [3] a étudié la convergence des suites monotones en moyenne au sens arithmétique, géométrique et harmonique. Dans la présente note, on généralise les résultats de [3] en considérant les moyennes quasi-arithmétiques et quasi-arithmétiques pondérées. F. Leja dans [3] a posé la définition suivante :

Définition 1
On dit qu'une suite $\{a_n\}$ est décroissante en moyenne

1° au sens arithmétique si

$$a_{n+2} \leq \frac{1}{2}(a_n + a_{n+1}), \qquad n \in \mathbb{N}_1 = \{1, 2, \ldots\},$$

2° au sens géométrique si $a_n \geq 0$ et

$$a_{n+2} \leq \sqrt{a_n a_{n+1}}, \qquad n \in \mathbb{N}_1,$$

3° au sens harmonique si $a_n > 0$ et

$$a_{n+2} \leq \frac{2a_n a_{n+1}}{a_n + a_{n+1}}, \qquad n \in \mathbb{N}_1.$$

La définition de la croissance en moyenne est analogue.

Puisque $\frac{x+y}{2} \geq \sqrt{xy} \geq \frac{2xy}{x+y}$ quels que soient $x, y > 0$ donc toute suite décroissante en moyenne au sens harmonique est décroissante en moyenne au sens géométrique et toute suite décroissante au sens géometrique est décroissan-

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te au sens arithmétique. Les réciproques sont fausses. Par exemple, la suite donnée par

$$a_1 = 1, \quad a_2 = 2, \quad a_{n+2} = \frac{a_n + a_{n+1}}{2}$$

décroît en moyenne au sens arithmétique et elle ne décroît pas en moyenne au sens géométrique car $a_{n+1} \geq \sqrt{a_n a_{n+1}}$ pour tout $n \in \mathbb{N}_1$ et $a_3 = 1,5 > \sqrt{a_1 a_2} = \sqrt{2}$.

Dans [3] on trouve le théorème.

Théorème 1
Toute suite monotone en moyenne tend vers une limite finie ou infinie.

2. Suites monotones en moyenne quasi-arithmétique

Soit I un intervalle non dégénéré de $\mathbb{R}$ (borné ou non borné, fermé, ouvert ou semi-ouvert, mais non réduit à un point). Soit $f\colon I \to \mathbb{R}$ une fonction strictement monotone et continue. J. Aczél et J. Dhombres dans [1] ont donné la définition :

Définition 2
On appelle moyenne quasi-arithmétique (associée à f) de deux nombres x et y de I l'expression

$$M(x, y) = f^{-1}\left(\frac{f(x) + f(y)}{2}\right).$$

Si

$f(x) = x,\ I \subset \mathbb{R}$ alors $M(x, y) = \frac{x+y}{2}$;
$f(x) = \ln x,\ I \subset (0, +\infty)$ donc $M(x, y) = \sqrt{xy}$;
$f(x) = \frac{1}{x},\ 0 \notin I \subset \mathbb{R}$ donc $M(x, y) = \frac{2xy}{x+y}$;
$f(x) = e^{ax},\ a \neq 0,\ I \subset \mathbb{R}$ donc $M(x, y) = \ln \frac{e^{ax}+e^{ay}}{2}$.

La dérnière moyenne est nommée moyenne exponentielle.

En appliquant la moyenne quasi-arithmétique on propose la définition suivante :

Définition 3
On dit qu'une suite $\{a_n\}$ est décroissante en moyenne au sens quasi-arthmétique (associée à f) si $a_n \in I$ et

$$a_{n+2} \leq M(a_n, a_{n+1})$$

quel que soit $n \in \mathbb{N}_1$.

La définition de la croissance en moyenne au sens quasi-arithmétique est analogue.

On va démontrer le théorème.

Théorème 2
Soit $f\colon I \to \mathbb{R}$ une fonction strictement monotone et continue et telle que f^{-1} est convexe (concave) dans I. Toute suite décroissante (croissante) en moyenne au sens quasi-arithmétique est décroissante (croissante) en moyenne au sens arithmétique.

Démonstration. Supposons f^{-1} convexe dans I. On a

$$M(x,y) = f^{-1}\left(\frac{f(x)+f(y)}{2}\right) \leq \frac{f^{-1}(f(x)) + f^{-1}(f(y))}{2} = \frac{x+y}{2}$$

quels que soient $x, y \in I$. D'où et de l'inégalité

$$a_{n+2} \leq M(a_n, a_{n+1}), \qquad n \in \mathbb{N}_1,$$

on obtient

$$a_{n+2} \leq \frac{a_n + a_{n+1}}{2}, \qquad n \in \mathbb{N}_1.$$

Si f^{-1} est concave dans I le raisonnement est analogue.

De même on obtient :

Théorème 3
Soit $f\colon I \to \mathbb{R}$ une fonction strictement monotone et continue et telle que f^{-1} est convexe (concave) dans I. Toute suite croissante (décroissante) en moyenne au sens arithmétique est croissante (décroissante) en moyenne au sens quasi-arithmétique.

L'exemple donné avant montre qu'en général les réciproques sont fausses. Notons que si la fonction f vérifie l'égalité de Jensen :

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}; \qquad x, y \in I$$

on a l'équivalence. Mais ce cas est trivial car l'égalité de Jensen et la monotonie stricte de f implique que $f(x) = ax + b$ avec $a, b \in \mathbb{R}$, $a \neq 0$ et la moyenne quasi-arithmétique associée à f nous donne la moyenne arithmétique.

Passons maintenant à la convergence des suites monotones en moyenne au sens quasi-arithmétique.

Théorème 4
Soit $f\colon I \to \mathbb{R}$ une fonction strictement monotone et continue. Toute suite monotone en moyenne au sens quasi-arithmétique tend vers une limite (finie ou infinie).

Démonstration. Supposons f croissante dans I et soit $\{a_n\}$ une suite décroissante en moyenne au sens quasi-arithmétique et considérons la suite $\{b_n\}$ définie par $b_n = f(a_n)$, $n \in \mathbb{N}_1$. On voit que $a_n = f^{-1}(b_n)$ et

$$a_{n+2} = f^{-1}(b_{n+2}) \leq f^{-1}\left(\frac{f(a_n) + f(a_{n+1})}{2}\right) = f^{-1}\left(\frac{b_n + b_{n+1}}{2}\right), \quad n \in \mathbb{N}_1.$$

La croissance de f implique que $b_{n+2} \leq \frac{b_n+b_{n+1}}{2}$ pour $n \in \mathbb{N}_1$. D'où et du théorème 1 de Leja on obtient la convergence de la suite $\{b_n\}$. Puisque f est continue dans I donc la suite $\{a_n\}$ est convergente. La démonstration dans les autres cas est analogue.

Remarque 1
Puisque les notions de croissance de la suite en moyenne considérées dans cette note sont analogues à celles de décroissance nous ne donnons dans la suite que des définitions de la décroissance.

3. Suites monotones en moyenne par rapport à la somme des indices.

Dans [3] on trouve :

Définition 4
Soit p un nombre entier, $p \geq 2$. On dit qu'une suite $\{a_n\}$ est décroissante en moyenne (au sens artithmétique) par rapport à la somme de p indices si

$$a_{|\mu|} \leq \frac{1}{p}(a_{\mu_1} + a_{\mu_2} + \cdots + a_{\mu_p}),\ \mu = (\mu_1, \mu_2, \ldots, \mu_p) \text{ et } |\mu| = \mu_1 + \mu_2 + \cdots + \mu_p$$

quels que soient des nombres entiers positifs $\mu_1, \mu_2, \ldots, \mu_p$.

Théorème 5
Toute suite monotone au sens de la définition 4 est convergente (vers une limite finie ou infinie).

On peut généraliser la notion de suites monotones donnée par la définition 4 comme le suit.

Définition 5
Soit $f\colon I \to \mathbb{R}$ une fonction strictement monotone est continue. Soit p un nombre entier x, $p \geq 2$. On dit qu'une suite $\{a_n\}$ est décroissante (au sens quasi-arithmétique) par rapport à la somme de p indices si $a_n \in I$ et

$$a_{|\mu|} \leq f^{-1}\left(\frac{f(a_{\mu_1}) + f(a_{\mu_2}) + \cdots + f(a_{\mu_p})}{p}\right)$$

quels que soient des nombres positifs entiers $\mu_1, \mu_2, \ldots, \mu_p$.

En appliquant le raisonnement fait dans la démonstration du théorème 4 on obtient le théorème suivant :

Théorème 6
Toute suite monotone en moyenne au sens de la définition 5 est convergente.

4. Suites monotones en moyenne pondérée.

F. Leja en [3] a consideré la monotonie en moyenne pondérée des suites à savoir :

Définition 6
On dit qu'une suite $\{a_n\}$ est décroissante en moyenne avec le poids θ, $0 < \theta < 1$, si

$$a_{n+2} \leq \theta a_n + (1-\theta)a_{n+1}, \qquad n \in \mathbb{N}_1. \tag{1}$$

On y trouve le théorème :

Théorème 7
Toute suite monotone en moyenne au sens de la définition 6 tend vers une limite.

On va généraliser la monotonie en moyenne avec le poids en deux directions. D'abord nous donnerons la définition de la moyenne quasi-arithmétique pondérée ([2]).

Définition 7
Soit $f\colon I \to \mathbb{R}$ une fonction strictement monotone et continue. Soit $\theta \in (0,1)$. Posons

$$M_\theta(x,y) = f^{-1}(\theta f(x) + (1-\theta)f(y))$$

pour x et y de I. L'expression $M_\theta(x,y)$ est dite moyenne quasi-arithmétique pondérée.

Dans la suite admettons la définition suivante :

Définition 8
Soit $\theta \in (0,1)$. On dit qu'une suite $\{a_n\}$ est décroissante en moyenne au sens quasi-arithmétique pondérée si $a_n \in I$ et

$$a_{n+2} \leq M_\theta(a_n, a_{n+1})$$

pour $n \in \mathbb{N}_1$.

Dans la même façon qu'avant on a

Théorème 8
Toute suite monotone en moyenne au sens de la définition 8 est convergente.

Il est possible de généraliser la définition 6 et le théorème 7 en considérant la moyenne quasi-arithmétique pondérée par rapport à p indices. Posons la définition suivante :

Définition 9
Soit p un nombre entier, $p \geq 2$ et soient $\theta_1, \theta_2, \ldots, \theta_p$ des nombres de l'intervalle $(0, 1)$ tels que $\theta_1 + \theta_2 + \cdots + \theta_p = 1$. On dit qu'une suite $\{a_n\}$ est décroissante en moyenne (au sens arithmétique pondérée) par rapport à p indices si

$$a_{|\mu|} \leq \theta_1 a_{\mu_1} + \theta_2 a_{\mu_2} + \cdots + \theta_p a_{\mu_p} \tag{2}$$

quels que soient des nombres entiers positifs $\mu_1, \mu_2, \ldots, \mu_p$,

On va démontrer le théorème.

Théorème 9
Toute suite monotone en moyenne au sens de la définition 9 est convergente.

Démonstration. Supposons $\{a_n\}$ décroissante en moyenne au sens de la définition 9. On remarque que $\{a_n\}$ est bornée supérieurement. En effet, par récurrence on démontre que

$$a_n \leq A = \sup\{a_1, a_2, \ldots, a_p\}, \qquad n \in \mathbb{N}_1.$$

Posons

$$\alpha = \liminf_{n \to +\infty} a_n \leq \limsup_{n \to +\infty} a_n = \beta$$

et supposons que α soit fini. Soit $\varepsilon > 0$. Il existe $m \in \mathbb{N}_1$ tel que $a_m < \alpha + \varepsilon$, Par (2) on a

$$\begin{aligned} a_{mp} &= a_{\underbrace{m + m + \cdots + m}_{p \text{ fois}}} \\ &\leq \theta_1 a_m + \theta_2 a_m + \cdots + \theta_p a_m \\ &= a_m. \end{aligned}$$

Et donc $a_{mp} < a + \varepsilon$.

Pour $n \in \mathbb{N}_1$ et $n > mp$ on a $n = km(p-1) + r$ avec $k, r \in \mathbb{N}_1$ et $1 \leq r \leq m(p-1)$. On voit que

$$\begin{aligned} a_{m(p-1)+r} &= a_{\underbrace{m + \cdots + m}_{(p-1) \text{ fois}} + r} \\ &\leq (1 - \theta_p) a_m + \theta_p a_r, \end{aligned}$$

$$\begin{aligned} a_{2m(p-1)+r} &= a_{\underbrace{m+\cdots+m}_{(p-1)\text{ fois}}+m(p-1)+r} \\ &\le (1-\theta_p)a_m + \theta_p a_{m(p-1)+r} \\ &\le (1-\theta_p)a_m + \theta_p(1-\theta_p)a_m + \theta_p^2 a_r \\ &= (1-\theta_p^2)a_m + \theta_p^2 a_r, \end{aligned}$$

et en général

$$a_{km(p-1)+r} \le (1-\theta_p^k)a_m + \theta_p^k a_r, \qquad k \in \mathbb{N}_1.$$

Il en résulte que

$$\limsup_{n\to+\infty} a_n = \beta \le a_m < \alpha + \varepsilon.$$

Puisque ε a été choisi arbitrairement donc $\alpha = \beta$, ce qui prouve que $\{a_n\}$ converge vers une limite finie. Dans le cas $\alpha = -\infty$ le raisonnement analogue prouve que $\beta = -\infty$.

Maintenant posons la définition suivante :

Définition 10
On dit qu'une suite $\{a_n\}$ est décroissante en moyenne pondérée (au sens quasi-arithmétique) par rapport à p indices si $a_n \in I$ et

$$a_{|\mu|} \le f^{-1}(\theta_1 f(a_{\mu_1}) + \theta_2 f(a_{\mu_2}) + \cdots + \theta_p f(a_{\mu_p})) \tag{3}$$

quels que soient des nombres entiers positifs $\mu_1, \mu_2, \ldots, \mu_p$.

En posant $b_n = f(a_n)$ et en appliquant le théorème 9 par le même raisonnement comme dans la démonstration du théorème 4 on a le théorème suivant :

Théorème 10
Toute suite monotone en moyenne au sens de la définition 10 tend vers une limite.

5. Certaine modification de la monotonie en moyenne pondérée des suites

F. Leja a considéré dans [3] encore une autre sorte de la monotonie en moyenne des suites en remplaçant la condition (1) par la suivante :

$$a_{\mu+\nu} \le \theta^{\mu}_{\mu+\nu} a_\mu + \theta^{\nu}_{\mu+\nu} a_\nu, \qquad \mu, \nu \in \mathbb{N}_1 \tag{4}$$

où $\theta^{\mu}_{\mu+\nu}$ et $\theta^{\nu}_{\mu+\nu}$ sont des nombres positifs quelconques satisfaisant à l'égalité $\theta^{\mu}_{\mu+\nu} + \theta^{\nu}_{\mu+\nu} = 1$.

On y trouve le théorème suivant.

Théorème 11

Toute suite monotone au sens de la condition (4) *tend vers une limite pourvu que le produit*

$$\theta^{\nu}_{\mu+\nu} \cdot \theta^{\mu+\nu}_{2\mu+\nu} \cdot \ldots \cdot \theta^{(k-1)\mu+\nu}_{k\mu+\nu}$$

tende vers zéro quels que soient μ *et* ν *lorsque* $k \to +\infty$.

Soit p un nombre entier, $p \geq 2$. Remplaçons maintenant la condition (2) par la suivante :

$$a_{|\mu|} \leq \theta^{\mu_1}_{|\mu|} a_{\mu_1} + \cdots + \theta^{\mu_p}_{|\mu|} a_{\mu_p}, \qquad \mu_1, \mu_2, \ldots, \mu_p \in \mathbb{N}_1, \tag{5}$$

où $\theta^{\mu_1}_{|\mu|}, \ldots, \theta^{\mu_p}_{|\mu|}$ sont des nombres positifs quelconques satisfaisant à l'égalité

$$\theta^{\mu_1}_{|\mu|} + \theta^{\mu_2}_{|\mu|} + \cdots + \theta^{\mu_p}_{|\mu|} = 1. \tag{6}$$

Posons

$$\lambda_k = \theta^{\mu_p}_{|\nu|+\mu_p} \cdot \theta^{|\nu|+\mu_p}_{2|\nu|+\mu_p} \cdot \ldots \cdot \theta^{(k-1)|\nu|+\mu_p}_{k|\nu|+\mu_p}, \tag{7}$$

où $|\nu| = \mu_1 + \mu_2 + \cdots + \mu_{p-1}$.

On va démontrer le théorème suivant :

Théorème 12

Toute suite monotone en moyenne au sens de la condition (5) *(avec* (6)*) est convergente pourvu que* $\lim_{k\to+\infty} \lambda_k = 0$.

Démonstration. Supposons que la suite $\{a_n\}$ vérifie la condition (5). Par récurrence on a

$$a_n \leq A \leq \sup\{a_1, a_2, \ldots, a_p\} \qquad \text{pour } n \in \mathbb{N}_1.$$

Alors $\{a_n\}$ est bornée supérieurement. Posons

$$\alpha = \liminf_{n\to+\infty} a_n \leq \limsup_{n\to+\infty} a_n = \beta$$

et supposons α fini. Soit $\varepsilon > 0$. Il existe $m \in \mathbb{N}_1$ tel que $a_m < \alpha + \varepsilon$. Pour $n \in \mathbb{N}_1$ et $n > mp$ on a $n = km(p-1) + r$ avec $k, r \in \mathbb{N}_1$ et $1 \leq r \leq m(p-1)$. D'après (5), (6) et (7) on obtient

$$\begin{aligned}
a_{m(p-1)+r} &= a_{\underbrace{m + \cdots + m}_{(p-1)\text{ fois}} + r} \\
&\leq (1 - \theta^{r}_{m(p-1)+r}) a_m + \theta^{r}_{m(p-1)+r} a_r \\
&= (1 - \lambda_1) a_m + \lambda_1 a_r, \\
a_{2m(p-1)+r} &= a_{\underbrace{m + \cdots + m}_{(p-1)\text{ fois}} + m(p-1) + r} \\
&\leq \left(1 - \theta^{m(p-1)+r}_{2m(p-1)+r}\right) a_m + \theta^{m(p-1)+r}_{2m(p-1)+r} a_{m(p-1)+r} \\
&\leq (1 - \lambda_2) a_m + \lambda_2 a_r.
\end{aligned}$$

et en général

$$a_{km(p-1)+r} \leq (1-\lambda_k)a_m + \lambda_k a_r, \qquad k \in \mathbb{N}_1.$$

Faisons tendre k vers l'infinie. Par supposition $\lambda_k \to 0$ et donc

$$\beta = \limsup_{n\to+\infty} a_n = \limsup_{k\to+\infty} a_{km(p-1)+r} \leq a_m < \alpha + \varepsilon.$$

Puisque ε a été choisi arbitrairement donc $\alpha = \beta$. Si $\alpha = -\infty$, le raisonnement est similaire.

