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Curvature properties of some submanifolds in space forms

Dedicated to Professor Dr. Andrzej Zajtz on his seventieth birthday

Abstract. Curvature properties of pseudosymmetry type of some submanifolds of codimension greater then 1 immersed isometrically in semi-Riemannian spaces of constant curvature are given.

1. Introduction

Theorem 3.1 of [20] states that if at every point x of a hypersurface M immersed isometrically in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 3$, its second fundamental tensor H has the form

$$H = \alpha v \otimes v + \beta w \otimes w, \qquad v, w \in T_x^* M,\ \alpha, \beta \in \mathbb{R}, \tag{1}$$

then on M we have

$$R \cdot R = \frac{\tilde{\kappa}}{n(n+1)}\, Q(g, R), \tag{2}$$

which means that M is a pseudosymmetric hypersurface. In particular, if the ambient space is a non-flat manifold then M is non-semisymmetric. Evidently, if the ambient space is a semi-Euclidean space $\mathbb{E}_s^{n+1}$ then (1) reduces to

$$R \cdot R = 0, \tag{3}$$

which means that M is a semisymmetric hypersurface. In this paper we prove, that under some additional assumptions, the mentioned above results remain also true when the codimension of a submanifold M in a semi-Riemannian space of constant curvature is greater than 1.

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2. Conditions of pseudosymmetry type

In this section we give a review on manifolds of pseudosymmetry type. We refer to [2], [17] and [33] for a survey of results related to this subject.

Let (M, g), $n = \dim M \geq 3$, be a connected semi-Riemannian manifold of class C^∞. We define on M the endomorphisms $\mathcal{R}(X, Y)$ and $X \wedge_A Y$ by

$$\mathcal{R}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y,$$

respectively, where A is a symmetric $(0, 2)$-tensor, ∇ is the Levi-Civita connection of (M, g) and $X, Y, Z \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M. Furthermore, we define the Riemann-Christoffel curvature tensor R and the $(0, 4)$-tensor G of (M, g) by

$$R(X_1, X_2, X_3, X_4) = g(\mathcal{R}(X_1, X_2)X_3, X_4),$$

$$G(X_1, X_2, X_3, X_4) = g((X_1 \wedge_g X_2)X_3, X_4),$$

respectively. We denote by S and κ the Ricci tensor and the scalar curvature of (M, g), respectively. For a $(0, k)$-tensor field T on M, $k \geq 1$ and a symmetric $(0, 2)$-tensor A we define the $(0, k + 2)$-tensors $R \cdot T$ and $Q(A, T)$ by

$$\begin{aligned}
(R \cdot T)(X_1, \ldots, X_k; X, Y) &= (\mathcal{R}(X, Y) \cdot T)(X_1, \ldots, X_k)\\
&= -T(\mathcal{R}(X, Y)X_1, X_2, \ldots, X_k)\\
&\quad - \ldots - T(X_1, \ldots, X_{k-1}, \mathcal{R}(X, Y)X_k),\\
Q(g, T)(X_1, \ldots, X_k; X, Y) &= ((X \wedge_A Y) \cdot T)(X_1, \ldots, X_k)\\
&= -T((X \wedge_A Y)X_1, X_2, \ldots, X_k)\\
&\quad - \ldots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k).
\end{aligned}$$

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be a *pseudosymmetric manifold* ([17, Section 3.1], [33]) if at every point of M the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent. Thus the manifold (M, g) is pseudosymmetric if and only if

$$R \cdot R = L_R\, Q(g, R) \tag{4}$$

on $U_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)} G \neq 0 \text{ at } x\}$, where L_R is some function on U_R. It is clear that every *semisymmetric manifold* ($R \cdot R = 0$, [32]) is pseudosymmetric. The condition (4) arose during the study on totally umbilical submanifolds of semisymmetric manifolds as well as when considering geodesic mappings of semisymmetric manifolds ([17, Sections 10 and 13], [33]). There exist pseudosymmetric manifolds which are non-semisymmetric. For instance, in [18] (see Example 3.1 and Theorem 4.1) it was shown that the warped product $S^p \times_F S^{n-p}$, $p \geq 2$, $n - p \geq 1$, of the standard spheres S^p and S^{n-p}, with some function F, is pseudosymmetric.

