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Strong maximum principles for implicit parabolic functional-differential problems together with initial inequalities

Dedicated to Professor Andrzej Zajtz on the occasion of his 70th birthday

Abstract. The aim of the paper is to give strong maximum principles for implicit non-linear parabolic functional-differential problems together with initial inequalities in relatively arbitrary $(n+1)$-dimensional time - space sets more general than cylindrical domain.

1. Introduction

In this paper we consider implicit diagonal systems of non-linear parabolic functional-differential inequalities of the form

$$\begin{aligned} &F^i(t,x,u(t,x),u^i_t(t,x),u^i_x(t,x),u^i_{xx}(t,x),u) \\ &\qquad \geq F^i(t,x,v(t,x),v^i_t(t,x),v^i_x(t,x),v^i_{xx}(t,x),v) \\ &\qquad\qquad (i=1,\dots,m) \end{aligned} \tag{1.1}$$

for $(t,x)=(t,x_1,\dots,x_n)\in D$, where $D\subset(t_0,t_0+T]\times\mathbb{R}^n$ is one of three relatively arbitrary sets more general than the cylindrical domain $(t_0,t_0+T]\times D_0\subset\mathbb{R}^{n+1}$. The symbol w $(=u$ or $v)$ denotes the mapping

$$w\colon \tilde{D}\ni(t,x)\longrightarrow w(t,x)=(w^1(t,x),\dots,w^m(t,x))\in\mathbb{R}^m,$$

where $\tilde{D}$ is an arbitrary set contained in $(-\infty,t_0+T]\times\mathbb{R}^n$ such that $\bar{D}\subset\tilde{D}$; F^i $(i=1,\dots,m)$ are functionals of w; $w^i_x(t,x)=\mathrm{grad}_x w^i(t,x)$ $(i=1,\dots,m)$ and $w^i_{xx}(t,x)$ $(i=1,\dots,m)$ denote the matrices of second order derivatives with respect to x of $w^i(t,x)$ $(i=1,\dots,m)$. We give a lemma and a theorem on strong maximum principles for problems together with inequalities of types (1.1) and with initial inequalities.

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The results obtained are a generalization of some results given by R. Redheffer and W. Walter [4], by J. Szarski [5] and [6], by P. Besala [1], by W. Walter [8], by N. Yoshida [9], by the author [2] and [3], and base on those results. To prove the results of this paper we use the theorem on a strong maximum principle from [2].

2. Preliminaries

The notation and definitions given in this section are valid throughout this paper. Some of them are similar to those applied by J. Szarski [7], [6], by R. Redheffer and W. Walter [4], by P. Besala [1], by N. Yoshida [9] and by the author [3].

We use the following notation:

$$\mathbb{R} = (-\infty, \infty), \qquad \mathbb{N} = \{1, 2, \ldots\}, \qquad x = (x_1, \ldots, x_n)\ (n \in \mathbb{N}).$$

For any vectors $z = (z^1, \ldots, z^m) \in \mathbb{R}^m$, $\tilde{z} = (\tilde{z}^1, \ldots, \tilde{z}^m) \in \mathbb{R}^m$ we write

$$z \leq \tilde{z} \qquad \text{if } z^i \leq \tilde{z}^i \ (i = 1, \ldots, m).$$

Let t_0 be a real finite number and let $0 < T < \infty$. A set

$$D \subset \{(t, x) : \ t > t_0,\ x \in \mathbb{R}^n\}$$

(bounded or unbounded) is called a *set of type* (P) if:

(a) The projection of the interior of D on the t-axis is the interval $(t_0, t_0 + T)$.

(b) For every $(\tilde{t}, \tilde{x}) \in D$ there is a positive r such that

$$\left\{(t, x) : \ (t - \tilde{t})^2 + \sum_{i=1}^{n} (x_i - \tilde{x}_i)^2 < r,\ t < \tilde{t}\right\} \subset D.$$

We define the following sets:

$$S_{t_0} = \operatorname{int}\{x \in \mathbb{R}^n : \ (t_0, x) \in \bar{D}\} \qquad \text{and} \qquad \sigma_{t_0} = \operatorname{int}[\bar{D} \cap (\{t_0\} \times \mathbb{R}^n)].$$

Let $\tilde{D}$ be a set contained in $(-\infty, t_0 + T] \times \mathbb{R}^n$ such that $\bar{D} \subset \tilde{D}$. We introduce the following sets:

$$\partial_p D := \tilde{D} \setminus D \qquad \text{and} \qquad \Gamma := \partial_p D \setminus \sigma_{t_0}.$$

For an arbitrary fixed point $(\tilde{t}, \tilde{x}) \in D$ we denote by $S^-(\tilde{t}, \tilde{x})$ the set of points $(t, x) \in D$ that can be joined to $(\tilde{t}, \tilde{x})$ by a polygonal line contained in D along which the t-coordinate is weakly increasing from (t, x) to $(\tilde{t}, \tilde{x})$.