En appliquant la méthode de la démonstration du théorème 4 on généralise le résultat du théorème 12 comme suit :

Théorème 13
Soit $f\colon I \to \mathbb{R}$ une fonction strictement monotone et continue. Soit p un nombre entier ≥ 2. Toute suite $\{a_n\}$ satisfaisant à la condition $a_n \in I$, $n \in N_1$ et

$$a_{|\mu|} \leq f^{-1}(\theta_{|\mu|}^{\mu_1} f(a_{\mu_1}) + \cdots + \theta_{|\mu|}^{\mu_p} f(a_{\mu_p}))$$

tend vers une limite où $\theta_{|\mu|}^{\mu_1}, \ldots, \theta_{|\mu|}^{\mu_p}$ sont des nombres positifs quelconques vérifiant la condition (6) *ainsi que la condition* $\lim_{k\to+\infty} \lambda_k = 0$*, où λ_k est donnée par* (7)*.*

6. Sur le problème de F. Leja

F. Leja dans [3] a posé le problème suivant :
Soient p et q deux nombres entiers positifs quelconques fixes et $\{a_n\}$ suite remplissant ou bien la condition

$$\frac{1}{p}(a_{n+1} + \cdots + a_{n+p}) \leq \frac{1}{q}(a_{n+p+1} + \cdots + a_{n+p+q}) \tag{8}$$

pour $n \in \mathbb{N}_1$ ou bien la condition

$$\frac{1}{p}(a_{\mu_1} + \cdots + a_{\mu_p}) \leq \frac{1}{q}(a_{\nu_1} + a_{\nu_2} + \cdots + a_{\nu_q}) \tag{9}$$

où $\mu_1, \mu_2, \ldots, \mu_p$ et $\nu_1, \nu_2, \ldots, \nu_q$ sont des indices quelconques de $\mathbb{N}_1$ mais tels qu'on ait

$$\mu_1 + \mu_2 + \cdots + \mu_p = \nu_1 + \nu_2 + \cdots + \nu_p.$$

Une telle suite tend vers une limite ?

F. Leja a remarqué ensuite que la réponse est positive dans le cas où l'un des nombres p et q est équal à 1 et l'autre est supérieur à 1. Il est facile de trouver un contrexemple qui montre que cette affirmation est fausse. En effet, en posant $a_n = (-2)^n$ on voit que cette suite remplit la condition (8) pour $p = 1$ et $q = 2$ si bien qu'elle ne converge pas. De plus, elle n'est ni bornée supérieurement ni bornée inférieurement. Quand même, le théorème 5 nous donne la réponse affirmative si $p > 1$ et $q = 1$.

Notons encore que la réponse est aussi négative dans le cas où p et q sont des nombres pairs et la suite remplit la condition (8). Il suffit de considérer la suite définie par $a_n = (-1)^n$.

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On a class of Riemannian manifolds with harmonic Weyl conformal curvature tensor

Dedicated to Professor Dr. Andrzej Zajtz on his seventieth birthday

Abstract. The paper deals with the local structure of those n-dimensional ($n \geq 5$) Riemannian manifolds of harmonic conformal curvature (M, g) which are not conformally flat and admit a non-homothetic conformal change of metric $g \mapsto \bar{g}$ such that $(M, \bar{g})$ is locally symmetric.

1. Introduction

An n-dimensional ($n \geq 4$) pseudo-Riemannian manifold (M, g) is called conformally symmetric [2] if its Weyl conformal curvature tensor

$$\begin{aligned} C_{hijk} = R_{hijk} &- \frac{1}{n-2}(g_{ij}S_{hk} - g_{ik}S_{hj} + g_{hk}S_{ij} - g_{hj}S_{ik}) \\ &+ \frac{K}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{hj}g_{ik}) \end{aligned} \tag{1}$$

is parallel, i.e., $C_{hijk,l} = 0$. Herewith and in the sequel we denote the curvature tensor, Ricci tensor and scalar curvature by R, S and K respectively, while the comma stands for covariant differentiation with respect to the Levi-Civita connection.

Clearly, the class of conformally symmetric manifolds contains all locally symmetric ones ($n \geq 4$) as well as all conformally flat manifolds of dimension $n \geq 4$. In the Riemannian case there are no more examples ([4], Theorem 2).

But in general, for each $n \geq 4$, there exist ([5]) conformally symmetric manifolds with metrics of indices from the range $\{1, 2, \dots, n-1\}$ which are neither conformally flat nor locally symmetric.

It is not hard to check (see (5)) that for every conformally symmetric manifold the condition

$$S_{ij,l} - S_{il,j} = \frac{1}{2(n-1)}(K_{,l}\, g_{ij} - K_{,j}\, g_{il}) \tag{2}$$

holds.

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An n-dimensional ($n \geq 2$) pseudo-Riemannian manifold is said to be nearly conformally flat [3] (or nearly conformally symmetric [10]) if its Ricci tensor satisfies condition (2). Any conformally symmetric manifold is therefore nearly conformally flat. Moreover, condition (2) shows that any n-dimensional ($n \geq 2$) manifold of harmonic curvature ($S_{ij,k} = S_{ik,j}$) is also nearly conformally flat.

The existence of essentially nearly conformally flat metrics, i.e. nearly conformally flat metrics which are neither conformally flat nor of harmonic curvature, can be stated as follows:

Example 1 ([10], Example 1)
Let $M = \mathbb{R}^{n-1} \times \mathbb{R}^1_+$, ($n \geq 5$) be endowed with the metric g given by

$$g_{\lambda\mu} dx^\lambda dx^\mu = ((n-1)x^n)^{\frac{2}{n-1}} f_{ij} dx^i dx^j + (dx^n)^2,$$

where $\lambda, \mu = 1, 2, \ldots, n$, $i, j = 1, 2, \ldots, n-1$, and f is an arbitrary non-flat Ricci-flat metric on $\mathbb{R}^{n-1}$ (which evidently exists since $n \geq 5$). Then (M, g) is essentially nearly conformally flat.

From Theorem 7 of [5] it follows that essentially nearly conformally flat manifolds cannot be conformally symmetric ones. Nearly conformally flat manifolds ($n \geq 4$) with positive definite metrics are also said to have harmonic Weyl tensor (i.e., $\delta C = 0$, see [1], p. 435) or to be of harmonic conformal curvature. Throughout this paper we shall use the latter name.

Let M be a manifold of of class C^∞ endowed with a (not necessarily positive definite) metric g. If $\bar{g}$ is another metric on M and there exists a smooth function p on M such that $\bar{g} = (\exp 2p)g$, then g and $\bar{g}$ are said to be conformally related or conformal to each other, and such a change of metric $g \mapsto \bar{g}$ is called a conformal change. If $p = constant$, then the conformal change of the metric is called a homothety.

Nickerson initiated [8] investigations of Riemannian manifolds (M, g) admitting a conformal change of metric $g \mapsto \bar{g}$ such that $(M, \bar{g})$ is locally symmetric.

The present paper deals with similar problems. It contains at generic points (Theorem 2) a full description of the local structure of those n-dimensional ($n \geq 5$) (Riemannian) manifolds of harmonic conformal curvature (M, g) which are not conformally flat and admit a non-homothetic conformal change of metric $g \mapsto \bar{g}$ such that $(M, \bar{g})$ is locally symmetric. Theorem 2 bases on the following result:

Theorem 1
Let (M, g), $\dim M \geq 4$, be of harmonic conformal curvature. If (M, g) is not conformally flat and it admits a non-homothetic conformal change of metric $g \mapsto \bar{g} = (\exp 2p)g$ such that $(M, \bar{g})$ is conformally symmetric, then $\dim M \geq 5$ and $(M, \bar{g})$ is a locally reducible locally symmetric manifold.

Throughout this paper, all manifolds under consideration are assumed to be connected and of class C^∞. Their metrics, unless stated otherwise, are assumed to be positive definite.

2. Preliminaries

In the sequel we need the following results:

Lemma 1
The Weyl conformal curvature tensor satisfies the well-known equations:

$$C_{hijl} = -C_{ihjl} = -C_{hilj} = C_{jlhi}\,, \tag{3}$$

$$C_{hijl} + C_{hjli} + C_{hlij} = 0, \qquad C^r{}_{ijr} = C^r{}_{irl} = C^r{}_{rjl} = 0, \tag{4}$$

$$C^r{}_{ijl,r} = \frac{n-3}{n-2}(S_{ij,l} - S_{il,j} - \frac{1}{2(n-1)}(K_{,l}g_{ij} - K_{,j}g_{il})). \tag{5}$$

Lemma 2 ([6], p. 89-90)
Let $\bar{g}_{ij} = (\exp 2p)g_{ij}$. Then we have:

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j p_k + \delta^i_k p_j - p^i g_{jk}\,, \tag{6}$$

$$\bar{C}^h{}_{ijl} = C^h{}_{ijl}\,, \tag{7}$$

where Γ denotes Christoffel symbols, $p_i = p_{,i}$ and $p^h = g^{hr}p_r$.

Lemma 3
Let $\bar{g}_{ij} = (\exp 2p)g_{ij}$. Then we have:

$$\bar{C}^r{}_{ijk;r} = C^r{}_{ijk,r} + (n-3)p_r C^r{}_{ijk}\,, \tag{8}$$

where the semicolon denotes covariant differentiation with respect to $\bar{g}$.

Proof. Differentiating (7) covariantly, using (6) and Lemma 1, we obtain

$$\begin{aligned}\bar{C}^h{}_{ijk;l} = C^h{}_{ijk,l} &+ \delta^h_l p_r C^r{}_{ijk} - 2p_l C^h{}_{ijk} - p^h C_{lijk} - p_i C^h{}_{ljk} - p_j C^h{}_{ilk} \\ &- p_k C^h{}_{ijl} + g_{il}p^r C^h{}_{rjk} + g_{jl}p^r C^h{}_{irk} + g_{kl}p^r C^h{}_{ijr}\,.\end{aligned} \tag{9}$$

Equation (8) follows from (9) and Lemma 1. This completes the proof.

Lemma 4 ([4], Theorem 2)
Let (M, g) be a Riemannian conformally symmetric manifold. If it is not conformally flat, then (M, g) is locally symmetric.

Lemma 5 ([11], Theorem 3)
Let (M, g) be a pseudo-Riemannian conformally symmetric manifold. If it admits a conformal change of metric $g \mapsto \bar{g}$ such that $(M, \bar{g})$ is conformally symmetric, then both (M, g) and $(M, \bar{g})$ are conformally flat or the conformal change of metric is a homothety.

Remark 1
It is known ([12], p. 286) that a Riemannian manifold is locally decomposable if and only if it admits a symmetric parallel tensor field of type $(0, 2)$ which is not a multiple of the metric tensor.

If (M, g) is locally decomposable and $\dim M = n$, then coordinates $(x^1, \ldots, x^{r_1}, x^{r_1+1}, \ldots, x^{r_1+r_2}, \ldots, x^n)$ can be locally chosen so (see [13], p. 414) that its metric takes the form:

$$\begin{bmatrix} g_{i_1 j_1} & & & \\ & g_{i_2 j_2} & & \\ & & \ddots & \\ & & & g_{i_t j_t} \end{bmatrix}, \tag{10}$$

where $i_1, j_1 = 1, \ldots, r_1$, $i_2, j_2 = r_1+1, \ldots, r_1+r_2, \ldots$, $i_t, j_t = 1+\sum_{l=1}^{t-1} r_l, \ldots, n$, and the tensors g_s $(s = 1, \ldots, t)$ given by $g_1 = [g_{i_1 j_1}(x^1, \ldots, x^{r_1})]$, $g_2 = [g_{i_2 j_2}(x^{r_1+1}, \ldots, x^{r_1+r_2})], \ldots$ are irreducible. M can be therefore locally written in the form $M_1 \times \ldots \times M_t$ and its metric is the direct sum of the metrics on M_i's. Obviously, if one or more of the M_i's are 1-dimensional, then, by a renumeration of coordinates, (10) can be modified so that (M_1, g_1) is Euclidean and that all g_k $(k = 2, \ldots, m \leq t)$ are irreducible and no one of the M_k's is 1-dimensional. Moreover, (10) implies

$$[g_{ij}] = \begin{bmatrix} g_{ab} & \\ & g_{AB} \end{bmatrix}, \tag{11}$$

where $a, b = 1, \ldots, r$, $A, B = r+1, \ldots, n$, g_{ab} are functions of $x^1, \ldots, x^r$ only, and g_{AB} depend on $x^{r+1}, \ldots, x^n$ only. Clearly, in a matrix of the form (11) the tensors g_1 and g_2 can be reducible.

Lemma 6
In the metric (11)*, the only components of the Weyl conformal curvature tensor and its covariant derivative which may not vanish are those related to*

$$\begin{aligned} C_{abcd} = R_{abcd} &- \frac{1}{n-2}(g_{bc}S_{ad} - g_{bd}S_{ac} + g_{ad}S_{bc} - g_{ac}S_{bd}) \\ &+ \frac{Q+N}{(n-1)(n-2)}(g_{bc}g_{ad} - g_{ac}g_{bd}), \end{aligned} \tag{12}$$

$$
\begin{aligned}
C_{ABCD} = R_{ABCD} - \frac{1}{n-2}(g_{BC}S_{AD} - g_{BD}S_{AC} + g_{AD}S_{BC} \\
-g_{AC}S_{BD}) + \frac{Q+N}{(n-1)(n-2)}(g_{BC}g_{AD} - g_{AC}g_{BD}),
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
C_{aABc} = -\frac{1}{n-2}\left(\left(S_{ac} - \frac{1}{n-1}Qg_{ac}\right)g_{AB}\right. \\
\left. + \left(S_{AB} - \frac{1}{n-1}Ng_{AB}\right)g_{ac}\right),
\end{aligned}
\tag{14}
$$

$$
C_{ABCD,a} = \frac{1}{(n-1)(n-2)}Q_{,a}(g_{BC}g_{AD} - g_{AC}g_{BD}), \tag{15}
$$

$$
C_{abcd,A} = \frac{1}{(n-1)(n-2)}N_{,A}(g_{bc}g_{ad} - g_{ac}g_{bd}), \tag{16}
$$

$$
C_{aABc,d} = -\frac{1}{n-2}(S_{ac,d} - \frac{1}{n-1}Q_{,d}g_{ac})g_{AB}, \tag{17}
$$

$$
C_{aABc,D} = -\frac{1}{n-2}(S_{AB,D} - \frac{1}{n-1}N_{,D}g_{AB})g_{ac}, \tag{18}
$$

$$
\begin{aligned}
C_{abcd,e} = R_{abcd,e} - \frac{1}{n-2}(g_{bc}S_{ad,e} - g_{bd}S_{ac,e} + g_{ad}S_{bc,e} - g_{ac}S_{bd,e}) \\
+\frac{1}{(n-1)(n-2)}Q_{,e}(g_{bc}g_{ad} - g_{ac}g_{bd}),
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
C_{ABCD,E} = R_{ABCD,E} - \frac{1}{n-2}(g_{BC}S_{AD,E} - g_{BD}S_{AC,E} + g_{AD}S_{BC,E} \\
-g_{AC}S_{BD,E}) + \frac{1}{(n-1)(n-2)}N_{,E}(g_{BC}g_{AD} - g_{AC}g_{BD}),
\end{aligned}
\tag{20}
$$

where $a, b, c, d, e = 1, 2, \ldots, r$, $A, B, C, D, E = r+1, \ldots, n$, *and* Q *and* N *denote the scalar curvatures of the metrics* $[g_{ab}]$ *and* $[g_{AB}]$, *respectively.*

The proof is obvious.

Lemma 7

Let (M, g) *be conformally symmetric with a possibly indefinite metric. If the metric can be locally written in the form* (11), *then* (M, g) *is locally symmetric.*

The proof follows easily from equations (15)-(20).

The following two results seem to be well known:

Lemma 8
Let $M = M_1 \times M_2$ be an n-dimensional $(n \geq 4)$ pseudo-Riemannian manifold, where M_1 and M_2 are of constant sectional curvature, $\dim M_1 = r \geq 1$, $\dim M_2 = s \geq 1$ *and $r + s = n$.*[1] *Denote by Q and N the scalar curvatures of M_1 and M_2, respectively. Then M is conformally flat if and only if the condition*

$$s(s-1)Q + r(r-1)N = 0 \tag{21}$$

holds.

Lemma 9
Let $M = M_1 \times M_2 \times \ldots \times M_t$ be a pseudo-Riemannian manifold, M_i ($\dim M_i = q_i \geq 1$, $i = 1, 2, \ldots, t$) being Einstein manifolds with scalar curvatures Q_i. Denote by Q the scalar curvature of M and let $q = \dim M$. Then M is Einsteinian if and only if

$$\frac{1}{q} Q = \frac{1}{q_i} Q_i \qquad (i = 1, 2, \ldots, t).$$

Lemma 10
Let (M, g) be an n-dimensional $(n \geq 2)$ Riemannian locally symmetric manifold whose curvature tensor satisfies the condition

$$v_r R^r{}_{ijl} = B_l g_{ij} - B_j g_{il} \tag{22}$$

for some covector fields v and B. If (M, g) is locally irreducible and if at least one of the covector fields v or B does not identically vanish, then (M, g) is of constant sectional curvature.