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be a *Ricci-pseudosymmetric manifold* ([17, Section 3.4]) if at every point of M the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent. Thus the manifold (M, g) is Ricci-pseudosymmetric if and only if

$$R \cdot S = L_S \, Q(g, S) \tag{5}$$

on $U_S = \{x \in M \mid \; S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$, where L_S is some function on U_S. It is clear that if (4) is satisfied at a point x of a manifold (M, g) then also (5) holds at x. The converse statement is not true. E.g. every warped product $M_1 \times_F M_2$, $\dim M_1 = 1$, $\dim M_2 = n - 1 \geq 3$, of a manifold $(M_1, \overline{g})$ and a non-pseudosymmetric Einstein manifold $(M_2, \tilde{g})$ is a non-pseudosymmetric, Ricci-pseudosymmetric manifold. It is also known that the Cartan hypersurfaces of dimensions 6, 12 or 24 are non-pseudosymmetric Ricci-pseudosymmetric manifolds ([24]).

For any $X, Y \in \Xi(M)$ we define the endomorphism $\mathcal{C}(X, Y)$ by

$$\mathcal{C}(X, Y) \;=\; \mathcal{R}(X, Y) - \frac{1}{n-2} \left(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y \right).$$

The Ricci operator $\mathcal{S}$ and the Weyl conformal curvature tensor C of (M, g) are defined by

$$g(\mathcal{S}X, Y) = S(X, Y),$$
$$C(X_1, X_2, X_3, X_4) = g(\mathcal{C}(X_1, X_2)X_3, X_4),$$

respectively. Now we define the $(0, 6)$-tensor $C \cdot C$ by

$$\begin{aligned}(C \cdot C)(X_1, X_2, X_3, X_4; X, Y) &= (\mathcal{C}(X, Y) \cdot C)(X_1, X_2, X_3, X_4) \\ &= -C(\mathcal{C}(X, Y)X_1, X_2, X_3, X_4) \\ &\qquad - \ldots - C(X_1, X_2, X_3, \mathcal{C}(X, Y)X_4).\end{aligned}$$

A semi-Riemannian manifold (M, g), $n \geq 4$, is said to be a *manifold with pseudosymmetric Weyl tensor* ([17, Section 12.6]) if at every point of M the tensors $C \cdot C$ and $Q(g, C)$ are linearly dependent. Thus the manifold (M, g) is a manifold with pseudosymmetric Weyl tensor if and only if

$$C \cdot C = L_C \, Q(g, C) \tag{6}$$

on $U_C = \{x \in M \mid \; C \neq 0 \text{ at } x\}$, where L_C is some function on U_C. It is known that (6) is fulfilled at every point of the warped product $M_1 \times_F M_2$,

$\dim M_1 = \dim M_2 = 2$ ([15], Theorem 2). An example of a 4-dimensional Riemannian manifold satisfying (6), which is not a warped product, was found in [25].

A semi-Riemannian manifold (M, g), $n \geq 4$, is said to be a *Weyl-pseudosymmetric manifold* if then at every point of M the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent. Thus the manifold (M, g) is a Weyl-pseudosymmetric manifold if and only if

$$R \cdot C = L\, Q(g, C) \tag{7}$$

on U_C, where L is some function on U_C. Every pseudosymmetric manifold is Weyl-pseudosymmetric. The converse statement is not true ([13]). Evidently, any *Weyl-semisymmetric manifold* ($R \cdot C = 0$) is Weyl-pseudosymmetric. We refer to [1] for a review of results on Weyl-pseudosymmetric manifolds.

It is easy to see that at every point of a pseudosymmetric Einstein manifold the tensors $R \cdot R - Q(S, R)$ and $Q(g, C)$ are linearly dependent. We also mention that any 3-dimensional semi-Riemannian manifold fulfils ([14], Theorem 3.1)

$$R \cdot R = Q(S, R). \tag{8}$$

Moreover, every hypersurface M immersed isometrically in an $(n+1)$-dimensional semi-Euclidean space $\mathbb{E}^{n+1}_s$ with signature $(n+1-s, s)$, $n \geq 3$, satisfies (8) ([21], Corollary 3.1). A review of results on manifolds satisfying (8) is given in Section 5 of [17].