Let $Z_m(\tilde{D})$ denote the space of mappings

$$w\colon \tilde{D} \ni (t,x) \longrightarrow w(t,x) = (w^1(t,x),\dots,w^m(t,x)) \in \mathbb{R}^m$$

continuous in $\bar{D}$.

In the set of mappings bounded from above in $\tilde{D}$ and belonging to $Z_m(\tilde{D})$ we define the functional

$$[w]_t = \max_{i=1,\dots,m} \sup\{0, w^i(\tilde{t},x) : \ (\tilde{t},x) \in \tilde{D},\ \tilde{t} \le t\}, \qquad \text{where } t \le t_0 + T.$$

By $M_{n\times n}(\mathbb{R})$ we denote the space of real square symmetric matrices $r = [r_{jk}]_{n\times n}$.

A mapping $w \in Z_m(\tilde{D})$ is called *regular* in D if

$$w^i_t, \quad w^i_x = \operatorname{grad}_x w^i, \quad w^i_{xx} = [w^i_{x_j x_k}]_{n\times n} \qquad (i = 1,\dots,m)$$

are continuous in D.

Let the mappings

$$F^i : D \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times M_{n\times n} \times Z_m(\tilde{D}) \ni (t,x,z,p,q,r,w) \longrightarrow F^i(t,x,z,p,q,r,w) \in \mathbb{R} \qquad (i = 1,\dots,m)$$

be given and let for an arbitrary regular in D function $w \in Z_m(\tilde{D})$

$$F^i[t,x,w] := F^i(t,x,w(t,x),w^i_t(t,x),w^i_x(t,x),w^i_{xx}(t,x),w), \qquad (t,x) \in D \quad (i = 1,\dots,m).$$

Each two regular in D mappings $u, v \in Z_m(\tilde{D})$ are said to be *solutions* of the system

$$F^i[t,x,u] \ge F^i[t,x,v] \qquad (i = 1,\dots,m) \tag{2.1}$$

in D, if they satisfy (2.1) for all $(t,x) \in D$.

For a given regular mapping w in D and for an arbitrary fixed $i \in \{1,\dots,m\}$, the mapping F^i is called *uniformly parabolic* with respect to w in a subset $E \subset D$ if there is a constant $\kappa > 0$ (depending on E) such that for any two matrices $\tilde{r} = [\tilde{r}_{jk}], \hat{r} = [\hat{r}_{jk}] \in M_{n\times n}(\mathbb{R})$ and for all $(t,x) \in E$ we have

$$\begin{aligned}\tilde{r} \le \hat{r} \Longrightarrow F^i(t,x,w(t,x),w^i_t(t,x),w^i_x(t,x),\hat{r},w)& \\ - F^i(t,x,w(t,x),w^i_t(t,x),w^i_x(t,x),\tilde{r},w)& \\ \ge \kappa \sum_{j=1}^{n} (\hat{r}_{jj} - \tilde{r}_{jj}),&\end{aligned} \tag{2.2}$$

where $\tilde{r} \le \hat{r}$ means that $\sum_{j,k=1}^{n} (\tilde{r}_{jk} - \hat{r}_{jk})\lambda_j\lambda_k \le 0$ for every $(\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n$.

If (2.2) is satisfied for $\tilde{r} = w^i_{xx}(t,x)$, $\hat{r} = w^i_{xx}(t,x) + r$, $r \geq 0$ and $\kappa = 0$, then F^i is called *parabolic* with respect to w in E.

An unbounded set D of type (P) is called a *set of type* (P_Γ) if

$$\Gamma \cap \overline{\sigma}_{t_0} \neq \emptyset. \tag{2.3}$$

A bounded set D of type (P) is called a *set of type* (P_B).

It is easy to see that each set D of type (P_B) satisfies condition (2.3). Moreover, it is obvious that if D_0 is a bounded subset [D_0 is an unbounded proper subset] of $\mathbb{R}^n$, then $D = (t_0, t_0 + T] \times D_0$ is a set of type (P_B) [(P_Γ), respectively].