Proof. Obviously, (M, g) is an Einstein manifold with constant scalar curvature and its curvature tensor satisfies the condition

$$R^{hij}{}_k R_{hijl} = \tau g_{kl} \tag{23}$$

where $\tau = constant$. Transvecting (22) with $R^{lji}{}_q$ and making use of (23), we easily obtain

$$\tau v_q = \frac{2}{n} K B_q \,. \tag{24}$$

On the other hand, condition (22) yields

$$\frac{1}{n} K v_l = (n-1) B_l$$

which, together with (24), implies

[1] Throughout this paper, 1-dimensional manifolds are ass med to be of constant sectional curvature.

$$\left(\tau - \frac{2}{n^2(n-1)}K^2\right) v_l = 0. \tag{25}$$

Assume that v_i does not vanish at least at some point of M. Since τ and K are *constants*, (25) gives

$$\tau = \frac{2}{n^2(n-1)}K^2.$$

But, by (23), we have $R^{hihk}R_{hijk} = \parallel R \parallel^2 = n\tau$, which, together with the last result, implies

$$\parallel R \parallel^2 = \frac{2}{n(n-1)}K^2. \tag{26}$$

Now, let T be given by

$$T_{hijl} = R_{hijl} - \frac{K}{n(n-1)}(g_{ij}g_{hl} - g_{hj}g_{il}).$$

Then, in view of (26), we get $\|T\|^2 = 0$. Thus, $T = 0$, which completes the proof in the case $v \neq 0$. If B_i does not identically vanish, then the proof is quite similar.

3. Locally decomposable manifolds

We are now in a position to prove Theorem 1.

Proof of Theorem 1. By (2) and (5), we get

$$C^r{}_{ijl,r} = 0, \tag{27}$$

which, in view of (8) and (7), yields

$$p_r\bar{C}^r{}_{ijl} = 0. \tag{28}$$

But (28), together with

$$\bar{C}^{hij}{}_l\bar{C}_{hijk} = \mu\bar{g}_{lk}\,.$$

which holds for every 4-dimensional manifold [9], implies $\mu p_l = 0$. Hence, $\bar{C} = 0$ at some point. Since $\bar{C}$ is parallel, it vanishes therefore everywhere, a contradiction. Thus $\dim M \geq 5$.

Assume that $(M, \bar{g})$ is locally irreducible. By (7) and Lemma 4, $(M, \bar{g})$ is locally symmetric and, in consequence, it must be Einsteinian. Thus, in view of (28), we have

$$p_r\bar{R}^r{}_{ijl} = \frac{1}{n(n-1)}\bar{K}(p_l\bar{g}_{ij} - p_j\bar{g}_{il}),$$

which, by Lemma 10, shows that $(M, \bar{g})$ is of constant sectional curvature. Consequently, $(M, \bar{g})$ is conformally flat, a contradiction. The last remark completes the proof.

Lemma 11
Let (M, g), $\dim M \geq 5$, be of harmonic conformal curvature admitting a non-homothetic conformal change of metric $g \mapsto \bar{g} = (\exp 2p)g$ such that $(M, \bar{g})$ is conformally symmetric. Assume that $(M, \bar{g})$ is in some coordinate neighbourhood U decomposable into $M_1 \times M_2$, $\dim M_1 = r \geq 1$, $\dim M_2 = s = n - r \geq 1$. (The metric of $(M, \bar{g})$ is therefore in U of the form (11)). If one of the M_i's, say M_1, is either irreducible or Euclidean and there exists a point in U at which the gradient of p does not vanish in the direction of M_1 (i.e. $p_a = p_{,a} \neq 0$ for some $a \in \{1, 2, \ldots, r\}$), then

(i) *M_1 is of constant sectional curvature (with constant scalar curvature) and M_2 is Einsteinian with parallel curvature tensor.*

(ii) *Condition (21) holds, where Q and N denote the scalar curvatures of M_1 and M_2, respectively.*

Proof. Both M_1 and M_2 are locally symmetric since $M_1 \times M_2$ does so (cf. Lemma 7). On the other hand, because of (28), equations (12)-(14) imply

$$p_a \bar{C}^a{}_{BDd} = 0, \quad p_a \bar{C}^a{}_{bcd} = 0, \quad p_A \bar{C}^A{}_{BCD} = 0, \quad p_A \bar{C}^A{}_{bdE} = 0, \tag{29}$$

whence, by (14), $\bar{S}_{ab} = \frac{1}{r} Q \bar{g}_{ab}$, $\bar{\nabla} \bar{C} = 0$ and $p_a \neq 0$, we get

$$\bar{S}_{BD} = \frac{1}{n-1} \left(N - \frac{s-1}{r} Q \right) \bar{g}_{BD} \,. \tag{30}$$

The last result shows that M_2 is Einsteinian and that condition (21) holds. Moreover, (12) yields

$$\bar{C}_{abcd} = \bar{R}_{abcd} - \frac{1}{n-2} \left(\frac{2}{r} Q - \frac{Q+N}{n-1} \right) (\bar{g}_{ad} \bar{g}_{bc} - \bar{g}_{ac} \bar{g}_{bd}),$$

which, because of (29), implies

$$p_a \bar{R}^a{}_{bcd} - \frac{1}{n-2} \left(\frac{2}{r} Q - \frac{Q+N}{n-1} \right) (p_d \bar{g}_{bc} - p_c \bar{g}_{bd}) = 0. \tag{31}$$

But from the last result, we have

$$\frac{1}{r} Q - \frac{r-1}{n-2} \left(\frac{2}{r} Q - \frac{Q+N}{n-1} \right) = 0,$$

which shows that in the case $\dim M > 1$ equation (31) takes the form

$$p_a \bar{R}^a{}_{bcd} = \frac{1}{r(r-1)} Q(p_d \bar{g}_{bc} - p_c \bar{g}_{bd}).$$

Since p_a does not identically vanish, the assertion is therefore an immediate consequence of Lemma 10. This completes the proof.

LEMMA 12
Let (M, g), $\dim M \geq 5$ be of harmonic conformal curvature admitting a conformal change of metric $g \mapsto \bar{g} = (\exp 2p)g$ such that $(M, \bar{g})$ is conformally symmetric. Assume that $(M, \bar{g})$ is in some coordinate neighbourhood U decomposable into $M_1 \times M_2 \times \ldots \times M_t$, where M_1 is Euclidean or irreducible and the others of the $M_j^\prime s$ $(j = 2, \ldots, t)$ are irreducible and no one of them is 1-dimensional (the metric of $(M, \bar{g})$ is therefore in U of the form (10), cf. Remark 1). If $p \neq$ constant on U, then $(M, \bar{g})$ is conformally flat or there is only one of the $M_i^\prime s$ $(i = 1, \ldots, t)$ in the direction of which the gradient of p does not identically vanish (i.e., there exists $s \in \{1, 2, \ldots, t\}$ such that for any $q \neq s$ the condition $p_{i_q} = p_{,i_q} = 0$ holds everywhere on U, where (x^{i_q}) denote coordinates in M_i.)

Proof. Let $t \geq 3$. Suppose that among $M_i^\prime s$ there exist at least two such in the direction of which the gradient of p does not identically vanish on U. Without loss of generality, we may assume (it is enough to change the numeration of coordinates if it is necessary) that the metric is in U of the form (10) and that among (x^{i_1}) and (x^{i_t}) there exist at least two coordinates x^{λ_1} and x^{λ_t} such that both componets $p_{\lambda_1} = p_{,\lambda_t}$ and $p_{\lambda_t} = p_{,\lambda_t}$ do not vanish identically on U. Lemma 11 shows that both M_1 and M_t are of *constant* sectional curvature (with constant scalar curvatures), $N_1 = M_2 \times \ldots \times M_t$ and $N_2 = M_1 \times \ldots \times M_{t-1}$ are Einsteinian and condition (21) holds.

Denote by Q_i the scalar curvature of M_i and let $q_i = \dim M_i$. Then, in view of (21), we have

$$q_1(1-q_1)\sum_{i=2}^{t} Q_i + Q_1 \sum_{i=2}^{t} q_i \left(1 - \sum_{i=2}^{t} q_i\right) = 0, \tag{32}$$

$$Q_t \sum_{i=1}^{t-1} q_i \left(1 - \sum_{i=1}^{t-1} q_i\right) + q_t(1-q_t)\sum_{i=1}^{t-1} Q_i = 0. \tag{33}$$

On the other hand, Lemma 9 implies

$$Q_i = \frac{1}{q_t} Q_t q_i \quad (i = 2, \ldots, t), \qquad Q_i = \frac{1}{q_1} Q_1 q_i \quad (i = 1, \ldots, t-1), \tag{34}$$

which, together with (32) and (33), yields

$$q_1(1-q_1)Q_t + q_tQ_1\left(1-\sum_{i=2}^{t} q_i\right) = 0,$$

$$q_t(1-q_t)Q_1 + q_1Q_t\left(1-\sum_{i=1}^{t-1} q_i\right) = 0.$$

Consequently, we have

$$Q_1\left(\left(1-\sum_{i=2}^{t} q_i\right)\left(1-\sum_{i=1}^{t-1} q_i\right) - (1-q_1)(1-q_t)\right) = 0,$$

whence it follows $Q_1 = Q_t = 0$. Thus, both M_1 and M_t are Euclidean, a contradiction. Assume now that $t = 2$ and that there exist p_b, $b \in \{1, 2, \dots, r\}$ and p_B, $B \in \{r+1, \dots, n\}$, which do not vanish on U (cf. Remark 1). Then, by Lemma 11, both M_1 and M_2 are of constant sectional curvature and condition (21) holds. Hence, in view of Lemma 8, $M_1 \times M_2$ is conformally flat. Since $\bar{C}$ is parallel and it vanishes on U, it does so everywhere. Consequently, $(M, \bar{g})$ is conformally flat. The last remark completes the proof.

4. A local structure result

We are now in a position to prove the following result:

Theorem 2

(i) *Let (M_1, g_1) be of constant sectional curvature K (K = constant), F a positive non-constant function on M_1, and (M_2, g_2) a locally symmetric Einstein manifold whose scalar curvature $N = -s(s-1)K$, where $s = \dim M_2$. If $(M_2,\, g_2)$ is not of constant sectional curvature and $\dim M_1 + \dim M_2 \geq 5$, then $M = M_1 \times M_2$ with the warped product metric $g = F^2g_1 + F^2g_2 = (\exp 2\log F)(g_1 \oplus g_2)$ is of harmonic conformal curvature and it admits a non-homothetic conformal change of metric $g \mapsto \bar{g}$ such that $(M, \bar{g})$ is locally symmetric. Moreover, (M, g) is neither conformally flat nor locally symmetric.*

(ii) *Let (M, g), $\dim M \geq 4$ be of harmonic conformal curvature admitting a non-homothetic conformal change of metric $g \mapsto \bar{g} = (\exp 2p)g$ such that $(M, \bar{g})$ is conformally symmetric. If (M, g) is not conformally flat, then $\dim M \geq 5$ and for each point $x \in M$ satisfying $(\operatorname{grad} p)(x) \neq 0$ coordinates can be chosen in a neighbourhood of x so that the metric of M takes the above stated warped product form with properties described in* (i).

Proof. (i) Obviously, coordinates can be locally chosen so that the metric of M can be written as

$$[g_{ij}] = e^{-2p}\begin{bmatrix} \bar{g}_{ab} & \\ & \bar{g}_{AB} \end{bmatrix}, \tag{35}$$

where $\bar{g}_{ab}(x^1, \ldots, x^r)$ denote the components of g_1, $\bar{g}_{AB}(x^{r+1}, \ldots, x^n)$ the components of g_2, $r = \dim M_1 = n - s$ and $p = -\log F \neq constant$ is a function of $x^1, \ldots, x^r$ only. Thus,

$$[\bar{g}_{ij}] = e^{2p}[g_{ij}]. \tag{36}$$

Since (M_1, g_1) is of constant sectional curvature (with constant scalar curvature), (M_2, g_2) is a locally symmetric Einstein manifold and condition (21) holds, $(M, \bar{g})$ is locally symmetric and equations (12)-(14) imply $\bar{C}_{abcd} = \bar{C}_{aABc} = 0$ and

$$\bar{C}_{ABCD} = \bar{R}_{ABCD} - \frac{1}{s(s-1)} N(\bar{g}_{BC}\bar{g}_{AD} - \bar{g}_{BD}\bar{g}_{AC}). \tag{37}$$

Moreover, one can easily check that condition (28) is satisfied. Hence, in view of (36), (7), (8) and $\bar{\nabla}\bar{C} = 0$, we obtain (27), which shows that (M, g) is of harmonic conformal curvature. On the other hand, by virtue of (36) and the definition of g it follows that (M, g) admits a non-homothetic conformal change of metric $g \mapsto \bar{g}$ such that $(M, \bar{g})$ is locally symmetric. Because of (37) and (7), (M, g) is not conformally flat and, by Lemma 5, it cannot be locally symmetric. The last remark completes the proof of (i).

(ii) Theorem 1 shows that $\dim M \geq 5$ and $(M, \bar{g})$ is a locally reducible locally symmetric manifold. Consequently, its metric has locally the form (10), where (M_1, g_1) is either Euclidean or irreducible, and any other (M_i, g_i) is neither reducible nor of dimension 1.

Denote by U a coordinate neighourhood in which $\operatorname{grad} p$ does not vanish identically. Whithout loss of generality we may assume (it is enough to renumerate the coordinates if it is necassary) that (M_1, g_1) with g_1 given by $[\bar{g}_{ab}]$, $a, b = 1, 2, \ldots, r$, is either Euclidean or irreducible, and that $p_c(x) \neq 0$ for some $x \in U$ and $c \in \{1, \ldots, r\}$. Obviously, the metric (10) can be written in the form (11), whence, by Lemma 11, it follows that (M_1, g_1) is of constant sectional curvature (with constant scalar curvature), (M_2, g_2) is Einsteinian with parallel curvature tensor and condition (21) holds. Moreover, Lemma 12 shows that p is a non-constant function of $x^1, \ldots, x^r$ only.

Assume now that (M_2, g_2) is of constant sectional curvature. Then, because of (37) and $\bar{C}_{abcd} = \bar{C}_{aABc} = 0$, the Weyl conformal curvature tensor $\bar{C}$ would vanish on U. Since $\bar{C}$ is parallel by assumption, it would vanish everywhere on M, a contradiction. Hence, in view of (35), (M, g) has in U the required warped product form with $F \neq constant$ and properties described in (i). This completes the proof.

Remark 2

The main part of Theorem 2(i) is due to A. Derdziński (cf. [1], p. 442).

Remark 3
In the case $B \neq 0$ the assertion of Lemma 10 follows also from a result of Grycak (cf. [7], Theorem 1).

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Local analytic solutions of a functional equation

Dedicated to Professor Andrzej Zajtz
on the occasion of his 70th birthday

Abstract. All analytic solutions of the functional equation

$$|f(r\exp(i\theta))|^2 + |f(1)|^2 = |f(r)|^2 + |f(\exp(i\theta))|^2$$

in the annulus

$$P := \{z \in \mathbb{C} :\ 1-\epsilon < |z| < 1+\epsilon\}$$

and in the domain

$$D := \{z = re^{i\theta} \in \mathbb{C} :\ 1-\epsilon < r < 1+\epsilon,\ \theta \in (-\delta, \delta)\},$$

are found.

1. Introduction

Hiroshi Haruki in [1] studied the following functional equations

$$|f(r\exp(i\theta))|^2 + |f(1)|^2 = |f(r)|^2 + |f(\exp(i\theta))|^2, \tag{1}$$

and

$$|f(r\exp(i\theta))| = |f(r)|, \tag{2}$$

where $r > 0$, θ are real. Equation (1) can be obtained from (2). In fact, let us put $r = 1$ in (2). Then we have

$$|f(\exp(i\theta))| = |f(1)| \tag{3}$$

for $\theta \in \mathbb{R}$. Next squaring (2) and (3) and adding them together we infer (1). Thus (1) is a generalization of (2), i.e., if f is a solution of (2), then it is a solution of (1). In paper [1] H. Haruki showed that all analytic solutions in $\mathbb{C}\setminus\{0\}$ of (1) which are analytic at 0 or have a pole at this point can be written as follows

$$f(z) = Az^p + Bz^{-p}, \tag{4}$$

where A, B are complex constants and p is an integer.

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We are going to prove that the functions of the form (4) are unique analytic solutions of (1) in the annulus

$$P := \{z \in \mathbb{C} : \ 1 - \epsilon < |z| < 1 + \epsilon\},$$

where $0 < \epsilon \leq 1$ is a constant. We shall also find all analytic solutions of (1) in the domain

$$D := \{z = re^{i\theta} \in \mathbb{C} : \ 1 - \epsilon < r < 1 + \epsilon, \ \theta \in (-\delta, \delta)\},$$

where $0 < \epsilon \leq 1$ and $0 < \delta \leq \pi$ are given constants. Moreover, we shall determinate all analytic solutions in P and in D of (2) and of the equation

$$|f(r \exp(i\theta))| = |f(\exp(i\theta))| \,. \tag{5}$$

Of course, (1) is also a generalization of (5).

2. Solutions of (1), (2) and (5) in P

In this section we will be concerned with analytic solutions of equations (1), (2) and (5) in the annulus P.

THEOREM 1

If f is an analytic solution of (1) in P, then there exist complex constants A, B and an integer p such that (4) is valid. Conversely, for every complex constants A, B and for every integer p, f given by (4) is a solution of (1).