Semi-Riemannian manifolds fulfilling the above presented conditions or other conditions of this kind are called *manifolds of pseudosymmetry type* ([17], [33]). Recently, a review of results on pseudosymmetry type manifolds was given in [2].

Further, for a symmetric $(0, 2)$-tensor fileds A and B on M we define their Kulkarni-Nomizu product $A \wedge B$ by

$$\begin{aligned}(A \wedge B)(X_1, X_2, X_3, X_4) = {} & A(X_1, X_4)B(X_2, X_3) + A(X_2, X_3)B(X_1, X_4) \\ & - A(X_1, X_3)B(X_2, X_4) - A(X_2, X_4)B(X_1, X_3).\end{aligned}$$

Further, for a symmetric $(0, 2)$-tensor field A on M we define the endomorphism $\mathcal{A}$ of $\Xi(M)$ and the $(0, 2)$-tensors A^2 and A^3 by

$$\left.\begin{aligned} g(\mathcal{A}X, Y) &= A(X, Y), \\ A^2(X, Y) &= A(\mathcal{A}X, Y), \\ A^3(X, Y) &= A^2(\mathcal{A}X, Y), \end{aligned}\right\} \tag{9}$$

respectively. We end this section with the following statement.

Lemma 2.1 ([20])
Let at a point x of a semi-Riemannian manifold (M, g), $n \geq 3$, be given a $(0,2)$-tensor A having the form

$$A = \alpha v \otimes v + \beta w \otimes w, \qquad v, w \in T_x^* M,\ \alpha, \beta \in \mathbb{R}. \tag{10}$$

Then the following relations are fulfilled at x

$$Q(A^2, A \wedge A) = 0, \tag{11}$$

$$A^3 = tr(A) A^2 + \lambda A, \quad \lambda = \alpha\beta((g(V,W))^2 - g(V,V) g(W,W)), \tag{12}$$

where the vectors $V, W \in T_x M$ are related to the covectors v, w by $v(X) = g(V,X)$ and $w(X) = g(W,X)$, respectively, and $X \in T_x M$.

3. Quasi-umbilical hypersurfaces

Let M be a connected submanifold immersed isometrically in a semi-Riemannian manifold $(N, \tilde{g})$, $3 \leq n = \dim M < n + k = \dim N$, $k \geq 1$. We denote by g the metric tensor induced on M from the metric tensor $\tilde{g}$. We denote by $\tilde{\nabla}$ and ∇ the Levi-Civita connections correspondig to the metric tensors $\tilde{g}$ and g, respectively. The *Gauss formula* of M in N is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{13}$$

where h is the second fundamental form of M in N and X, Y are vector fields tangent to M. Further, for any vector field ξ normal to M and for any vector field X tangent to M we have the *Weingarten formula* of M in N

$$\tilde{\nabla}_X \xi = -\mathcal{A}_\xi X + D_X \xi\,, \tag{14}$$

where D denotes the normal connection induced in the normal bundle $N(M)$ of M in N and $\mathcal{A}$, defined by $\mathcal{A}(\xi, X) = \mathcal{A}_\xi X$, is the *Weingarten map* (the *shape operator*) of M in N. We have

$$g(\mathcal{A}_\xi X, Y) \ = \ \tilde{g}(h(X,Y), \xi). \tag{15}$$

A submanifold M in a semi-Riemannian manifold $(N, \tilde{g})$ is said to be *quasi-umbilical* with respect to the normal direction ξ at a point $x \in M$ (cf. [21], [22]) if at x its second fundamental tensor H_ξ satisfies the equality

$$H_\xi = \alpha_\xi\, g + \beta_\xi\, v_\xi \otimes v_\xi\,, \qquad v_\xi \in T_x^* M,\ \alpha_\xi, \beta_\xi \in \mathbb{R}. \tag{16}$$