3. Lemma

As a consequence of Theorem 3.1 from [2] we obtain the following:

Lemma 3.1
Assume that:

1° *D is a set of type (P).*

2° *The mappings F^i $(i = 1, \ldots, m)$ are weakly increasing with respect to $z^1, \ldots, z^{i-1}, z^{i+1}, \ldots, z^m$ $(i = 1, \ldots, m)$. Moreover, there is a positive constant $L > 0$ such that*

$$\begin{aligned} &F^i(t,x,z,p,q,r,w) - F^i(t,x,\tilde{z},p,\tilde{q},\tilde{r},\tilde{w}) \\ &\quad \leq L\Bigg(\max_{k=1,\ldots,m} \mid z^k - \tilde{z}^k \mid + \mid x \mid \sum_{j=1}^{n} \mid q^j - \tilde{q}^j \mid \\ &\qquad + \mid x \mid^2 \sum_{j,k=1}^{n} \mid r_{jk} - \tilde{r}_{jk}] + [w - \tilde{w}]_t \Bigg) \end{aligned}$$

for all $(t,x) \in D$, $z, \tilde{z} \in \mathbb{R}^m$, $p \in \mathbb{R}$, $q, \tilde{q} \in \mathbb{R}^n$, $r, \tilde{r} \in M_{n\times n}(\mathbb{R})$, $w, \tilde{w} \in Z_m(\tilde{D})$, where $\sup_{(t,x)\in\tilde{D}}(w(t,x) - \tilde{w}(t,x)) < \infty$ $(i = 1, \ldots, m)$.

3° *There are constants $C_i > 0$ $(i = 1, 2)$ such that*

$$F^i(t,x,z,p,q,r,w) - F^i(t,x,z,\tilde{p},q,r,w) < C_1(\tilde{p} - p) \qquad (i = 1, \ldots, m)$$

for all $(t,x) \in D$, $z \in \mathbb{R}^m$, $p > \tilde{p}$, $q \in \mathbb{R}^n$, $r \in M_{n\times n}(\mathbb{R})$, $w \in Z_m(\tilde{D})$ and

$$F^i(t,x,z,p,q,r,w) - F^i(t,x,z,\tilde{p},q,r,w) < C_2(\tilde{p} - p) \qquad (i = 1, \ldots, m)$$

for all $(t,x) \in D$, $z \in \mathbb{R}^m$, $p < \tilde{p}$, $q \in \mathbb{R}^n$, $r \in M_{n\times n}(\mathbb{R})$, $w \in Z_m(\tilde{D})$.

4° *The mapping* $u \in Z_m(\tilde{D})$ *is regular in* D, *and* $\sup_{(t,x)\in D} u(t,x) < \infty$.

5° $u(t,x) \le K$ *for* $(t,x) \in \partial_p D$, *where* $K = (K^1, \dots, K^m)$ *is a constant mapping.*

6° *The mappings* u *and* K *are solutions of the system*

$$F^i[t,x,u] \ge F^i[t,x,K] \qquad (i = 1, \dots, m)$$

in D.

7° *The mappings* F^i $(i = 1, \dots, m)$ *are parabolic with respect to* u *in* D *and uniformly parabolic with respect to* K *in any compact subset of* D.

Then

$$u(t,x) \le K \qquad \text{for } (t,x) \in \tilde{D}.$$

Moreover, if there is a point $(\tilde{t},\tilde{x}) \in D$ *such that* $u(\tilde{t},\tilde{x}) = K$ *then*

$$u(t,x) = K \qquad \text{for } (t,x) \in S^-(\tilde{t},\tilde{x}).$$

4. Strong maximum principles together with initial inequalities in sets of types (P_Γ) and (P_B)

Now, we shall give the following theorem on strong maximum principles together with initial inequalities in sets of types (P_Γ) and (P_B):

THEOREM 4.1
Assume that:

(i) D *is a set of type* (P_Γ) *or* (P_B) *and assumptions* 2^0 *and* 3^0 *of Lemma 3.1 are satisfied.*

(ii) *The mapping* $u \in Z_m(\tilde{D})$ *is regular in* D *and the maximum of* u *on* Γ *is attained. Moreover,*

$$K := \max_{(t,x)\in\Gamma} u(t,x). \tag{4.1}$$

(iii) *The inequality*

$$u(t_0,x) \le K \qquad \text{for } x \in S_{t_0} \tag{4.2}$$

is satisfied.