Proof. It is easy to check that f given by (4) satisfies (1). The function $f(z) \equiv 0$ in P is a solution of (1) of the form (4). Suppose that an analytic function f is a solution of (1) and $f \not\equiv 0$. Of course,

$$f(re^{i\theta})\overline{f(re^{i\theta})} + |f(1)|^2 = |f(r)|^2 + |f(e^{i\theta})|^2 \tag{6}$$

for $\theta \in \mathbb{R}$ and $r \in (1 - \epsilon, 1 + \epsilon)$. Differentiating (6) at first with respect to r and then with respect to θ we successively infer

$$e^{i\theta} f'(re^{i\theta})\overline{f(re^{i\theta})} + e^{-i\theta} f(re^{i\theta})\overline{f'(re^{i\theta})} = \frac{d}{dr}|f(r)|^2$$

and

$$\begin{aligned} re^{2i\theta} f''(re^{i\theta})\overline{f(re^{i\theta})} - re^{-2i\theta} f(re^{i\theta})\overline{f''(re^{i\theta})} + e^{i\theta} f'(re^{i\theta})\overline{f(re^{i\theta})} \\ - e^{-i\theta} f(re^{i\theta})\overline{f'(re^{i\theta})} \\ = 0. \end{aligned}$$

Let us multiply the obtained equality by r and replace $re^{i\theta}$ by z. Then

$$z^2 f''(z)\overline{f(z)} - \overline{z}^2 f(z)\overline{f''(z)} + zf'(z)\overline{f(z)} - \overline{z}f(z)\overline{f'(z)} = 0,$$

i.e.,

$$\Im[z^2 f''(z)\overline{f(z)} + zf'(z)\overline{f(z)}\,] = 0 \tag{7}$$

for all $z \in P$. Since $f \not\equiv 0$, we can find a disc $V \subset P$ such that $f(z) \neq 0$ for all $z \in V$. The equality $\overline{f(z)} = \frac{|f(z)|^2}{f(z)}$, valid in this disc, and (7) imply

$$\Im\left[\frac{z^2 f''(z) + zf'(z)}{f(z)}\right] = 0$$

for all $z \in V$. Since an analytic function preserves domains, there exists a real constant k such that

$$z^2 f''(z) + zf'(z) - kf(z) = 0 \tag{8}$$

for all $z \in V$. By the Identity Theorem formula (8) remains valid in P. (The above part of the proof is due to H. Haruki, see [1], pp. 130-131). We can find complex numbers a_n, $n \in \mathbb{Z}$ such that for all $z \in P$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

Since

$$f'(z) = \sum_{n=-\infty}^{\infty} n a_n z^{n-1}, \qquad f''(z) = \sum_{n=-\infty}^{\infty} n(n-1) a_n z^{n-2}$$

we conclude that

$$0 = z^2 f''(z) + zf'(z) - kf(z) = \sum_{n=-\infty}^{\infty} [n(n-1) + n - k] a_n z^n,$$

whence

$$(n^2 - k)a_n = 0 \qquad \text{for all } n \in \mathbb{Z}. \tag{9}$$

We choose $p \in \mathbb{Z}$ such that $a_p \neq 0$. It is possible as $f \neq 0$. From (9) we get that $p^2 = k$ and

$$\left(n^2 - p^2\right) a_n = 0 \qquad \text{for all } n \in \mathbb{Z}.$$

So, if $n^2 \neq p^2$, then $a_n = 0$, whence it follows that $a_n = 0$ for all $n \neq p$ and $n \neq -p$. Thus

$$f(z) = a_p z^p + a_{-p} z^{-p}$$

for $z \in P$, as desired.

The following two lemmas are quite obvious.

Lemma 1
If the equality

$$Ae^{ia\theta} + \overline{A}e^{-ia\theta} = A + \overline{A}$$

holds true for all $\theta \in (-\delta, \delta)$, where A is a complex constant, $a \neq 0$ is a real one, then $A = 0$.

Lemma 2
If the equality

$$\alpha e^{a\theta} + \beta e^{-a\theta} = \alpha + \beta$$

holds true for all $\theta \in (-\delta, \delta)$, where $a \neq 0$, α, β are real constants, then $\alpha = \beta = 0$.

Now we will consider equation (2). As we mentioned above, every solution of (2) is a solution of (1). Thus if f is an analytic solution of (2), then f has to be of form (4) for some complex constants A, B and some integer p. Assume that $p \neq 0$. Substituting (4) to (2) we get

$$A\overline{B}e^{2ip\theta} + \overline{A}Be^{-2ip\theta} = A\overline{B} + \overline{A}B, \qquad \theta \in \mathbb{R}.$$

Lemma 1 yields $A = 0$ or $B = 0$. Thus we have

Theorem 2
If f is an analytic solution of (2) in the annulus P, then there exist a complex constant A and an integer p such that

$$f(z) = Az^p. \tag{10}$$

Conversely, for every complex constant A and for every integer p, the function f given by (10) is a solution of (2).

Theorem 3
Every analytic solution of (5) in the annulus P is a constant function.

Proof. Suppose that f is a solution of (5). Then f has to be of form (4). We may assume that $p \neq 0$. Combining (4) with (5) we obtain

$$|A|^2 r^{2p} + |B|^2 r^{-2p} = |A|^2 + |B|^2 \qquad \text{for all } r \in (1-\epsilon, 1+\epsilon).$$

Lemma 2 shows that $A = B = 0$, which completes the proof.

3. Solutions of (1), (2) and (5) in D

In this part of the paper we shall find all analytic solutions of equations (1), (2) and (5) in the domain $D := \{re^{i\theta} :\ 1-\epsilon < r < 1+\epsilon,\ \theta \in (-\delta, \delta)\}$, where $0 < \epsilon \leq 1$ and $0 < \delta \leq \pi$. In the sequel z^a denotes the principal branch of the power in D and $\log z$ is the principal branch of the logarithm of z, i.e., $z^a = \exp(a \log z)$ and $\log z = \log|z| + i \arg z$ for $z \in D$, where $\arg z \in (-\delta, \delta)$.

THEOREM 4
If an analytic function f satisfies (1) in D, then there exist complex constants A, B and $a \in \mathbb{R}$ or $a \in i\mathbb{R}$ such that

$$f(z) = Az^a + Bz^{-a}. \tag{11}$$

Conversely, every function f of form (11) with arbitrary complex constants A, B and arbitrary real or purely imaginary constant a is a solution of (1).

Proof. We may repeat the argument of the proof of Theorem 1. Thus we observe that if an analytic function f satisfies (1) in D , then it has to be a solution of the differential equation

$$z^2 f''(z) + zf'(z) - kf(z) = 0, \qquad z \in D, \tag{12}$$

where k is a real constant. Let

$$G = \{\log z : \ z \in D\}.$$

Of course, G is a domain. We define a function $g : G \longrightarrow \mathbb{C}$ as follows

$$g(u) := f(e^u).$$

g is analytic, $f(z) = g(\log z)$ for $z \in D$ and

$$e^u f'(e^u) = g'(u), \quad e^{2u} f''(e^u) = g''(u) - g'(u), \qquad u \in G. \tag{13}$$

It follows from (12) that

$$e^{2u} f''(e^u) + e^u f'(e^u) - kf(e^u) = 0 \qquad \text{for all } u \in G,$$

whence by (13)

$$g''(u) - kg(u) = 0, \qquad u \in G.$$

Solving this differential equation we get

$$g(u) = Ae^{au} + Be^{-au},$$

where A, B are suitable complex constants and $a^2 = k$. So a is a real constant or $a = ic$, where $c \in \mathbb{R}$. Putting $u = \log z$ we obtain (11). The first assertion of the theorem follows.

For the second conclusion, let us take arbitrarily $a \in \mathbb{R}$, $A, B \in \mathbb{C}$ and let f be given by (11). We observe that

$$f(re^{i\theta}) = Ar^a e^{i\theta a} + Br^{-a} e^{-i\theta a}, \qquad f(e^{i\theta}) = Ae^{i\theta a} + Be^{-i\theta a},$$
$$f(r) = Ar^a + Br^{-a}, \qquad f(1) = A + B.$$

Thus

$$
\begin{aligned}
&|f(re^{i\theta})|^2 + |f(1)|^2 \\
&\quad = (Ar^a e^{i\theta a} + Br^{-a} e^{-i\theta a})(\overline{A} r^a e^{-i\theta a} + \overline{B} r^{-a} e^{i\theta a}) + (A+B)(\overline{A} + \overline{B}) \\
&\quad = |A|^2 r^{2a} + |B|^2 r^{-2a} + A\overline{B} e^{2i\theta a} + \overline{A} B e^{-2i\theta a} + |A|^2 + |B|^2 + A\overline{B} + \overline{A} B
\end{aligned}
$$

and

$$
\begin{aligned}
&|f(e^{i\theta})|^2 + |f(r)|^2 \\
&\quad = (Ae^{i\theta a} + Be^{-i\theta a})(\overline{A} e^{-i\theta a} + \overline{B} e^{i\theta a}) + (Ar^a + Br^{-a})(\overline{A} r^a + \overline{B} r^{-a}) \\
&\quad = |A|^2 + |B|^2 + A\overline{B} e^{2i\theta a} + \overline{A} B e^{-2i\theta a} + |A|^2 r^{2a} + |B|^2 r^{-2a} + A\overline{B} + \overline{A} B.
\end{aligned}
$$

Now we assume that $a = ic$, where $c \in R$. Then

$$
\begin{aligned}
f\left(re^{i\theta}\right) &= Ae^{ic(\log r + i\theta)} + Be^{-ic(\log r + i\theta)} \\
&= Ae^{-c\theta} e^{ic\log r} + Be^{c\theta} e^{-ic\log r}, \\
f\left(e^{i\theta}\right) &= Ae^{-c\theta} + Be^{c\theta}, \\
f(r) &= Ae^{ic\log r} + Be^{-ic\log r}, \\
f(1) &= A + B.
\end{aligned}
$$

These formulas lead to

$$
\begin{aligned}
&|f(re^{i\theta})|^2 + |f(1)|^2 \\
&\quad = (Ae^{-c\theta} e^{ic\log r} + Be^{c\theta} e^{-ic\log r})(\overline{A} e^{-c\theta} e^{-ic\log r} + \overline{B} e^{c\theta} e^{ic\log r}) + |A+B|^2 \\
&\quad = |A|^2 e^{-2c\theta} + |B|^2 e^{2c\theta} + A\overline{B} e^{2ic\log r} + \overline{A} B e^{-2ic\log r} \\
&\qquad + |A|^2 + |B|^2 + A\overline{B} + \overline{A} B
\end{aligned}
$$

and

$$
\begin{aligned}
|f(e^{i\theta})|^2 + |f(r)|^2 &= (Ae^{-c\theta} + Be^{c\theta})(\overline{A} e^{-c\theta} + \overline{B} e^{c\theta}) \\
&\quad + (Ae^{ic\log r} + Be^{-ic\log r})(\overline{A} e^{-ic\log r} + \overline{B} e^{ic\log r}) \\
&= |A|^2 e^{-2c\theta} + |B|^2 e^{2c\theta} + A\overline{B} + \overline{A} B + |A|^2 + |B|^2 \\
&\quad + A\overline{B} e^{2ic\log r} + \overline{A} B e^{-2ic\log r}.
\end{aligned}
$$

So in both cases the function f given by (11) satisfies (1), as required.

Theorem 5
All analytic solutions of (2) *in* D *are of the form*

$$f(z) = Az^a, \tag{14}$$

where A *is a complex constant and* a *is a real one.*

Proof. Suppose that f is a non-constant analytic solution of (2) in D. Since (1) is a generalization of (2) we can apply Theorem 4. Thus there exist complex constants A, B and real or purely imaginary $a \neq 0$ such that f is given by (11). At first we assume that a is real. Substituting (11) in (2) after some easy calculations we obtain

$$\overline{A}B\exp(-2ia\theta)+A\overline{B}\exp(2ia\theta)=\overline{A}B+A\overline{B}$$

for $\theta\in(-\delta,\delta)$. Lemma 1 yields $A=0$ or $B=0$ and f is of the form (14), as required.

Now, we assume that $a=ic$, where c is real. Replacing in (2), $f(z)$ by (11) we infer the equality

$$|A|^2\exp(-2c\theta)+|B|^2\exp(2c\theta)=|A|^2+|B|^2.$$

This together with Lemma 2 yields $A=B=0$.

Theorem 6
All analytic solutions of (5) *in* D *are given by the formula*

$$f(z)=Az^{ic}, \tag{15}$$

where A *is a complex constant and* c *is a real one.*

Proof. We argue as in the preceding proof. Suppose that f is a non-constant analytic solution of (5) in D. f has to be given by (11). Assume that a is a real constant. Substituting (11) in (5) we get

$$|A|^2r^{2a}+|B|^2r^{-2a}=|A|^2+|B|^2$$

for all $r\in(1-\epsilon,1+\epsilon)$. From Lemma 2 we infer that $A=B=0$. It remains to consider $a=ic$, where c is real. Again substituting (11) in (5) we can obtain

$$A\overline{B}\exp(2ic\log r)+\overline{A}B\exp(-2ic\log r)=A\overline{B}+\overline{A}B.$$

The above formula and Lemma 1 yield (15).

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Seshadri constants of unisecant line bundles on ruled surfaces

Dedicated to Professor Dr. A. Zajtz

Abstract. The aim of this paper is to show that for any ruled surface X with a unisecant polarization $L \equiv C_0 + \mu_0 f$ the Seshadri constant of L at every point of X is equal 1.

1. Introduction

We investigate Seshadri constants of ample line bundles on ruled surfaces. In general it is difficult to calculate Seshadri constants and their values are known only in few examples. As for lower bounds, if a line bundle L is very ample, then we have always $\varepsilon(L; x) \geq 1$. Ein and Lazarsfeld proved that on any smooth surface X with arbitrary polarization L, the previous bound is somewhat surprisingly valid in almost every point of X, more exactly: we have $\varepsilon(L; x) \geq 1$ for x very general (see [2]). On the other hand Bauer proved that if L is very ample and there is a line passing through a point x, then $\varepsilon(L; x) = 1$ (see [1]). Here we investigate unisecant polarizations on ruled surfaces. In this situation there is a line passing through every point of x but the polarization is usually not very ample. Our main result states that nevertheless the Seshadri constant at every point is equal 1.

Main Theorem
If X is a ruled surface with a unisecant polarization $L \equiv C_0 + \mu_0 f$, then the Seshadri constant of L at every point of X is equal 1.

We follow the notation and terminology used by R. Hartshorne in [4]. All facts recalled in the introduction to theory of ruled surfaces are taken from [4] V section 2, and we use them here without proofs.

Throughout this paper we work over the field $\mathbb{C}$ of complex numbers. For any coherent sheaf on a (smooth, projective) variety X, we write $H^i(\mathcal{F})$ instead of $H^i(X, \mathcal{F})$, and we denote by $h^i(\mathcal{F})$ the dimension of the cohomology group

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$H^i(\mathcal{F})$. As customarily we use additive notation for tensor powers of line bundles.

2. Ruled surfaces

2.1. Basic definitions and properties

Definition 1
A geometrically ruled surface, or simply a ruled surface, is a surface X, together with a surjective morphism $\pi\colon X \longrightarrow C$ to a (nonsingular) curve C, such that for every point $y \in C$, the fibre X_y is isomorphic to $\mathbb{P}^1$, and such that π admits a section (i.e. a morphism $s\colon C \longrightarrow X$ such that $\pi \circ s = \mathrm{id}_C$).

Example 1
If C is a nonsingular curve, then $C \times \mathbb{P}^1$ with the first projection is a ruled surface.

Example 2
Let $\mathcal{E}$ be a vector bundle of rank 2 over a curve C. The associated projective space bundle $\mathbb{P}(\mathcal{E})$ with the projection morphism $\pi\colon \mathbb{P}(\mathcal{E}) \longrightarrow C$ is a ruled surface.

The following proposition shows that all ruled surfaces arise as in the above example.

Proposition 1 ([4] V, 2.2)
If $\pi\colon X \longrightarrow C$ is a ruled surface, then there exists a vector bundle $\mathcal{E}$ of rank 2 on C such that $X \cong \mathbb{P}(\mathcal{E})$ over C. If $\mathcal{E}$ and $\mathcal{E}'$ are two vector bundles of rank 2 on C, then $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{E}')$ are isomorphic as ruled surfaces over C if and only if there is an invertible sheaf $\mathcal{L}$ on C such that $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{L}$.

Remark 1
A surface X is called a birationally ruled surface if is birationally equivalent to $C \times \mathbb{P}^1$ for some curve C. Since $\mathbb{P}^2$ is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, this means that every rational surface is a birationally ruled surface.

Let $\pi\colon X \longrightarrow C$ be a ruled surface over a curve C of a genus g. By Proposition 1, we can choose $\mathcal{E}_0$ a locally free sheaf of rank 2 on C such that $X \cong \mathbb{P}(\mathcal{E}_0)$. Moreover we can assume that $\mathrm{H}^0(\mathcal{E}_0) \neq 0$ but for all invertible sheaves $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$, we have $\mathrm{H}^0(\mathcal{E}_0 \otimes \mathcal{L}) = 0$. A sheaf $\mathcal{E}_0$ with this property is called *normalized*.

In general $\mathcal{E}_0$ is not necessarily determined uniquely, but its invariant $e = -\deg(\mathcal{E}_0)$ is fixed.

Example 3

Let C be a curve with positive genus, and $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$ where $\deg(\mathcal{L}) = 0$ but $\mathcal{L} \not\cong \mathcal{O}_C$. In this case we have two choices of normalized $\mathcal{E}_0$, namely $\mathcal{E}$ and $\mathcal{E} \otimes \mathcal{L}^{-1}$.

Let $\mathfrak{e}$ be the divisor on C corresponding to the invertible sheaf $\bigwedge^2 \mathcal{E}_0$, then $e = -\deg(\mathfrak{e})$. Moreover, there exists a section $s_0 \colon C \longrightarrow X$ with the image C_0, such that $\mathcal{O}_X(C_0) \cong \mathcal{O}_X(1)$, where $\mathcal{O}_X(1)$ is the Serre line bundle on X (for more details see [4] V, 2.8).

Proposition 2 ([4] V, 2.3)

Under above assumptions we have:

$$\operatorname{Pic}(X) \cong \mathbb{Z} \cdot C_0 \oplus \pi^* \operatorname{Pic}(C).$$

Also

$$\operatorname{Num}(X) \cong \mathbb{Z} \cdot C_0 \oplus \mathbb{Z} \cdot f,$$

where f is the class of a fiber. Moreover $C_0.f = 1$, $f^2 = 0$ and $C_0^2 = -e$ (see Proposition 3).

If $\mathfrak{b}$ is any divisor on C, then we denote the divisor $\pi^*\mathfrak{b}$ on X by $\mathfrak{b}f$. Thus from Proposition 2 we have that, any element of $\operatorname{Pic}(X)$ can be written as $aC_0 + \mathfrak{b}f$ with $a \in \mathbb{Z}$ and $\mathfrak{b} \in \operatorname{Pic}(C)$. Any element of $\operatorname{Num}(X)$ can be written as $aC_0 + bf$ with $a, b \in \mathbb{Z}$.

Lemma 1 ([4] V, 2.20 and 2.11)

Using above notations

(1) *the canonical divisor K on X is given by*

$$K \sim -2C_0 + (\mathfrak{k} + \mathfrak{e})f$$

where $\mathfrak{k}$ is the canonical divisor on C.