If $\alpha_\xi = 0$ (resp., $\beta_\xi = 0$ or $\alpha_\xi = \beta_\xi = 0$) holds at x then it is called *cylindrical* (resp., *umbilical* or *geodesic*) w.r.t. ξ at p. If (16) is fulfilled at every point of M then M is called a *quasi-umbilical hypersurface* w.r.t. ξ. Let now M be a submanifold immersed isometrically in a Riemannian manifold $(N, \tilde{g})$ and let ξ be a local unit normal vector field on M in N. In this case we can prove that the above notion of quasi-umbilicity equivalent to the following definitions ([5], [8], [9]): The submanifold M immersed isometrically in a Riemannian manifold $(N, \tilde{g})$ is said to be *quasi-umbilical* w.r.t. ξ at a point $x \in M$ when it has a principal curvature with multiplicity $\geq n-1$, i.e. when the principal curvatures of M at x w.r.t. ξ are given by μ_ξ, $\lambda_\xi, \ldots, \lambda_\xi$, where λ_ξ occurs $(n-1)$-times. In particular, when $\mu_\xi = \lambda_\xi$ (resp., $\mu_\xi = \lambda_\xi = 0$), then M is *umbilical* (resp., *geodesic*) at x w.r.t. ξ. If we have $\mu_\xi, 0, \ldots, 0$, where 0 occurs $(n-1)$-times then M is *cylindrical* at x w.r.t. ξ.

The following statement gives a curvature characterization of quasi-umbilical hypersurfaces in Euclidean spaces.

THEOREM 3.1 ([3])
A hypersurface M immersed isometrically in a Euclidean space $\mathbb{E}^{n+1}$, $n \geq 4$, is quasi-umbilical if and only if it is conformally flat.

By the invariance of the multiplicity of principal curvatures of submanifolds under conformal changes of the metric of the ambient space we obtain the following result.

THEOREM 3.2 ([31])
A hypersurface M, $n \geq 4$, immersed isometrically in a Riemannian conformally flat manifold is quasi-umbilical if and only if it is conformally flat.

The assertion of Theorem 3.2 is not true when $n = 3$. Namely, there exist conformally flat hypersurfaces in $\mathbb{E}^4$, which are not quasi-umbilical, i.e. hypersurfaces with three distinct principal curvatures ([26]). A generalization of Theorem 3.2, for the case when the ambient space is a semi-Riemannian manifold, was given in [21].

THEOREM 3.3 ([21], Theorem 4.1)
A hypersurface M, $n \geq 4$, immersed isometrically in a semi-Riemannian conformally flat manifold is quasi-umbilical if and only if it is conformally flat.

A submanifold M immersed isometrically in a semi-Riemannian manifold $(N, \tilde{g})$ is said to be *2-quasi-umbilical* w.r.t. ξ at a point $x \in M$ (cf. [22], [23]) if at x the second fundamental tensor H_ξ of M satisfies the equality

$$H_\xi = \alpha_\xi g + \beta_\xi v_\xi \otimes v_\xi + \gamma_\xi w_\xi \otimes w_\xi\,, \quad v_\xi, w_\xi \in T_x^* M,\ \alpha_\xi, \beta_\xi, \gamma_\xi \in \mathbb{R}, \tag{17}$$

where $U_\xi, V_\xi \in T_x M$, $g(U_\xi, V_\xi) = 0$, $u_\xi(X) = g(U_\xi, X)$, $v_\xi(X) = g(V_\xi, X)$ for any vector $X \in T_x M$. If (17) is fulfilled at every point of M then it is called a *2-quasi-umbilical submanifold* w.r.t. ξ. It is clear that if the ambient space $(N, \tilde{g})$ is a Riemannian manifold then the above definition of a 2-quasi-umbilical submanifold M w.r.t. ξ at a point x is equivalent to the following definition (cf. [6]): The submanifold M, $n \geq 4$, immersed isometrically in a Riemannian manifold $(N, \tilde{g})$ is said to be *2-quasi-umbilical* w.r.t. ξ at a point $x \in M$ when it has a principal curvature w.r.t. ξ with multiplicity $\geq n-2$, i.e. when the principal curvatures of M at x w.r.t. ξ are given by μ_ξ, ν_ξ, λ_ξ, $\ldots$, λ_ξ, where λ_ξ occurs $(n-2)$-times. Hypersurfaces with pseudosymmetric Weyl tensor immersed isometrically in Euclidean spaces were considered in [12]. The main result of [12] is the following