(iv) *The maximum of* u *in* $\tilde{D}$ *is attained. Moreover,*

$$M := \max_{(t,x)\in\tilde{D}} u(t,x). \tag{4.3}$$

(v) *The mappings* u *and* M *are solutions of the system*

$$F^i[t,x,u] \geq F^i[t,x,M] \qquad (i=1,\dots,m)$$

in D.

(vi) *The mappings* F^i $(i=1,\dots,m)$ *are parabolic with respect to* u *in* D *and uniformly parabolic with respect to* M *in any compact subset of* D.

Then

$$\max_{(t,x)\in\tilde{D}} u(t,x) = \max_{(t,x)\in\Gamma} u(t,x). \tag{4.4}$$

Moreover, if there is a point $(\tilde{t},\tilde{x}) \in D$ *such that* $u(\tilde{t},\tilde{x}) = \max_{(t,x)\in\tilde{D}} u(t,x)$ *then*

$$u(t,x) = \max_{(t,x)\in\Gamma} u(t,x) \qquad \textit{for } (t,x)\in S^-(\tilde{t},\tilde{x}).$$

Proof. We shall prove Theorem 4.1 for a set of type (P_Γ) only since the proof for a set of type (P_B) is analogous.

We shall argue by contradiction. Suppose

$$M \neq K. \tag{4.5}$$

From (4.1) and (4.3), we have

$$K \leq M. \tag{4.6}$$

Consequently

$$K < M. \tag{4.7}$$

Observe, from assumption (iv), that

$$\text{there is } (t^*,x^*)\in\tilde{D} \text{ such that } u(t^*,x^*) = M := \max_{(t,x)\in\tilde{D}} u(t,x). \tag{4.8}$$

By (4.8), by assumption (ii) and by (4.7), we have

$$(t^*,x^*)\in\tilde{D}\setminus\Gamma = D\cup\sigma_{t_0}. \tag{4.9}$$

Suppose that

$$(t^*,x^*)\in D. \tag{4.10}$$

From assumptions (ii) and (v), and from (4.8), we get

$$\begin{cases} u\in Z_m(\tilde{D}) \text{ and } u^i_t,\ u^i_x,\ u^i_{xx}\ (i=1,\dots,m) \text{ are continuous in } D,\\ F^i[t,x,u]\geq F^i[t,x,M] \text{ for } (t,x)\in D\ (i=1,\dots,m),\\ u(t,x)\leq M \text{ for } (t,x)\in\tilde{D},\\ u(t^*,x^*)=M. \end{cases} \tag{4.11}$$

The assumption that D is a set of type (P), assumptions 2° and 3° (see assumption (i)), formulas (4.10) and (4.11), and assumption (vi) imply, by Lemma 3.1, the equation

$$u(t,x) = M \qquad \text{for } (t,x) \in S^-(t^*,x^*). \tag{4.12}$$

On the other hand, from the definition of a set of type (P_Γ), there is a polygonal line $\gamma \subset S^-(t^*,x^*)$ such that

$$\overline{\gamma} \cap \Gamma \neq \emptyset. \tag{4.13}$$

Since $u \in C(\bar{D}, \mathbb{R}^m)$, we have a contradiction of formulas (4.12) and (4.13) with formulas (4.1) and (4.7). Therefore, $(t^*,x^*) \notin D$ and, consequently, from (4.9), $(t^*,x^*) \in \sigma_{t_0}$. But this leads, by (4.7), to a contradiction of (4.2) with (4.8). The proof of (4.4) is complete.

The second part of Theorem 4.1 is a consequence of equality (4.4) and of Lemma 3.1. Therefore, the proof of Theorem 4.1 is complete.

Remark 4.1
If D is a set of type (P_B) and if $\tilde{D} = \bar{D}$ then the first part of assumption (ii) of Theorem 4.1 relative to the maximum of u and the first part of assumption (iv) of this theorem are trivially satisfied since $u, v \in C(\bar{D}, \mathbb{R}^m)$ and Γ is bounded and closed set in this case.

Remark 4.2
If the mappings F^i $(i = 1, \ldots, m)$ do not depend on the functional argument w then Lemma 3.1 and Theorem 4.1 reduce to the lemma and the theorem, respectively, on parabolic differential inequalities including terms

$$F^i(t,x,u(t,x),u^i_t(t,x),u^i_x(t,x),u^i_{xx}(t,x)) \qquad (i = 1, \ldots, m)$$

and in this case we can put $\tilde{D} = \bar{D}$.

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