(2) *For numerical equivalence, we have*

$$K \equiv -2C_0 + (2g - 2 - e)f$$

and therefore

$$K^2 = 8(1 - g).$$

Proposition 3 ([4] V, 2.6 and 2.9)

Let $\mathcal{E}$ be a locally free sheaf of rank two on a curve C, and let X be the ruled surface $\mathbb{P}(\mathcal{E})$. Let $\mathcal{O}_X(1)$ be the invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Then there exists a one-to-one correspondence between sections $s \colon C \longrightarrow X$ and surjections $\mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$, where $\mathcal{L}$ is an invertible sheaf on C, given by $\mathcal{L} = s^*\mathcal{O}_X(1)$.

Furthermore, if D is any section of X corresponding to a surjection $\mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$, and if $\mathcal{L} = \mathcal{O}_C(\mathfrak{d})$, for some divisor $\mathfrak{d}$ on C, then $\deg(\mathfrak{d}) = C_0.D$, and $D \sim C_0 + (\mathfrak{d} - \mathfrak{e})f$. In particular, we have that $C_0^2 = \deg(\mathfrak{e}) = -e$.

From Proposition 2 and Proposition 3 it follows that C_0 is a curve on X with the minimum self-intersection. Next lemma gives us more information about a number of such curves.

Lemma 2 ([3], 2.8)
Let $\pi\colon X = \mathbb{P}(\mathcal{E}_0) \longrightarrow C$ be a ruled surface. Then $\mathrm{h}^0(\mathcal{O}_X(C_0)) = 2$ if $X \cong C \times \mathbb{P}^1$ and $\mathrm{h}^0(\mathcal{O}_X(C_0)) = 1$ in all other cases.

This means that the curve C_0 is unique in its class of linear equivalence, except when the ruled surface is the product $C \times \mathbb{P}^1$.

Decomposable ruled surfaces

Definition 2
A ruled surface $X \cong \mathbb{P}(\mathcal{E}_0)$ is called *decomposable* if $\mathcal{E}_0$ is a direct sum of two invertible sheaves.

Theorem 1 ([4] V, 2.12)
Let X be a ruled surface over a curve C of genus g, determined by a normalized locally free sheaf $\mathcal{E}_0$.

(1) *If $\mathcal{E}_0$ is decomposable, then $\mathcal{E}_0 \cong \mathcal{O}_C \oplus \mathcal{L}$ for some $\mathcal{L}$ with $\deg(\mathcal{L}) \leq 0$. Therefore $e \geq 0$. All values of $e \geq 0$ are possible.*

(2) *If $\mathcal{E}_0$ is indecomposable, then $-g \leq e \leq 2g - 2$.*

Let $X \cong \mathbb{P}(\mathcal{E}_0)$ be a decomposable ruled surface. Geometrically it means that X has two disjoint unisecant curves C_0 and C_1 (i.e. $C_i.f = 1$ for each fiber f). These curves are given by surjections $\mathcal{E}_0 \cong \mathcal{O}_C(\mathfrak{e}) \oplus \mathcal{O}_C \longrightarrow \mathcal{O}_C \longrightarrow 0$ and $\mathcal{E}_0 \cong \mathcal{O}_C(\mathfrak{e}) \oplus \mathcal{O}_C \longrightarrow \mathcal{O}_C(\mathfrak{e}) \longrightarrow 0$, respectively. Moreover from Proposition 3, it follows that $C_1 \sim C_0 - \mathfrak{e}f$.

3. Linear systems on ruled surfaces

We start by recalling some basic facts.

Theorem 2 ([4] V, 2.20 and 2.21)
Let X be a ruled surface over a curve C of genus g, with a fiber f, the section C_0 and $e = -\deg(\mathfrak{e}) = -C_0^2$.

(1) *If $Y \equiv aC_0 + bf$ is an irreducible curve different from C_0 and a fiber, then*

(a) $a > 0$ *and* $b \geq ae$ *for* $e \geq 0$,

(b) $(a = 1$ *and* $b \geq 0)$ *or* $(a \geq 2$ *and* $b \geq \frac{1}{2}ae)$ *for* $e < 0$.

(2) *A divisor* $D \equiv aC_0 + bf$ *is ample if and only if*

(a) $a > 0$ *and* $b > ae$ *for* $e \geq 0$,

(b) $a > 0$ *and* $b > \frac{1}{2}ae$ *for* $e < 0$.

Remark 2
There are no better numerical conditions characterizing irreducible curves on ruled surfaces as this property does not depend only on the numerical equivalence class of the considered line bundle.

3.1. Elementary transformation of a ruled surface

Let $\pi: X \longrightarrow C$ be a geometrically ruled surface and let x be a point on X with $\pi(x) = P$. We denote by Pf the fiber through the point x.

Let $\sigma: X_x \longrightarrow X$ be the blow-up of X at x with the exceptional divisor $E = \sigma^{-1}(x)$. We have $\sigma^*(Pf) = \widetilde{Pf} + E$, where $\widetilde{Pf} = \overline{\sigma^{-1}(Pf \setminus \{x\})}$ is the *strict transform* of the fiber Pf. Since $\widetilde{Pf} \cong \mathbb{P}^1$, and $\widetilde{Pf}^2 = -1$, this means that we can blow-down the surface X_x along $\widetilde{Pf}$ (this follows from the Castelnuovo's criterion). We denote by $\tau: X_x \longrightarrow X'$ the blow-down of X_x along the exceptional curve $E' = \widetilde{Pf}$.

Definition 3
An elementary transformation of X at the point x is the birational map $\nu: X' \longrightarrow X$ where $\nu = \sigma \circ \tau^{-1}$. The surface X' is called the elementary transform of X at x. For a curve C on the surface X we define its strict transform as $C' = \tau_*(\widetilde{C})$.

Note that $(Pf)'$ is zero as τ contracts $\widetilde{PF}$ to a point. We observe furter that:

Remark 3
If X' is an elementary transform of X at x, then X is the elementary transform of X' at $\tau(y)$, where y is the intersection of the exceptional divisors E and E' on X_x. Moreover, if Pf' is the fiber through the point $\tau(y)$, then $\widetilde{Pf'} = E$.

Assume that $\pi: X \longrightarrow C$ is a geometrically ruled surface over a curve C of genus g with the invariant e. Let $\nu: X' \longrightarrow X$ be the elementary transformation of the surface X at a point x with $\pi(x) = P$. The question is: how the elementary transformation ν changes properties of X?

Proposition 4 ([3], 4.4)
With above assumptions we have:

(1) *If* $\mathfrak{b}$ *is a divisor on* C, *then* $\nu^*(\mathfrak{b}f) = \mathfrak{b}f'$.

(2) *If* D *is a curve on* X, *then* $\nu^* D = D' + (\operatorname{mult}_x D) \cdot Pf'$.

(3) *If* D *is a* n*-secant curve on* X *(i.e.* $D \equiv nC_0 + bf$ *for some* $b \in \mathbb{Z}$*) and* G *is a* m*-secant curve on* X, *then*

$$D'.G' = D.G + nm - n \cdot \operatorname{mult}_x G - m \cdot \operatorname{mult}_x D.$$

Therefore, if D *and* G *are unisecant curves on* X *then:*

(a) *if* $x \in D \cap G$, *then* $D'.G' = D.G - 1$.
(b) *if* $x \notin D \cup G$, *then* $D'.G' = D.G + 1$.
(c) *if* $x \in D$ *but* $x \notin G$, *then* $D'.G' = D.G$.

(4) *If* D *is a unisecant curve on* X, *then* $\nu_* \nu^* D = D + Pf$.

Let C_0 be the minimum self-intersection curve on X. We know that $C_0^2 = -e$ and for any other curve D on X, we have $D^2 \geq -e$. Moreover assume that $x \in C_0$. Let C_0' denote the strict transform of C_0 by the elementary transformation X at x. From Proposition 4 it follows that $C_0'^2 = C_0^2 - 1$, but for any other unisecant curve D we have $D'^2 \geq D^2 - 1$. It means that $D'^2 \geq C_0'^2$ and C_0' is the minimum self-intersection curve on X'. Since $C_0'^2 = -e-1$, then $e' = e + 1$.

In this way we gave the idea of the following

Theorem 3 ([3], 4.9)
Let $\pi\colon \mathbb{P}(\mathcal{E}_0) \longrightarrow C$ *be a ruled surface. Fix a point* x *on the minimum self-intersection curve* C_0 *on* X, *with* $\pi(x) = P$. *Let* X' *denote the elementary transform of* X *at* x. *Then* X' *is a ruled surface corresponding to a normalized sheaf* $\mathcal{E}_0'$ *with* $\bigwedge^2 \mathcal{E}_0' \cong \mathcal{O}_C(\mathfrak{e}')$ *satisfying* $\mathfrak{e}' \sim \mathfrak{e} - P$ $(e' = e+1)$. *Furthermore, the minimum self-intersection curve on* X' *is* C_0'.

Let X_0 be an indecomposable ruled surface over a curve C of genus g and invariant e. If we apply an elementary transformation to X at a point on C_0, then we obtain a ruled surface X_1 with invariant $e_1 = e+1$ (from Theorem 3). We can take n such transformations so that $e_n = e + n > 2g - 2$. This means that after n steps the surface X_n is decomposable (see Theorem 1). Applying Remark 3 to surfaces X and X_n, we have that X can be obtained from X_n by elementary transformations. We proved the following

Remark 4 ([3], 4.10)
Any indecomposable ruled surface is obtained from a decomposable one by a finite number of elementary transformations.

We can say more, namely

Remark 5 ([3], 4.11)
Any ruled surface over the curve C is obtained from $C \times \mathbb{P}^1$ applying a finite number of elementary transformations.

From Remark 5 follows that every ruled surface is birationally ruled surface (compare with Remark 1).

Remark 3 and Theorem 3 give us useful tools to study numerical properties of transformed divisors.

Proposition 5
Let $\nu: X' \longrightarrow X$ be the elementary transformation at $x \in C_0$.

(a) *Let D be a divisor on X. If $D \equiv aC_0 + bf$ with integers a and b, then $\nu^* D \equiv aC_0' + (a+b)f'$, where C_0' and f' generate $Num(X')$.*

(b) *Let Y be a divisor on X'. If $Y \equiv pC_0' + qf'$ with integers p and q, then $\nu_* Y \equiv pC_0 + qf$.*

Proof. We are using the notation introduced in the definition of an elementary transformation and in the previous propositions.

Part (a). Let

$$\nu^* D \equiv pC_0' + qf', \qquad \text{with } p, q \in \mathbb{Z}. \tag{1}$$

From Proposition 2 we have that for any fiber f'

$$(\nu^* D).f' = p,$$

but

$$(\nu^* D).f' = (\tau_* \sigma^* D).f' = (\sigma^* D).(\tau^* f').$$

Let $\tau(y) \in f'$. In our notation it means that $f' = Pf'$. Then

$$\begin{aligned}(\nu^* D).f' &= (\sigma^* D).(\widetilde{Pf'} + E') = (\sigma^* D).E + (\sigma^* D).\widetilde{Pf} \\ &= (\sigma^* D).(\sigma^*(Pf)) - (\sigma^* D).E = D.(Pf) \\ &= a.\end{aligned}$$

If $\tau(y) \notin f'$, then

$$(\nu^* D).f' = (\sigma^* D).(\tau^* f') = (\sigma^* D).\widetilde{f'} = (\sigma^* D).\widetilde{f} = (\sigma^* D).(\sigma^* f) = D.f = a.$$

In this way we proved $p = a$.

To show that it holds $q = a + b$, it is enough to test the intersection product $(\nu^* D).C_0'$.

Since $x \in C_0$, then $\tau(y) \notin C_0'$. Moreover from Theorem 3 it follows that

$$C_0'^2 = C_0^2 - 1. \tag{2}$$

By Proposition 2, conditions (1) and (2)

$$(\nu^* D).C_0' = pC_0'^2 + q = pC_0^2 - p + q. \tag{3}$$

On another hand

$$\begin{aligned}(\nu^* D).C_0' &= (\sigma^* D).(\tau^* C_0') = (\sigma^* D).\widetilde{C_0'} = (\sigma^* D).\widetilde{C_0}\\ &= (\sigma^* D).(\sigma^* C_0) - (\sigma^* D).E = D.C_0 \\ &= aC_0^2 + b.\end{aligned} \tag{4}$$

Applying the equality $p = a$ for conditions (3) and (4) we have $q = a + b$.

Part (b). Let Pf' denote, as before, the fiber through $\tau(y)$. Moreover assume that

$$\nu_* Y \equiv aC_0 + bf. \tag{5}$$

The idea of the proof for this part is the same as in the part (a). In particular, it is not difficult to see that $a = p$. We concentrate more on the second intersection product i.e. $(\nu_* Y).C_0$.

From conditions (2) and (5) it follows

$$(\nu_* Y).C_0 = aC_0^2 + b = aC_0'^2 + a + b. \tag{6}$$

We have also

$$\begin{aligned}(\nu_* Y).C_0 &= (\sigma_*(\tau^* Y)).C_0 = (\tau^* Y).(\sigma^* C_0) = (\tau^* Y).(\widetilde{C_0} + E)\\ &= (\tau^* Y).\widetilde{C_0'} + (\tau^* Y).\widetilde{Pf'}\\ &= (\tau^* Y).(\tau^* C_0') + (\tau^* Y).(\tau^* Pf' - E')\\ &= Y.C_0' + Y.Pf'\\ &= pC_0'^2 + q + p.\end{aligned} \tag{7}$$

Applying the equality $a = p$ to (6) and (8) we see that $b = q$.

Proposition 6

For any n-secant curve D on X its strict transform D' on X' is still an n-secant curve.

Proof. Let $D \equiv nC_0 + bf$ be an n-secant curve on X and let $\nu\colon X' \longrightarrow X$ be an elementary transformation at a point x. From Proposition 4 it follows that

$$D' = \nu^* D - (\operatorname{mult}_x D) \cdot Pf'.$$

Hence by Proposition 5 we have:

(a) if $x \in C_0$, then $D' \equiv nC_0' + (n + b - \operatorname{mult}_x D)f'$;

(b) if $x \notin C_0$, then $D' \equiv nC_0' + (b - \operatorname{mult}_x D)f'$.

Let $G \equiv C_0 + \mu_0 f$ be an ample divisor on X. The question is: what happens with ampleness of the strict transform G'? Is G' still ample? In general G' need not to be ample. More precisely we can formulate the following

PROPOSITION 7
With above assumptions, the strict transform G' is ample except when

(i) *in the case $e > 0$ we have $G \equiv C_0 + (e+1)f$ and we apply an elementary transformation at a point $x \in C_0$ which is also a base point of $|G|$,*

(ii) *in the case $e < 0$ and e odd we have $G \equiv C_0 + \frac{1}{2}(e+1)f$ and we apply an elementary transformation at a base point of $|G|$.*

Proof. Let X be a ruled surface with invariant e, and let $D \in |G|$. As before, by $\nu : X' \longrightarrow X$ we denote the elementary transformation at a point $x \in X$.

Case (1). If $x \in C_0$, then by Theorem 3 the surface X' is ruled with invariant $e' = e + 1$. Moreover by Proposition 5 the strict transform $D' \equiv C_0' + (\mu_0 + 1 - \operatorname{mult}_x D)f'$.

From Theorem 2 it follows that:

(a) for $e \geq 0$ we have $\mu_0 \geq e + 1$ and
$$\mu_0 + 1 - \operatorname{mult}_x D \geq e' + 1 - \operatorname{mult}_x D,$$
hence D' is not ample if $\mu_0 = e + 1$ and $x \in D$;

(b) for $e < 0$ and e even, $\mu_0 \geq \frac{1}{2}e + 1$ and
$$\mu_0 + 1 - \operatorname{mult}_x D \geq \frac{1}{2}(e' + 1) + 1 - \operatorname{mult}_x D,$$
then D' always is ample;

(c1) for $e = -1$ we have $\mu_0 \geq 0$ and
$$\mu_0 + 1 - \operatorname{mult}_x D \geq 1 - \operatorname{mult}_x D;$$

(c2) for $e < -1$ and e odd, $\mu_0 \geq \frac{1}{2}(e + 1)$ and
$$\mu_0 + 1 - \operatorname{mult}_x D \geq \frac{1}{2}e' + 1 - \operatorname{mult}_x D.$$

It means that D' is not ample if $\mu_0 = \frac{1}{2}(e + 1)$ and $x \in D$.

Case (2). If $x \notin C_0$, then by Theorem 3 and Remark 3 the surface X' is the ruled surface with invariant $e' = e - 1$. By Proposition 5 the strict transform $D' \equiv C_0' + (\mu_0 - \operatorname{mult}_x D)f'$.

Using the same technique we have:

(a1) for $e > 0$ the coefficient

$$\mu_0 - \operatorname{mult}_x D \geq e' + 2 - \operatorname{mult}_x D,$$

(a2) for $e = 0$ we have $\mu_0 \geq 1$ and $\mu_0 - \operatorname{mult}_x D > -\frac{1}{2}$,

it means that for $e \geq 0$ the strict transform D' is always ample;

(b) for $e < 0$ and e even,

$$\mu_0 - \operatorname{mult}_x D \geq \frac{1}{2}e' + \frac{3}{2} - \operatorname{mult}_x D,$$

and D' is always ample;

(c) for $e < 0$ and e odd

$$\mu_0 - \operatorname{mult}_x D \geq \frac{1}{2}e' + 1 - \operatorname{mult}_x D,$$

and D' is not ample if $\mu_0 = \frac{1}{2}(e+1)$ and $x \in D$.

4. Seshadri constants

The concept of Seshadri constants was introduced by Damailly. He associated a real number $\varepsilon(L;x)$ with an ample line bundle L at a point x of an algebraic variety X. This number in effect measures how much of positivity of L can be concentrated at x.

In this section we calculate Seshadri constant for a ruled surface X with a unisecant polarization i.e. an ample line bundle of type $L \equiv C_0 + \mu_0 f$.

Let us recall the definition and some properties of Seshadri constants.

Definition 4
Let L be a nef line bundle on a smooth projective variety X. Fix a point x on X. Let $\sigma\colon X_x \longrightarrow X$ be the blowing-up of X at the point x with the exceptional divisor $E = \sigma^{-1}(x)$. The Seshadri constant of L at x is the non-negative real number

$$\varepsilon(L;x) = \sup\{\varepsilon \in \mathbb{R} \mid \sigma^* L - \varepsilon E \text{ is nef}\}.$$

From Kleiman's nefness criterion it follows that $\varepsilon(L;x) \leq \sqrt[\dim X]{L^{\dim X}}$. If the value of $\varepsilon(L;x)$ is less than the previous upper bound, then we say that the Seshadri constant is *L-submaximal* (or simply *submaximal*).