THEOREM 3.4 ([12], Theorem 1)
A hypersurface M immersed isometrically in a Euclidean space $\mathbb{E}^{n+1}$, $n \geq 4$, is a manifold with pseudosymmetric Weyl tensor if and only if at every point of the set $U_C \subset M$, M has at most three distinct principal curvatures. Moreover, if x is a point of U_C, at which M has exactly three distinct principal curvatures, then their multiplicities are the following: $1, 1, n-2$, *i.e., M is 2-quasi-umbilical at x.*

Examples of hypersurfaces in $\mathbb{E}^{n+1}$, $n \geq 4$, with pseudosymmetric Weyl tensor are also given in [12]. A review of results on hypersurfaces satisfying (6) is given in [23].

4. Quasiumbilical submanifolds of codimension two

Let M be a submanifold immersed isometrically in a semi-Riemannian manifold N, $n = \dim M \geq k = \operatorname{codim} M$. Let $\xi_1, \ldots, \xi_k$ be mutually orthogonal units normal local vector fields on M and let $\tilde{g}(\xi_y, \xi_y) = e_y$, $e_y = \pm 1$, $x, y, z = 1, \ldots, k$. From (15) we get

$$h(X,Y) = \sum_y H_y(X,Y)\, \xi_y\,. \tag{18}$$

The scalar valued form H_y is called the second fundamental tensor with respect to the normal section ξ_y. We denote by R and $\tilde{R}$ the Riemann-Christoffel curvature tensors of M and N, respectively. The *Gauss equation* of M in N has the following form

$$\begin{aligned} R(X_1, \ldots, X_4) = {} & \tilde{g}(h(X_1, X_4), h(X_2, X_3)) \\ & - \tilde{g}(h(X_1, X_3), h(X_2, X_4)) \\ & + \tilde{R}(X_1, \ldots, X_4)\,, \end{aligned} \tag{19}$$

where $X_1, \ldots, X_4$ are vector fields tangent to M.

The submanifold M in a semi-Riemannian manifold N, $n = \dim M \geq k = \operatorname{codim} M$, is said to be *quasi-umbilical* if at every point $x \in M$ there exist mutually orthogonal units normal vector fields $\xi_1, \ldots, \xi_k$, defined on a neighbourhood $\mathcal{U}$ of x such that on $\mathcal{U}$ we have

$$H_y = \alpha_y g + \beta_y \bar{u}_y \otimes \bar{u}_y \,, \tag{20}$$

where α_y and β_y are some functions and u_y is some 1-forms on $\mathcal{U}$, respectively, $x = 1, \ldots, k$, and the vector fields U_y related with $\bar{u}_y$ by $\bar{u}_y(X) = g(U_y, X)$, $X \in T_x\mathcal{U}$, satisfy

$$g(U_y, U_z) = 0, \quad y \neq z, \qquad \text{and} \qquad g(U_y, U_y) = \bar{e}_y, \quad \bar{e}_y = \pm 1. \tag{21}$$

Quasi-umbilical submanifolds were studied among others in: [6]-[9], [27]-[30] and [34].

From now we will assume that the ambient space $(N, \tilde{g})$ is a semi-Riemannian space of constant curvature $N_s^{n+k}(c)$ with signature $(n+k-s, s)$, $n \geq 4$. The Gauss equation (22) of M in $N_s^{n+k}(c)$ reads

$$\begin{aligned} R(X_1, \ldots, X_4) = {} & \tilde{g}(h(X_1, X_4), h(X_2, X_3)) - \tilde{g}(h(X_1, X_3), h(X_2, X_4)) \\ & + \frac{\tilde{\kappa}}{(n+k-1)(n+k)} \, G(X_1, \ldots, X_4), \end{aligned} \tag{22}$$

where $\tilde{\kappa}$ denotes the scalar curvature of the ambient space. Further, if M is quasi-umbilical with respect to $\xi_1, \ldots, \xi_k$, then (22) turns into

$$R(X_1, \ldots, X_4) = (g \wedge u)(X_1, \ldots, X_4) + \eta \, G(X_1, \ldots, X_4), \tag{23}$$

where

$$\left. \begin{aligned} \eta &= \frac{\tilde{\kappa}}{(n+k-1)(n+k)} + \sum_{y=1}^{k} \bar{e}_y \alpha_y^2 \\ \text{and} \qquad & \\ u(Y, Z) &= \sum_{y=1}^{k} \bar{e}_y \alpha_y \beta u_y(Y) u_y(Z). \end{aligned} \right\} \tag{24}$$