Remark 6
We can define the Seshadri constant of L at x as

$$\varepsilon(L;x) = \inf_{D \ni x} \left\{ \frac{L.D}{\mathrm{mult}_x D} \right\}$$

where the infimum is taken over all (irreducible) curves D (see [5] 5.1.5).

If $\frac{L.D}{\mathrm{mult}_x D} = \varepsilon(L;x)$, then we say that the curve D *computes* the Seshadri constant.

Assume moreover that L is an ample line bundle. For a fixed point $x \in X$, we denote by $\mathfrak{m}_x \subset \mathcal{O}_X$ its maximal ideal.

Definition 5
We say that the complete linear system $|L|$ separates s-jets at x, if the natural map

$$H^0(L) \longrightarrow H^0\left(L \otimes \mathcal{O}_X/\mathfrak{m}_x^{s+1}\right)$$

taking sections of L to their s-jets is surjective. By $s(L,x)$ we denote the maximal number such that $|L|$ separates s-jets at x.

Using above terminology we have the following

Proposition 8 ([5], 5.1.17)
For an ample line bundle L on X

$$\varepsilon(L;x) = \limsup_{k \to \infty} \frac{s(kL,x)}{k}.$$

Theorem 4 (Main theorem)
If X is a ruled surface with an invariant e and a polarization $L \equiv C_0 + \mu_0 f$, then for every point $x \in X$ the Seshadri constant $\varepsilon(L;x) = 1$.

Proof. Since $L \equiv C_0 + \mu_0 f$ is ample, then from Theorem 2 it follows that:

(a) $\mu_0 \geq e + 1$ for $e \geq 0$;

(b) $\mu_0 \geq \frac{1}{2}e + 1$ for $e < 0$ and e even;

(c) $\mu_0 \geq \frac{1}{2}(e+1)$ for $e < 0$ and e odd.

Fix a point $x \in X$. Let $D \equiv aC_0 + bf$ with $a, b \in \mathbb{Z}$, be an irreducible curve on X different from C_0 and a fiber f. By m we denote the multiplicity of D at the point x. Since D is a-secant it must be $m \leq a$.

To calculate the Seshadri constant in cases (a) and (b), it is enough to study the Seshadri quotients i.e. $\frac{L.D}{m} = \frac{b + a(\mu_0 - e)}{m}$.

By assumption D is an irreducible curve. By Theorem 2 we have two cases to consider.

Case $a > 0$ and $b \geq ae$.
This implies

$$\frac{L.D}{m} \geq \frac{a(e+1)}{m} \geq e+1 \geq 1.$$

Case ($a=1$ and $b>0$) or ($a \geq 2$ and $b \geq \frac{1}{2}ae$).
If $a=1$ then also $m=1$ and we have

$$\frac{L.D}{m} \geq 1 - \frac{1}{2}e > 1.$$

Also it is easy to check that for $a \geq 2$ and $b \geq \frac{1}{2}ae$ the Seshadri quotient satisfies

$$\frac{L.D}{m} \geq \frac{a}{m} \geq 1.$$

Thus we showed that $\varepsilon(L;x) \geq 1$.

For any ruled surface X and a unisecant line bundle L we have $\frac{L.Pf}{\mathrm{mult}_x Pf} = 1$, where Pf is the fiber through x. Thus $\varepsilon(L;x) = 1$ and Pf computes the Seshadri constant.

Using the same method in the case (c) we have:
for $a=1$ and $b>0$ the quotient

$$\frac{L.D}{m} \geq \frac{1}{2} - \frac{1}{2}e \geq 1,$$

but for $a \geq 2$ and $b \geq \frac{1}{2}ae$ it follows only that:

$$\frac{L.D}{m} \geq \frac{1}{2},$$

and it means that we still do not know the value of the Seshadri constant at the point x.

To prove that $\varepsilon(L;x) = 1$ for $e<0$ and e odd, we use a different method.

Let p be a point on X such that $p \neq x$ and p not a base point of $|L|$. Apply the elementary transformation at the point p. Since $x \neq p$, then $\nu^{-1}(x) = x$. Moreover from Proposition 7 it follows that L' is an ample line bundle on the surface X' with even invariant e'. By (2) we have that $\varepsilon(L';x) = 1$. Separating s-jets at x is a local property of L at x and the elementary transformation change the surface X only in the neighborhood of the fiber through p. It means that we can choose p such that $s(L;x) = s(L';x)$. Note that there is an obvious isomorphism $H^0(X,L) \cong H^0(X',L')$. Then Proposition 8 implies $\varepsilon(L;x) = \varepsilon(L';x) = 1$.

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Seshadri fibrations on algebraic surfaces

Dedicated to Professor Andrzej Zajtz, on occasion of his 70th birthday

Abstract. We show that small Seshadri constants in a general point of a surface have strong geometrical implications, the surface is fibered by curves computing the Seshadri constant. We give a sharp bound in terms of the selfintersection of the given ample line bundle and discuss some examples.

Introduction

Seshadri constants were introduced by Demailly [2] in the late 80's in connection with attempts to tackle the Fujita Conjecture. They express, roughly speaking, how ample a line bundle is locally. Seshadri constants quickly gained considerable interest on their own.

Recently Nakamaye [5] showed that these local invariants when studied at a general point of a variety carry interesting global geometric information. In particular he was interested in to which extend Seshadri constants capture existence of morphisms to lower dimensional varieties. This problem was considered also by Hwang and Keum [3]. In both papers it is shown that a small Seshadri constant in a generic point of a variety forces a fibration structure on the variety. Here we prove that in case of algebraic surfaces the bound from [3, Theorem 2] is in fact the optimal one. One could hope that on the contrary a large Seshadri constant in a generic point prohibits in turn a fibration structure on the variety. We show that this need not to be the case and we answer to the negative two related questions from [5].

1. Seshadri fibrations

Let us first recall the following

Definition
Let X be a smooth projective variety, let L be a nef line bundle on X and let $x \in X$ be a fixed point. Then the real number

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$$\varepsilon(L,x) := \inf\left\{\frac{L.C}{\operatorname{mult}_x C} \mid C \text{ an irreducible curve passing through } x\right\}$$

is the *Seshadri constant* of L at x.

In case of surfaces, it is well known that $\varepsilon(L,x) \le \sqrt{L^2}$. Moreover, if the Seshadri constant is *submaximal* $\varepsilon(L,x) < \sqrt{L^2}$, then a theorem of Campana and Peternell (see [1]) assures that there exists a *Seshadri curve* C_x computing $\varepsilon(L,x)$ i.e.

$$\varepsilon(L,x) = \frac{L.C_x}{\operatorname{mult}_x C_x}\,.$$

Nakamaye shows [5, Corollary 3] that if the Seshadri constant at every point of X is sufficienly small, namely $\varepsilon(L,x) < \sqrt{\frac{1}{3}L^2}$, then Seshadri curves form a fibration on X i.e. they are fibers of a non-trivial morphism $f\colon X \longrightarrow Y$ onto a curve Y. We speak in this situation of a *Seshadri fibration* on X. Our main result strengthens that of Nakamaye.

Theorem
Let (X,L) be a polarized surface and suppose that

$$\varepsilon(L,x) < \sqrt{\frac{3}{4}L^2} \qquad (1)$$

at every point $x \in X$. Then there exists a non-trivial morphism $f\colon X \longrightarrow Y$ to a curve Y whose fibers are Seshadri curves. Moreover the above bound is sharp.

The proof of our result builds upon the following Lemma due to Xu [7, Lemma 1].

Lemma
Let X be a smooth projective surface, let (C_t, x_t) be a one parameter family of pointed curves on X and let $m \ge 2$ be an integer such that $\operatorname{mult}_{x_t} C_t \ge m$. *Then*

$$C_t^2 \ge m(m-1) + 1.$$

Proof of Theorem. Let C_x be a Seshadri curve at $x \in X$ and suppose that for $x \in X$ general we have $\operatorname{mult}_x C_x \ge m$.

If $m \ge 2$, then the assumption (1), the Hodge index Theorem and the Lemma yield

$$\frac{3}{4}m^2L^2 > (L.C_x)^2 \ge L^2C_x^2 \ge (m(m-1)+1)L^2,$$

which is easily seen to be equivalent to $(m-2)^2 < 0$, a contradiction.

If $m = 1$ then again by the assumption (1) and the Hodge index Theorem we get

$$\frac{3}{4}L^2 > (L.C_x)^2 \geq C_x^2 L^2.$$

Since C_x moves in a family we have $C_x^2 \geq 0$ and thus the above inequality implies $C_x^2 = 0$. Then a standard argument (see e.g. [5]) shows that for some positive integer $k > 0$ the linear system $|kC_x|$ gives the desired fibration.

2. Examples

The following examples show that our Theorem is optimal.

Example 2.1
Let (X, L) be a smooth cubic in $\mathbb{P}^3$. Then

$$\varepsilon(L, x) = \frac{3}{2} = \sqrt{\frac{3}{4}L^2}$$

for $x \in X$ general. Indeed, the hyperlane tangent at x cuts on X a curve $C_x \in |L|$, which for x general is irreducible and has multiplicity 2. Of course, the curves C_x do not define a fibration on X as they belong to a very ample line series.

It may well happen that for $\varepsilon(L, x) = \sqrt{\frac{3}{4}L^2}$ one gets a Seshadri fibration on X.

Example 2.2
Let (X, L) be the Hirzebruch $\mathbb{F}_1$ surface with polarization $L = 6C_0 + 7f$, where as usual f denotes the class of the fiber in the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ and C_0 its zero section. In this case the fibers compute the Seshadri constant of L at every point of X and we have

$$\varepsilon(L, x) = 6 = \sqrt{\frac{3}{4}L^2}.$$

One might ask whether there is a converse to our Theorem i.e. if the existence of a Seshadri fibration imposes some constrains on the Seshadri constant at a general point. The following example shows that this is not the case, the Seshadri constant can be arbitrarily close to its maximal possible value $\sqrt{L^2}$.

Example 2.3
Let $f\colon X \longrightarrow \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at a point $P \in \mathbb{P}^2$ with the exceptional divisor E and let $L = H - \lambda E$ be a $\mathbb{Q}$-line bundle with $H = f^*\mathcal{O}_{\mathbb{P}^2}(1)$. Then

$$\varepsilon(L, x) = 1 - \lambda$$

for $x \in X \setminus E$. Indeed, the quotient $1 - \lambda$ is computed by the proper transform of the line through P and x.

Moreover, if $C \subset X$ is another irreducible curve through x, then C is of the form $C = dH - mE$ with $m \leq d-1$ and $m + \operatorname{mult}_x C \leq d$. Then

$$\frac{L.C}{\operatorname{mult}_x C} = \frac{d - m\lambda}{\operatorname{mult}_x C} \geq \left\{ \begin{array}{ll} \frac{m(1-\lambda)}{\operatorname{mult}_x C} & \text{if } m \geq \operatorname{mult}_x C \\ \frac{d-\lambda \operatorname{mult}_x C}{\operatorname{mult}_x C} & \text{if } m \leq \operatorname{mult}_x C \end{array} \right\} \geq 1 - \lambda.$$

Hence $\varepsilon(L,x) = \sqrt{\frac{(1-\lambda)^2}{1-\lambda^2} L^2}$.

3. Answers

Finally, we answer two questions from Nakamaye's paper. First we address the question if the existence on a surface X of a nef real class χ with $\chi^2 = 0$ implies that X admits a surjective morphism to a curve [5, Question 9].

Example 3.1
Let (X, Θ) be a principally polarized abelian surface with the endomorphism ring $\operatorname{End}(X) \cong \mathbb{Z}[\sqrt{d}]$, where d is a square free positive integer. Let $M \in \operatorname{NS}(X)$ be a line bundle with $M^2 = -2d$ corresponding to the endomorphism $\sqrt{d}$ under the group homomorphism

$$\operatorname{NS}(X) \ni N \longrightarrow \varphi_\Theta^{-1} \circ \varphi_N \in \operatorname{End}(X).$$

Then Θ and M form an orthogonal basis of $\operatorname{NS}(X)$ and a line bundle $a\Theta + bM$ is ample if and only if

$$(a\Theta + bM)^2 > 0 \quad \text{and} \quad (a\Theta + bM).\Theta > 0.$$

It follows that

$$\operatorname{Nef}(X) = \mathbb{R}_{\geq 0} \cdot (\sqrt{d}\Theta + M) + \mathbb{R}_{\geq 0} \cdot (\sqrt{d}\Theta - M)$$

and $\sqrt{d}L \pm M$ are the only real nef classes with selfintersection 0. This shows that there is no non-trivial morphism from the abelian surface X to a curve, which in turn answers the above question negatively.

The last problem concerns pairs (X, L) consisting of a smooth projective surface X and an ample line bundle L with selfintersection 1. On such a surface $\varepsilon(L,x) = 1$ for x very general by the result of Ein and Lazarsfeld [4]. Nakamaye asks, obviously motivated by $\mathbb{P}^2$, if the Seshadri curves computing $\varepsilon(L,x)$ on X can be forced to form a fibration when one passes to a blow up of X at a single point. The answer to this question is also negative.

Example 3.2
Let (X, Θ) be a principally polarized abelian surface with Picard number $\rho(X) = 1$. Let $f\colon Y \longrightarrow X$ be the blow up of X at a point $P \in X$ with

the exceptional divisor E. Then $L = f^*\Theta - E$ is ample and $L^2 = 1$. Its Seshadri constant at every point x away of E is computed by the proper transform of a translate of the Θ-divisor passing through x and P. Indeed, there are exactly two such translates and since $\rho(Y) = 2$ the claim follows from [6, Proposition 1.8]. Of course there doesn't exist any point $y \in Y$ such that the translates of the Θ-divisor form a fibration on the blow up of Y at y.

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Włodzimierz Waliszewski

Oriented angles in affine space

To Andrzej Zajtz, on the occasion of His 70th birthday

Abstract. The concept of a smooth oriented angle in an arbitrary affine space is introduced. This concept is based on a kinematics concept of a run. Also, a concept of an oriented angle in such a space is considered. Next, it is shown that the adequacy of these concepts holds if and only if the affine space, in question, is of dimension 2 or 1.

0. Preliminaries

Let us consider an arbitrary affine space, i.e. a triple

$$(E, V, \overrightarrow{}), \tag{0}$$

(see [B–B]), where E is a set, V is an arbitrary vector space over reals and $\overrightarrow{}$ is a function which to any points $p, q \in E$ assigns a vector $\overrightarrow{pq}$ of V in such a way that

1) $\overrightarrow{pq} + \overrightarrow{qr} = \overrightarrow{pr}$ for $p, q, r \in E$,

2) $\overrightarrow{pq} = 0$ iff $p = q$ for $p, q \in E$,

3) for any $p \in E$ and any vector x of V there exists $q \in E$ with $\overrightarrow{pq} = x$.

The unique point q for which $\overrightarrow{pq} = x$ will be denoted by $p + x$. The set of all vectors of the vector space V will be denoted by $\underline{V}$. The fact that W is a vector subspace of V will be written as $W \leq V$. For any sets M, N, X, Y, P such that $M \cup N \subset \mathbb{R}$, $X \cup Y \subset \underline{V}$, $P \subset E$, any $b \in \mathbb{R}$, $y \in \underline{V}$ and $p \in E$ we set

$$\begin{aligned} M + N &= \{a + b;\ a \in M \ \&\ b \in N\}, & M + b &= M + \{b\}, \\ MN &= \{ab;\ a \in M \ \&\ b \in N\}, & bM &= \{b\} M, \\ MX &= \{ax;\ a \in M \ \&\ x \in X\}, & bX &= \{b\} X, \\ X + Y &= \{x + y;\ x \in X \ \&\ y \in Y\}, & & \\ P + X &= \{p + x;\ p \in P \ \&\ x \in X\}, & p + X &= \{p\} + X. \end{aligned}$$

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A subset H of E is a hyperplane in an affine space (0) iff there exist $p \in E$ and $W \leq V$ such that

$$H = p + \underline{W}. \tag{1}$$

The subspace W of V for which (1) holds will be denoted by V_H. The affine space

$$(H, V_H, {}^{\to H}), \tag{2}$$

where ${}^{\to H}$ is the restriction of the function ${}^{\to}$ to the set $H \times H$, is called the *subspace of* (0) *determined by the hyperplane* H. The triple (2), where $H = \emptyset$, $V_H \leq V$, $\underline{V_H} = \{0\}$ and ${}^{\to H} = \emptyset$ is an affine space and will be treated as a subspace of (0) as well. Also, the set $\emptyset$ will be considered as a *hyperplane* in (0). We will write $W \leq_k V$ instead of to state that a vector subspace W of V is of codimension k in V. In particular, $W \leq_1 V$ means that W is of codimension 1 in V. We say that H is a *hyperplane of codimension* k in the affine space (0) iff $V_H \leq_k V$.

Any set P of points of the affine space (0), i.e. $P \subset E$, such that

$$P = H + \mathbb{R}_+ e, \tag{3}$$

where H is a hyperplane of codimension 1 in (0), $e \in \underline{V} \setminus \underline{V_H}$, $\mathbb{R}_+ = \langle 0; +\infty)$, is said to be a *halfspace* of (0). The hyperplane H in (3) uniquely determined by P is called the *shore* of the halfspace P and denoted by P^o. The set $P \setminus P^o$ will be called the *interior* of the halfspace P and denoted by P_+. It is easy to check that the set P^- of the form $E \setminus P_+$ is also a halfspace and the equalities

$$\left(P^-\right)^o = P^o \qquad \text{and} \qquad \left(P^-\right)_+ = E \setminus P \tag{4}$$

hold. The set $E \setminus P$ will be denoted by P_-. The halfspace P^- is called the *opposite* one to P. It is easy to verify that (3) yields also

$$P_+ = P^o + (0; +\infty)\, e, \quad P^- = P^o + \mathbb{R}_+ (-e)\,, \quad P_- = P^o + (-\infty; 0)\, e \tag{5}$$

where $e \in \underline{V} \setminus \underline{V_H}$ and $H = P^o$.