Using (23) we can present the Ricci tensor S of (M, g) in the form

$$\begin{aligned} S(X_1, X_4) = \rho \, g(X_1, X_4) + (n-2) \, u(X_1, X_4), \\ \rho \; = \; (n-1)\eta + tr_g u. \end{aligned} \tag{25}$$

We note that form (23), by an application of (25), it follows that the Weyl curvature tensor C of (M, g) vanishes identically on M (cf. [5]). From (25) we get easily

$$S(U_y \,, Z) = (\rho + (n-2) e_y \bar{e}_y \alpha_y \beta_y) g(U_y \,, Y), \qquad y = 1, \ldots, k. \tag{26}$$

We denote by $\mathcal{S}$ the Ricci operator of S. Now (26) turns into

$$\mathcal{S}U_y = \tilde{\tau}_y U_y\,, \qquad \tilde{\tau}_y = \rho + (n-2)\tau_y\,,\ \tau_y = e_y \bar{e}_y \alpha_y \beta_y\,,\ y = 1, \ldots, k. \quad (27)$$

We have the following generalizations of Theorem 3.1 for the case when codimension of a submanifold is ≥ 1.

Remark 4.1

(i) Let M be a n-dimensional submanifold in $\mathbb{E}^{n+k}$, $n \geq 4$.

   (a) ([7]) The submanifold M, with a flat normal connection and such that $1 \leq k \leq n-3$, is quasi-umbilical if and only if it is conformally flat.

   (b) ([28]) The submanifold M, such that $1 \leq k \leq \inf(4, n-3)$, is quasi-umbilical if and only if it is conformally flat.

(ii) An example of a non quasi-umbilical conformally flat submanifold of codimension 2 in a Euclidean space $\mathbb{E}^6$ is given in [34] (Chapter 5, p. 100).

Let M, $n = \dim M \geq 3$, be quasi-umbilical submanifold, with respect to the normal sections $\xi_1, \ldots, \xi_k$, in a Riemannian space of constant curvature $N^{n+k}(c)$, $k \geq 2$. From (27) we have $\mathcal{S}U_y = (\rho + (n-2)\alpha_y\beta_y)U_y$, $y = 1, \ldots, k$. Further, we note that if V is a vector such that $g(U_y, V) = 0$, then from (27) we have $\mathcal{S}V = \rho V$. Thus we see that ρ, $\rho + (n-2)\,\alpha_1\,\beta_1$, …, $\rho + (n-2)\,\alpha_k\,\beta_k$, are eigenvalues of the Ricci operator $\mathcal{S}$ of M. In [11] (Theorem 1.3) it was shown that a conformally flat Riemannian manifold (M, g) is pseudosymmetric if and only if at every point of M its Ricci operator $\mathcal{S}$ has at most two distinct eigenvalues. Thus we have

Theorem 4.1

Let M, $n \geq 3$, be quasi-umbilical submanifold, with respect to the normal sections $\xi_1, \ldots, \xi_p$, in a Riemannian space of constant curvature $N^{n+k}(c)$, $k \geq 2$. Then M is pseudosymmetric if and only if at every point of M the Ricci operator $\mathcal{S}$ of M has at most two distinct eigenvalues ρ_1, ρ_2, i.e., the set $\{\rho,\, \rho + (n-2)\alpha_1\beta_1, \ldots, \rho + (n-2)\alpha_k\beta_k\}$ has at most two distinct numbers.

From the last theorem it follows

Corollary 4.1

Let M, $n \geq 3$, be a quasi-umbilical submanifold, with respect to the normal sections ξ_1, ξ_2, in a Riemannian space of constant curvature $N^{n+2}(c)$. Then M is pseudosymmetric if and only if at every point $x \in M$ we have: M is umbilical or cylindrical with respect to ξ_1 or ξ_2 at x or M is non-umbilical and non-cylindrical quasi-umbilical with respect to ξ_1 or ξ_2 at x and $\alpha_1\beta_2 = \alpha_2\beta_2$.

Let M be a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 3$, and let ξ_1 be the normal sections of M in $N_s^{n+1}(c)$. Now the Gauss equation (22) reads

$$R - \frac{\tilde{\kappa}}{n(n+1)}\, G = \frac{\varepsilon}{2}\, H \wedge H, \tag{28}$$

where H is the second fundamental tensor of M in $N_s^{n+1}(c)$. From (28) we get immediately

$$S - \frac{(n-1)\tilde{\kappa}}{n(n+1)}\, g = \varepsilon\, (tr\,(H)H - H^2). \tag{29}$$

Further, applying Lemma 2.1 of [21] into (28) we obtain

$$\begin{aligned}&\left(R - \frac{\tilde{\kappa}}{n(n+1)}\, G\right) \cdot \left(R - \frac{\tilde{\kappa}}{n(n+1)}\, G\right)\\ &\quad = Q\left(S - \frac{(n-1)\kappa}{n(n+1)}\, g, R - \frac{\tilde{\kappa}}{n(n+1)}\, G\right),\end{aligned} \tag{30}$$

whence by making use of (28) and (29) we obtain

$$\left(R - \frac{\tilde{\kappa}}{n(n+1)} G\right) \cdot R = -\frac{1}{2}\, Q(H^2, H \wedge H). \tag{31}$$

Using Lemma 2.1 and (31) we can prove

Theorem 4.2 ([20], Theorem 3.1)
Let M be a hypersurface immersed isometrically in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 3$. If at every point $x \in M$ its second fundamental tensor H has the form

$$H = \alpha v \otimes v + \beta w \otimes w, \qquad v, w \in T_x^* M,\ \alpha, \beta \in \mathbb{R}, \tag{32}$$

then the following relation is satisfied on M

$$R \cdot R = \frac{\tilde{\kappa}}{n(n+1)}\, Q(g, R). \tag{33}$$

In particular, from this it follows that if at every point of a hypersurface M in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, M has three distinct principal curvatures $\lambda, \mu, 0, \ldots, 0$, where 0 occurs $(n-2)$-times, then M is a pseudosymmetric manifold.

We give now an extension of the last theorem on the case of the codimension greater then 1. Let M, $n \geq 3$, be a submanifold in a Riemannian space of constant curvature $N^{n+k}(c)$, $k > 1$, and let $\xi_1, \dots, \xi_k$ be the normal sections of M in $N_s^{n+k}(c)$. The Gauss equation (22) of M reads

$$R_{sijk} = \sum_x \varepsilon_x (H_{x\,sk} H_{x\,ij} - H_{x\,sj} H_{x\,ik}) + \frac{\tilde{\kappa}}{(n+k-1)(n+k)}\, G_{hijk}\,. \qquad (34)$$

Transvecting this with $R^s{}_{hef}$ and using (34) we get

$$\begin{aligned} R_{sijk} R^s{}_{hef} = \sum_x \big(&- H_{x\,ik}(H_{x\,he} H^2_{x\,jf} - H_{x\,hf} H^2_{x\,je}) \\ &+ H_{x\,ij}(H_{x\,he} H^2_{x\,kf} - H_{x\,hf} H^2_{x\,ke})\big) \\ + \sum_{x \neq y} \big(&- \varepsilon_x \varepsilon_y\, H_{x\,ik}(H_{y\,he} H_{xy\,jf} - H_{y\,hf} H_{xy\,je}) \\ &+ \varepsilon_x \varepsilon_y\, H_{x\,ij}(H_{y\,he} H_{xy\,kf} - H_{y\,hf} H_{xy\,ke})\big) \\ + \sum_{x \neq y} \big(&- \varepsilon_x \varepsilon_y\, H_{y\,ik}(H_{x\,he} H_{yx\,jf} - H_{x\,hf} H_{yx\,je}) \\ &+ \varepsilon_x \varepsilon_y\, H_{y\,ij}(H_{y\,he} H_{yx\,kf} - H_{y\,hf} H_{yx\,ke})\big) \\ + \frac{\tilde{\kappa}}{(n+k-1)(n+k)} &(g_{ij} R_{khef} - g_{ik} R_{jhef} \\ + g_{he} R_{fijk} - g_{hf} R_{eijk}&), \end{aligned} \qquad (35)$$