Let B be a base of a vector space V. For any $\mathrm{v} \in \underline{V}$ there exists a unique real function v_B defined on B such that $\{e;\ e \in B \ \&\ \mathrm{v}_B(e) \neq 0\}$ is finite and

$$\mathrm{v} = \sum_{e \in B} \mathrm{v}_B(e)\, e, \tag{6}$$

where the sign of addition in (6) denotes of course a finite operation. This formula will be very useful.

For any topology $\mathcal{T}$ (see [K]) the set of all points of $\mathcal{T}$ will be denoted by $\underline{\mathcal{T}}$, i.e. by definition we have

$$\underline{\mathcal{T}} = \bigcup \mathcal{T}. \tag{7}$$

For any set $A \subset \underline{\mathcal{T}}$ the induced to A topology from the topology $\mathcal{T}$ will be denoted by $\mathcal{T}|A$, i.e. $\mathcal{T}|A = \{B \cap A;\ B \in \mathcal{T}\}$.

For any affine space (0) the smallest topology containing the set of all sets P_+, where P is a halfspace of (0) will be called the *topology of the affine space* (0) and denoted by $\mathrm{top}(E, V, \rightarrow)$. It is easy to check that for any hyperplane H in (0) we have

$$\mathrm{top}(H, V_H, \rightarrow^H) = \mathrm{top}(E, V, \rightarrow)|H. \tag{8}$$

Let f be any function. The domain of f will be denoted by D_f. For any $A \subset D_f$ the restriction of the function f to the set A and the f-image of A will be denoted by $f|A$ and fA, respectively. Any function may be treated as a set of ordered pairs, and then

$$D_f = \{x;\ \exists y\ \ ((x, y) \in f)\}, \qquad f|A = \{\,(x, y)\,;\ (x, y) \in f\ \&\ x \in A\}$$

and

$$fA = \{y;\ \exists x \in A\ \ ((x, y) \in f)\}\,.$$

For any set B the f-preimage $f^{-1}B$ is defined by

$$f^{-1}B = \{x;\ \exists y \in B\ \ ((x, y) \in f)\}$$

or, equivalently, $f^{-1}B = \{x;\ x \in D_f\ \&\ f(x) \in B\}$.

Let f be a function with $D_f \subset \mathbb{R}$, $fD_f \subset E$, $t \in \mathbb{R}$ and $p \in E$. We say that f tends to p at t in the affine space (0) and we write

$$f(x) \underset{x \longrightarrow t}{\longrightarrow} p \qquad (\text{in } (E, V, \rightarrow)) \tag{9}$$

iff for any $U \in \mathrm{top}(E, V, \rightarrow)$ such that $p \in U$ there exists $\delta > 0$ for which $f(x) \in U$ whenever $0 < |x - t| < \delta$. It is easy to prove the following

Proposition 1

For any function f with $D_f \subset \mathbb{R}$, $fD_f \subset E$, any $t \in \mathbb{R}$ and $p \in E$ we have (9) *if and only if for any base B of vector space V and any $e \in B$ we have*

$$\overrightarrow{pf(x)}_B(e) \underset{x \longrightarrow t}{\longrightarrow} 0. \tag{10}$$

For any vector space V we have well defined the affine space $\mathrm{aff}\, V$ as $(\underline{V}, V, \rightarrow)$, where $\overrightarrow{\mathrm{vw}} = \mathrm{w} - \mathrm{v}$ for $\mathrm{v},\mathrm{w} \in \underline{V}$. Let $D_f \subset \mathbb{R}$ and $fD_f \subset E$. Setting

$$f' = \left\{(t, \mathrm{v});\ t \in D_f \cap (D_f)'\ \&\ \tfrac{1}{x-t}\,\overrightarrow{f(t)f(x)} \underset{x \longrightarrow t}{\longrightarrow} \mathrm{v}\ (\text{in } \mathrm{aff}\, V)\right\}, \tag{11}$$

where for any set $A \subset \mathbb{R}$, A' denotes the set of all cluster points of A, we have defined the derivative function f' of a function f. A function $f\colon D_f \to E$, $D_f \subset \mathbb{R}$, is differentiable iff

$$D_{f'} = D_f. \tag{12}$$

Denoting the natural topology of $\mathbb{R}$ by $\mathcal{R}$ we have the topology $\mathcal{R}|D_f$. The function f satisfying (12) and having the continuous derivative function f' from $\mathcal{R}|D_f$ to top aff V is said to be *smooth* in $(E, V, \to)$.

1. Runs, o-turns, and smooth oriented angles

Before introducing the concept of smooth oriented angle in an arbitrary affine space we introduce a concept of a run and a turn. Any function f smooth in $(E, V, \to)$ with $D_f = \langle a; b\rangle$, $a < b$, is said to be a *run* in $(E, V, \to)$. Let $o \in E$. Any run f satisfying one of the following conditions:

$$f(t) = f(u) \neq o \qquad \text{for } t, u \in D_f, \tag{$o1f$}$$

or

$$f'(t),\ \overrightarrow{of(t)} \text{ are linearly independent for } t \in D_f, \tag{$o2f$}$$

is said to be an *o-turn* in $(E, V, \to)$. The set of all o-turns in $(E, V, \to)$ will be denoted by $T_o(E, V, \to)$. In this set we introduce an equivalence $\equiv_o$ setting $f \equiv_o g$ iff $f, g \in T_o(E, V, \to)$ and there exist real smooth functions λ and φ such that

(i) $D_\varphi = D_\lambda = D_f$ and $\varphi D_\varphi = D_g$,

(ii) $\lambda(t) > 0$, $\varphi'(t) > 0$ and $\overrightarrow{og(\varphi(t))} = \lambda(t)\overrightarrow{of(t)}$ for $t \in D_f$.

Denoting by $T_o(E, V, \to)/\equiv_o$ the set of all cosets in $T_o(E, V, \to)$ given by the equivalence $\equiv_o$ we may define the set $\operatorname{soa}(E, V, \to)$ by the equality

$$\operatorname{soa}(E, V, \to) = \bigcup_{o \in E} T_o(E, V, \to)/\equiv_o .$$

Any element of this set is said to be a *smooth oriented angle* in the affine space $(E, V, \to)$.

PROPOSITION 2
For any $o \in E$, $\mathfrak{a} \in T_o(E, V, \to)/\equiv_o$ and $g \in \mathfrak{a}$ we have

$$\underline{\mathfrak{a}} = \bigcup_{p \in gD_g} (o\,p\,\infty),$$

where

$$\underline{\mathfrak{a}} = \bigcup_{f \in \mathfrak{a}} fD_f \qquad \text{and} \qquad (o\,p\,\infty) = \{o + t\,\overrightarrow{op};\ t > 0\}.$$

Proof. Let $f \in \mathfrak{a}$. We have then $f \equiv_o g$. Taking any $q \in fD_f$ we get $q = f(t)$, $t \in D_f$. Then there exist functions λ, φ such that (i) and (ii) hold. Setting $p = g(\varphi(t))$ we get $\overrightarrow{oq} = \frac{1}{\lambda(t)}\overrightarrow{op}$, which yields $q \in (o\,p\,\infty)$, where $p \in gD_g$. Now, let $q \in (o\,p\,\infty)$, where $p \in gD_g$. We have then $\overrightarrow{oq} = s\,\overrightarrow{op}$, where $p = g\,(u)$, $u \in D_g$ and $s > 0$. Setting $D_f = D_g$ and $f(t) = o + s\,\overrightarrow{og(t)}$ for $t \in D_f$ we get $f \equiv_o g$ and $q = o + s\,\overrightarrow{op} = o + s\,\overrightarrow{og(u)} = f(u) \in fD_f$, so $(o\,p\,\infty) \subset \underline{\mathfrak{a}}$.

PROPOSITION 3
For any $o \in E$ *and* $\mathfrak{a} \in T_o(E, V, \overrightarrow{\ })/\equiv_o$ *if* $o \in U \in \operatorname{top}(E, V, \overrightarrow{\ })$*, then there exists* $g \in \mathfrak{a}$ *such that* $gD_g \subset U$.

Proof. Let $f \in \mathfrak{a}$ and $s > 0$. Setting $D_{f_s} = D_f$ and

$$f_s(t) = o + s\,\overrightarrow{of}\,(t) \qquad \text{for } t \in D_f$$

we have, of course, $f_s \equiv_o f$, so $f_s \in \mathfrak{a}$. We will prove that

for any halfspace P with $o \in P_+$ there exists $\varepsilon > 0$ such that for any $s \in (0; \varepsilon)$ the relation $f_s D_f \subset P_+$ holds. $(\star)$

Let P be a halfspace such that $o \in P_+$. Then we have $P = o + \underline{W} + \langle -\beta; +\infty)\,e$, where $W \leq_1 V$, $e \in \underline{V} \setminus \underline{W}$ and $\beta > 0$. Then $P_+ = o + \underline{W} + (-\beta; +\infty)e$. For any $t \in D_f$ we have $\overrightarrow{of(t)} = \mathrm{w}(t) + \mu\,(t)\,e$. From continuity of f by Proposition 1 it follows that μ is continuous. Thus, μ is bounded. So, there exists $m > 0$ such that $|\mu(t)| < m$ for $t \in D_f$. Hence it follows that $\overrightarrow{of_s(t)} = s\,\mathrm{w}(t) + s\,\mu(t)e \in \underline{W} + (-sm; +\infty)e$, so $f_s(t) \in o + \underline{W} + (-sm; +\infty)e \subset P_+$ for $t \in D_f$, as $0 < s < \frac{\beta}{m}$.

Now, assume that $o \in U \in \operatorname{top}(E, V, \overrightarrow{\ })$. Then there exist halfspaces $P_1, \ldots, P_n$ such that $o \in P_{1+} \cap \ldots \cap P_{n+} \subset U$. By $(\star)$ for any $j \in \{1, \ldots, n\}$ we get $\varepsilon_j > 0$ such that $f_s D_f \subset P_{j+}$ as $s \in (0; \varepsilon_j)$. Setting $g = f_s$, where $0 < s < \min\{\varepsilon_1, \ldots, \varepsilon_n\}$, we get $gD_g \subset U$.

PROPOSITION 4
If $o, q \in E$ *and* $\mathfrak{a} \in T_o(E, V, \overrightarrow{\ })/\equiv_o \cap\, T_q(E, V, \overrightarrow{\ })/\equiv_q$*, then* $o = q$.

Proof. Let us suppose that $o \neq q$. Take any $U \in \operatorname{top}(E, V, \overrightarrow{\ })$ such that $q \in U$. Since $\mathfrak{a} \in T_q(E, V, \overrightarrow{\ })/\equiv_q$, by Proposition 3 there exists $g \in \mathfrak{a}$ such that $gD_g \subset U$. From the condition $\mathfrak{a} \in T_o(E, V, \overrightarrow{\ })/\equiv_o$ it follows that $\mathfrak{a} \subset T_o(E, V, \overrightarrow{\ })$. Therefore $g \in T_o(E, V, \overrightarrow{\ })$, so $gD_g \subset U \setminus \{o\}$, and by Proposition 2 we get

$$\underline{\mathfrak{a}} \subset A \qquad \text{where } A = \bigcap_{q \in U \in \operatorname{top}(E, V, \overrightarrow{\ })} \; \bigcup_{p \in U \setminus \{o\}} (o\,p\,\infty).$$

Now, we will prove that $A \subset (o\,q\,\infty)$. Assume that there exists a point $x \in A \setminus (o\,q\,\infty)$. Let us set $C = \{\overrightarrow{oq}, \overrightarrow{ox}\}$, whenever $\overrightarrow{ox}$ and $\overrightarrow{oq}$ are linearly independent and $C = \{\overrightarrow{oq}\}$ in the opposite case. Then there exists a base B of V with $C \subset B$. Let W be the vector subspace of V generated by $B \setminus \{e\}$, where $e = \overrightarrow{oq}$. Let us set

$$P = o + \underline{W} + \mathbb{R}_+ e.$$

So, we have $P^o = o + \underline{W}$ and $P_+ = o + \underline{W} + (0;+\infty)e$. First, we suppose that $\overrightarrow{ox}$ and $\overrightarrow{oq}$ are linearly independent. Then $x = o + \overrightarrow{ox} \in o + \underline{W} = P^o$. If we assume that $x \in \bigcup_{p\in P_+}(o\,p\,\infty)$, then we get $p \in P_+$ with $x \in (o\,p\,\infty)$. Then it should be in turn, $p = o + w + te$, $w \in \underline{W}$, $t > 0$, $x = o + u\,\overrightarrow{op}$, $u > 0$, $x = o + uw + ute \in P_+$, which is impossible. Therefore we have $x \notin \bigcup_{p\in P_+}(o\,p\,\infty) \supset A$. So, $\overrightarrow{ox}$ and $\overrightarrow{oq}$ should be linearly dependent. Thus, $\overrightarrow{ox} = a \cdot \overrightarrow{oq}$, $a \in \mathbb{R}$. Because of $x \notin (o\,q\,\infty)$ we get $a \leq 0$. Thus $x \in P_-$. By definition of P_- we have

$$P_- \cap \bigcup_{p\in P_+}(o\,p\,\infty) = \emptyset,$$

what yields $x \notin A$. So, we have $A \subset (o\,q\,\infty)$. Hence it follows that $\underline{\mathfrak{a}} \subset (o\,q\,\infty)$ and similarly $\underline{\mathfrak{a}} \subset (q\,o\,\infty)$. By Proposition 2 we get $(o\,p\,\infty) \subset \underline{\mathfrak{a}}$ for some $p \in gD_g$. This yields $(o\,p\,\infty) \subset (o\,q\,\infty) \cap (q\,o\,\infty)$, which is impossible.

The point $o \in E$ such that $\mathfrak{a} \in T_o(E, V, \overrightarrow{\ })/\equiv_o$ is called the *vertex* of $\mathfrak{a}$.

Notice that if $f, g \in \mathfrak{a} \in T_o(E, V, \overrightarrow{\ })/\equiv_o$, $D_f = \langle a; b\rangle$, and $D_g = \langle c; d\rangle$, then $\langle o\,f(a)\,\infty) = \langle o\,g(c)\,\infty)$ and $\langle o\,f(b)\,\infty) = \langle o\,g(d)\,\infty)$, where

$$\langle o\,p\,\infty) = \{o + s\,\overrightarrow{op};\ s \geq 0\} \qquad \text{for } p \in E. \tag{13}$$

The sets $\langle o\,f(a)\,\infty)$ and $\langle o\,f(b)\,\infty)$ we called the *former side* and the *latter one* of $\mathfrak{a}$, respectively.

2. Oriented angles

Consider any affine space (0) and any $o \in E$. The set of all functions L such that D_L is a closed segment in $\mathbb{R}$ and there exists a function f with $D_f = D_L$, continuous from $\mathcal{R}|D_f$ to $\operatorname{top}(E, V, \overrightarrow{\ })$ such that for any $t \in D_f$ we have

$$o \neq f(t) \qquad \text{and} \qquad L(t) = \langle o\,f(t)\,\infty)\,, \tag{L}$$

$\langle o\,f\,(t)\,\infty)$ is defined by (13), and one of the following two conditions

(1 L) $L(t) = L(u)$ for $t, u \in D_L$,

$(2\,L)$ for any $t \in D_L$ there exists $\delta > 0$ for which

$$L|D_L \cap (t-\delta; t+\delta) \text{ is 1–1},$$

is satisfied will be denoted by $\langle o; E, V, \overset{\rightarrow}{} \rangle$. We set

$$\langle E, V, \overset{\rightarrow}{} \rangle = \bigcup_{o \in E} \langle o; E, V, \overset{\rightarrow}{} \rangle$$

and $L \equiv M$ iff $L, M \in \langle E, V, \overset{\rightarrow}{} \rangle$ and there exists a real continuous increasing function φ such that $D_\varphi = D_L$, $\varphi D_\varphi = D_M$ and $M \circ \varphi = L$. It is easy to see that $\equiv$ is an eqiuvalence.

Elements of the set $\langle E, V, \overset{\rightarrow}{} \rangle / \equiv$ of all cosets of $\equiv$ will be called *oriented angles* in the affine space (0). The point o such that the equality in (L) is satisfied depending only on the oriented angle for which L belongs is called the *vertex* of this oriented angle. Any oriented angle for which constant function L belongs is said to be zero angle in the affine space (0).

Proposition 5
For any smooth oriented angle $\mathfrak{a}$ in the affine space (0) *we have the oriented angle $<\mathfrak{a}>$ well defined by the formula*

$$<\mathfrak{a}> = [f_o] \tag{14}$$

where $f_o(t) = \langle o\, f(t)\, \infty \rangle$ for $t \in D_f$, $f \in \mathfrak{a} \in T_o(E, V, \overset{\rightarrow}{})/ \equiv_o$, $L \in [L] \in \langle E, V, \overset{\rightarrow}{} \rangle / \equiv$ for $L \in \langle E, V, \overset{\rightarrow}{} \rangle$. The function

$$\operatorname{soa}(E, V, \overset{\rightarrow}{}) \ni \mathfrak{a} \longmapsto <\mathfrak{a}> \tag{15}$$

is 1–1. *If* $\dim V > 2$, *then there exists an oriented angle in* (0) *which is not of the form $<\mathfrak{a}>$, where $\mathfrak{a}$ is a smooth oriented angle in* (0).

Lemma
If l_1, l_2 are real functions, f_1, f_2 are vector ones with $D_{l_1} = D_{l_2} = D_{f_1} = D_{f_2} \subset \mathbb{R}$, $f_j(x) \xrightarrow[x \longrightarrow t]{} e_j$ (in $\operatorname{aff}(V)$), $j \in \{1, 2\}$, e_1, e_2 are linearly independent in V and

$$l_1(x) f_1(x) + l_2(x) f_2(x) \xrightarrow[x \longrightarrow t]{} \mathrm{v} \quad (\textit{in } \operatorname{aff} V),$$

then there exist reals c_1, c_2 such that $l_j(x) \xrightarrow[x \longrightarrow t]{} c_j$, $j \in \{1, 2\}$.

Proof. There exists a base B in V containing $\{e_1, e_2\}$. By Proposition 1 we have $g_i(x) \xrightarrow[x \longrightarrow t]{} \mathrm{v}_B(e_i)$ where

$$g_i(x) = l_1(x) f_1(x)_B(e_i) + l_2(x) f_2(x)_B(e_i) \tag{16}$$

and

$$f_j(x)_B(e_i) \xrightarrow[x \longrightarrow t]{} e_{jB}(e_i) = \delta_{ji} \qquad (\delta_{ji} \text{ — Kronecker's delta}),$$

so $\det[f_j(x)_B(e_i);\ i,j \le 2] \xrightarrow[x\longrightarrow t]{} 1$. Therefore, by (16),

$$l_1(x) = \begin{vmatrix} g_1(x) & f_2(x)_B(e_1) \\ g_2(x) & f_2(x)_B(e_2) \end{vmatrix} m(x) \xrightarrow[x\longrightarrow t]{} \begin{vmatrix} \mathrm{v}_B(e_1) & \delta_{21} \\ \mathrm{v}_B(e_2) & \delta_{22} \end{vmatrix} = c_1$$

and

$$l_2(x) = \begin{vmatrix} f_1(x)_B(e_1) & g_1(x) \\ f_1(x)_B(e_2) & g_2(x) \end{vmatrix} m(x) \xrightarrow[x\longrightarrow t]{} \begin{vmatrix} \delta_{11} & \mathrm{v}_B(e_1) \\ \delta_{12} & \mathrm{v}_B(e_2) \end{vmatrix} = c_2,$$

where $m(x) = 1/\det[f_j(x)_B(e_i);\ i,j \le 2]$ and $c_i = \mathrm{v}_B(e_i)$.

Proof of Proposition 5. Correctness of the definition of $<\mathfrak{a}>$ by (14) is evident. To prove that (15) is 1–1 assume that $<\mathfrak{a}> = <\mathfrak{b}>$, where $\mathfrak{a} \in T_o(E,V,\overset{\longrightarrow}{})/\equiv_o$ and $\mathfrak{b} \in T_q(E,V,\overset{\longrightarrow}{})/\equiv_q$. We have (14) and

$$<\mathfrak{b}> = [g_q], \qquad \text{where } g_q(u) = \langle q\, g(u)\, \infty\rangle \text{ for } u \in D_g,\ g \in \mathfrak{b}. \tag{14'}$$

By definition of $\equiv$ we get a continuous increasing function φ such that $D_\varphi = D_f$, $\varphi D_\varphi = D_g$ and $g_q \circ \varphi = f_o$, i.e. by (14) and (14′), $\langle q\, g(\varphi(t))\, \infty\rangle = \langle o\, f(t)\, \infty\rangle$ for $t \in D_f$. Hence $q = o$ and for any $t \in D_f$ there is

$$\lambda(t) > 0 \qquad \text{with } \overrightarrow{og(\varphi(t))} = \lambda(t)\,\overrightarrow{of(t)}. \tag{17}$$

This yields, in turn,

$$\lambda(t+s)\,\overrightarrow{of(t+s)} = \overrightarrow{og(\varphi(t+s))} \xrightarrow[s\longrightarrow 0]{} \overrightarrow{og(\varphi(t))} = \lambda(t)\,\overrightarrow{of(t)}$$

and

$$\overrightarrow{of(t+s)} \xrightarrow[s\longrightarrow 0]{} \overrightarrow{of(t)} \neq 0.$$

According to Lemma we get $\lambda(t+s) \xrightarrow[s\longrightarrow 0]{} \lambda(t)$. So, λ is continuous. We have also

$$\tfrac{1}{s}(\varphi(t+s)-\varphi(t)) \cdot \tfrac{1}{\varphi(t+s)-\varphi(t)}\,\overrightarrow{g(\varphi(t))g(\varphi(t+s))} - \tfrac{1}{s}(\lambda(t+s)-\lambda(t))\,\overrightarrow{of(t)}$$
$$= \lambda\,(t+s)\cdot \tfrac{1}{s}\,\overrightarrow{f(t)\ f(t+s)},$$

$$\tfrac{1}{\varphi(t+s)-\varphi(t)}\,\overrightarrow{g(\varphi(t))g(\varphi(t+s))} \xrightarrow[s\longrightarrow 0]{} g'(\varphi(t))$$

and

$$\tfrac{1}{s}\,\overrightarrow{f(t)f(t+s)} \xrightarrow[s\longrightarrow 0]{} f'(t).$$

First, we consider the case when o–turns f and g satisfy conditions $(o2f)$ and $(o2g)$, respectively. Then by Lemma we have

$$\frac{\varphi(t+s)-\varphi(t)}{s} \xrightarrow[s\longrightarrow 0]{} \varphi'(t) \qquad \text{and} \qquad \frac{\lambda(t+s)-\lambda(t)}{s} \xrightarrow[s\longrightarrow 0]{} \lambda'(t).$$

Thus,

$$\varphi'(t)g'(\varphi(t)) - \lambda'(t)\,\overrightarrow{of(t)} = \lambda(t)f'(t) \qquad \text{for } t\in D_f. \tag{18}$$

From the fact that φ is increasing it follows that $\varphi'(t)\geq 0$. By $(o2f)$ we have $\varphi'(t)>0$. According to Lemma by (18) and $(o2f)$ we conclude that the functions φ' and λ' are continuous. In other words, φ and λ are smooth. So, $f\equiv_o g$ and we have $\mathfrak{a}=\mathfrak{b}$.

Now, let us assume $(o1f)$. Setting $\overrightarrow{of(t)}=e$, by (17), we get $\overrightarrow{og(u)}=\mu(u)e$, where $\mu(u)=\lambda(\varphi^{-1}(u))$ for $u\in D_g$. Thus

$$\tfrac{1}{s}\left(\mu(u+s)-\mu(u)\right)\cdot e = \tfrac{1}{s}\,\overrightarrow{g(u)g(u+s)} \xrightarrow[s\longrightarrow 0]{} g'(u).$$

By Lemma we get $g'(u)=\mu'(u)e$. Hence it follows that $g'(u)$, $\overrightarrow{og(u)}$ are not linearly independent. Therefore $(o1g)$ holds. Thus, taking any $u,u_1\in D_g$ by (17) we get $\mu(u_1)e=\overrightarrow{og(u_1)}=\overrightarrow{og(u)}=\mu(u)e$, and $\mu(u)=\mu(u_1)$, which yields $g\equiv_o f$, i.e. $\mathfrak{a}=\mathfrak{b}$. Therefore (15) is 1–1.

Assuming that $\dim V>2$ we get three vectors e_1, e_2, e_3 linearly independent in V. Let us set

$$\overrightarrow{og(u)} = \begin{cases} e_1+u(e_2-e_1), & \text{when } 0\leq u\leq 1,\\ e_2+(u-1)(e_3-e_2), & \text{when } 1<u\leq 2,\end{cases}$$

and $L(u)=\langle o\,g(u)\,\infty)$ for $u\in\langle 0;2\rangle$. Let us suppose that there exists $f\in T_o(E,V,\overrightarrow{\ })$ such that $[L]=[f_o]$, where $f_o(t)=\langle o\,f\,(t)\,\infty)$ for $t\in D_f$. Then there exist a continuous and increasing function φ for which $D_\varphi=D_f$, $L\circ\varphi=f_o$, $\varphi D_\varphi=D_L=\langle 0;2\rangle$. Thus, for some function λ with $D_\lambda=D_\varphi$ (17) holds. Let us set $t_1=\varphi^{-1}(1)$. Hence it follows that $\overrightarrow{of(t)}=\alpha_1(t)e_1+\alpha_2(t)e_2$ as $t\in D_f$, $t\leq t_1$ and $\overrightarrow{of(t)}=\beta_2(t)e_2+\beta_3(t)e_3$ as $t\in D_f$, $t\geq t_1$, where α_1, α_2, β_2, β_3 are real functions. Thus, by Lemma we get

$$f'(t_1)=\alpha_1'(t_1)e_1+\alpha_2'(t_1)e_2=\beta_2'(t_1)e_2+\beta_3'(t_1)e_3.$$

Then $\alpha_1'(t_1)=0=\beta_3'(t_1)$. So, $f'(t_1)=\alpha_2'(t_1)e_2$. On the other hand,

$$\overrightarrow{of(t_1)} = \tfrac{1}{\lambda(t_1)}\,\overrightarrow{og(\varphi(t_1))} = \tfrac{1}{\lambda(t_1)}\,\overrightarrow{og(1)} = \tfrac{1}{\lambda(t_1)}e_2.$$

The vectors $f'(t_1)$ and $\overrightarrow{of(t_1)}$ are linearly dependent. So, $(o2f)$ does not hold. Therefore $(o1f)$ is satisfied, which yields $\overrightarrow{og(\varphi(t))}=\lambda(t)\,\overrightarrow{of(t_1)}$ for $t\in D_\varphi$, i.e. $\overrightarrow{og(u)}=\lambda(\varphi^{-1}(u))\,\overrightarrow{of(t_1)}$ for $u\in\langle 0;2\rangle$, which is impossible.

3. Oriented angles in an Euclidean plane

Let us consider an Euclidean plane, i.e. an affine space (0), $\dim V = 2$, together with a positively defined scalar product $\underline{V} \times \underline{V} \ni (\mathrm{v},\mathrm{w}) \mapsto \mathrm{v}\cdot\mathrm{w} \in \mathbb{R}$. For any $\mathrm{v} \in \underline{V}$ we set $|\mathrm{v}| = \sqrt{\mathrm{v}\cdot\mathrm{v}}$ and for any function f defined on the segment of $\mathbb{R}$ with values in E we set $D_f = \langle a; b\rangle$ and for $t \in D_f$

$$|f|\,(t) = \sup\left\{\sum_{i=0}^{k}\left|\overrightarrow{f(t_i)f(t_{i+1})}\right|;\ a = t_0 < \ldots < t_k = t \ \&\ k \in \mathbb{N}\right\}. \tag{19}$$

The function $|f|$ defined by (19) has values in $\mathbb{R} \cup \{+\infty\}$, in general.

PROPOSITION 6
In the Euclidean plane for any oriented angle $\mathcal{A} \in \langle E, V, ^{\rightarrow}\rangle/\equiv$ there exists a unique continuous function $f\colon D_f \to E$ such that $D_f = \langle 0; c\rangle$, $c > 0$, $\langle o\, f\,(\cdot)\ \infty\rangle \in \mathcal{A}$,

$$\left|\overrightarrow{of(s)}\right| = 1 \qquad \text{for } s \in D_f, \tag{20}$$

o is a vertex of $\mathcal{A}$, and one of the following conditions

$$|f|\,(s) = 0 \qquad \text{for } s \in D_f, \tag{0; f}$$

$$|f|\,(s) = s \qquad \text{for } s \in D_f \tag{1; f}$$

is satisfied. We have $f \in \mathfrak{a} \in T_o(E, V, ^{\rightarrow})/\equiv_o$ and $\mathcal{A} = <\mathfrak{a}>$, where $<\mathfrak{a}>$ is the oriented angle defined by (14).

Proof. Let $L \in \mathcal{A} \in \langle E, V, ^{\rightarrow}\rangle/\equiv$. Then there exists a continuous function h such that $D_L = D_h = \langle a; b\rangle$ and $L(t) = \langle o\, h(t)\, \infty\rangle$ for $t \in D_h$. We consider two cases. First, when $(1\,L)$ is satisfied. Then, setting $c = b - a$ and

$$f(s) = o + \frac{1}{\left|\overrightarrow{oh(a+s)}\right|}\,\overrightarrow{oh(a+s)} \qquad \text{for } s \in \langle 0; c\rangle$$

we see that

$$f(s) = f(t) \qquad \text{for } s, t \in D_f \tag{21}$$

and

$$\langle o\, f\,(\cdot)\ \infty\rangle = (s \mapsto L(a+s)) \in \mathcal{A}.$$

The condition $(0; f)$ holds in this case. From $(0; f)$ it follows (21). In the second case we assume $(2\,L)$. Thus, for any $t \in D_h$ we have $\delta_t > 0$ such that the function $L|D_L \cap (t - \delta_t; t + \delta_t)$ is 1–1. Then there exist $\tau_1, \ldots, \tau_l \in D_L$ such that $\tau_1 < \ldots < \tau_l$ and $D_L \subset \bigcup_{j=1}^{l}(a_j;b_j)$, where $a_j = \tau_j - \frac{\delta_{\tau_j}}{2}$, $b_j = \tau_j + \frac{\delta_{\tau_j}}{2}$. We have then 1–1 functions

$$L|D_L \cap \langle a_j;b_j\rangle\,, \qquad j \in \{1,\ldots,l\}\,.$$

Setting, $g(t) = o + \frac{1}{\left|\overrightarrow{oh(t)}\right|}\overrightarrow{oh(t)}$ we get $\left|\overrightarrow{og(t)}\right| = 1$ and $L(t) = \langle o\, g\,(t)\ \infty)$ for $t \in D_L$ and 1–1 functions $g|D_g \cap \langle a_j;b_j\rangle$, $D_g = D_L$. We may assume that $a_1 = a$ and $b_l = b$, so $D_L \cap \langle a_j;b_j\rangle = \langle a_j;b_j\rangle$ and setting $g_j = g|\,\langle a_j;b_j\rangle$ we get

$$|g_j|\,(t) \le 2\pi \qquad \text{for } t \in \langle a_j;b_j\rangle\,.$$

Hence it follows that for any $t \in D_g$ we have

$$|g|\,(t) \le |g|\,(b) \le \sum_{j=1}^{l} |g_j|\,(b_j) \le 2l\pi < +\infty.$$

Then the function $|g|$ is finite continuous and increasing. Taking the inverse function $|g|^{-1}$ to $|g|$ and setting $f = g \circ |g|^{-1}$ we get the continuous function f with $D_f = \langle 0;c\rangle$, where $c = |g|\,(b)$. It is easy to see that $|f|$ is continuous and increasing and $L\left(|g|^{-1}\,(s)\right) = \langle o\, f(s)\,\infty)$ for $s \in D_f$. Therefore, we have $(1;f)$ and $\langle o\, f\,(\cdot)\ \infty) = L \circ |g|^{-1} \equiv L$, so $\langle o\, f(\cdot)\,\infty) \in \mathcal{A}$. From (20) and $(1;f)$ it follows that there exist orthonormal vectors $e_1, e_2 \in \underline{V}$ such that

$$\overrightarrow{of(s)} = \cos s \cdot e_1 + \sin s \cdot e_2 \qquad \text{for } s \in D_f.$$

Thus f is smooth. Taking $\mathfrak{a} \in T_o(E,V,\rightarrow)/\equiv_o$ such that $f \in \mathfrak{a}$ we get $\mathcal{A} = <\mathfrak{a}>$.

To prove that f is uniquely determined we take a continuous function $f_1\colon D_{f_1} \to E$ with $D_{f_1} = \langle 0;c_1\rangle$, $c_1 > 0$, $\langle o\, f_1\,(\cdot)\ \infty) \in \mathcal{A}$, $\left|\overrightarrow{of_1(t)}\right| = 1$ for $t \in D_{f_1}$ and satisfying $(0;f_1)$ or $(1;f_1)$. Then there exists a real continuous increasing function φ such that $\underline{D_\varphi = D_f}$ and $\underline{\varphi D_\varphi = D_{f_1}}$ and $\langle o\, f_1(\varphi(s))\,\infty) = \langle o\, f(s)\,\infty)$ for $s \in D_f$. Thus, $\overrightarrow{of_1(\varphi(s))} = \lambda(s)\,\overrightarrow{of(s)}$, where $\lambda\,(s) > 0$ for $s \in D_f$. Hence it follows that $1 = \left|\overrightarrow{of_1(\varphi(s))}\right| = \lambda(s)\left|\overrightarrow{of(s)}\right| = \lambda(s)$, so $f_1 \circ \varphi = f$. This yields $|f_1| \circ |\varphi| = |f|$. If $(0;f_1)$ holds, then $|f_1| = 0$, so $|f| = 0$. If $(1;f_1)$ is satisfied, then $\varphi = |f| = \mathrm{id}_{\langle 0;c\rangle}$. Therefore $f_1 = f$.

Corollary

If (0) *is an affine plane, i.e.* $\dim V = 2$*, then the function in* (15) *is* 1–1 *and maps* $\mathrm{soa}(E,V,\rightarrow)$ *onto* $\langle E,V,\rightarrow)/\equiv$.

Indeed, taking any positively defined scalar product in V we get an Euclidean space and we may apply Proposition 6.

4. Conclusion

The case when the affine space is 1-dimensional is not of importance however from purely logical point of view the definition of the set $\langle E, V, \xrightarrow{}\rangle / \equiv$ is correct.

Remark
If the affine space (0) is 1-dimensional, then all elements of $\langle E, V, \xrightarrow{}\rangle / \equiv$ are zero angles and (15) is 1–1 and maps $\mathrm{soa}(E, V, \xrightarrow{})$ onto $\langle E, V, \xrightarrow{}\rangle / \equiv$.

Indeed, for any $\mathcal{A} \in \langle E, V, \xrightarrow{}\rangle / \equiv$ there is $L \in \mathcal{A}$, so $L(t) = \langle o\, f(t)\, \infty)$ and $o \neq f(t)$ for $t \in D_L$, where $f\colon D_L \to E$ is continuous and $(1\,L)$ or $(2\,L)$ holds. Let $0 \neq e \in \underline{V}$. Then $\overrightarrow{of(t)} = \lambda(t)e$, $0 \neq \lambda(t) \in \mathbb{R}$. According to Lemma λ is continuous. Thus $\lambda(t) > 0$ for $t \in D_L$ or $\lambda(t) < 0$ for $t \in D_L$. We may assume that $\lambda(t) > 0$. Therefore $L(t) = \langle o\, p\, \infty)$, where $p = o + e$. Setting $f_1(t) = p$ for $p \in D_L$ we get a smooth function f_1 for which $L(t) = \langle o\, f_1(t)\, \infty)$ as $t \in D_L$. Then we have $(1\,L)$. For $\mathfrak{a} \in T_o(E, V, \xrightarrow{})/\equiv_o$ such that $f_1 \in \mathfrak{a}$ we get $<\mathfrak{a}> = \mathcal{A}$.

Proposition 5, Corollary to Proposition 6 and the above Remark allows us to conclude our consideration by

Theorem
For any affine space (0) *the function* (15) *is* 1–1. *This function maps the set* $\mathrm{soa}(E, V, \xrightarrow{})$ *of all smooth oriented angles in the affine space* (0) *onto the set* $\langle E, V, \xrightarrow{}\rangle / \equiv$ *of all oriented angles in* (0) *if and only if* $\dim V = 2$ *or* $\dim V = 1$.

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