where

$$H_{xy\,ij} = H_{x\,is} g^{sr} H_{y\,jr}\,. \qquad (36)$$

Applying now (35) to the idendity

$$\begin{aligned} &(R \cdot R)_{hijklm} \\ &\quad = R_{sijk} R^s{}_{hef} - R_{shjk} R^s{}_{ief} + R_{skhi} R^s{}_{jef} - R_{sjhi} R^s{}_{kef}\,, \end{aligned} \qquad (37)$$

and using the definition of the tensor $Q(A,T)$, where A is a symmetric $(0,2)$-tensor and T a generalized curvature tensor, we find

$$\begin{aligned} R \cdot R = &\frac{\tilde{\kappa}}{(n+k-1)(n+k)}\, Q(g,R) - \frac{1}{2} \sum_x Q(H_x^2, H_x \wedge H_x) \\ &- \sum_{x \neq y} \varepsilon_x \varepsilon_y\, Q(H_{xy} + H_{yx}, H_x \wedge H_y). \end{aligned} \qquad (38)$$

Theorem 4.3

Let M, $n = \dim M \geq 3$, be a submanifold in a Riemannian space of constant curvature $N^{n+k}(c)$, $k \geq 2$, and let ξ_z, $z = 1, \dots, k$, be the normal sections of M in $N^{n+k}(c)$. If at every point $x \in M$ the second fundamental tensors H_z of M satisfy the following relations:

$$H_z = \alpha_z u_z \otimes u_z + \beta_z w_z \otimes w_z\,, \qquad u_z, w_z \in T_x^* M,\ \alpha_z, \beta_z, \in \mathbb{R}, \qquad (39)$$

and the subspaces $\mathrm{lin}\,\{U_x, W_x\}$ *and* $\mathrm{lin}\,\{U_y, W_y\}$, *for* $x \neq y$, *are mutually orthogonal then*

$$R \cdot R = \frac{\tilde{\kappa}}{(n+k-1)(n+k)}\, Q(g, R), \tag{40}$$

where the vectors $U_x, W_x \in T_xM$ *are related to the covectors* u_x, w_x *by* $u_z(X) = g(U_z, X)$, $w_z(X) = g(W_z, X)$ *and* $X \in T_xM$.

Proof. From Lemma 2.1 it follows immediately that $Q(H_z^2, H_z \wedge H_z) = 0$. Further, we have

$$\begin{aligned}(H_{xy} + H_{yx})_{ij} = \alpha_x\alpha_y g(U_x, U_y)\,(u_{x\,i}u_{y\,j} + u_{x\,i}u_{y\,j}) \\ + \beta_x\alpha_y g(W_x, U_y)\,(w_{x\,i}u_{y\,j} + w_{x\,j}u_{y\,i}) \\ + \alpha_x\beta_y g(U_x, W_y)\,(w_{x\,i}u_{y\,j} + w_{x\,j}u_{y\,i}) \\ + \beta_x\beta_y g(W_x, W_y)\,(w_{x\,i}w_{y\,j} + w_{x\,i}w_{y\,j}),\end{aligned} \tag{41}$$

where $u_{z\,j}$, $w_{z\,j}$ are the local components of the covectors u_z and w_z. By our assumptions, (41) reduces to $H_{xy} + H_{yx} = 0$, whence $Q(H_{xy} + H_{yx}, H_x \wedge H_y) = 0$. Now (38) turns into (40), completing the proof.

From the above theorem it follows immediately the following

Theorem 4.4
Let M, $n = \dim M \geq 3$, *be a submanifold in a semi-Euclidean space* $\mathbb{E}_s^{n+k}$ *and let* ξ_z, $z = 1, \ldots, k$, *be the normal sections of* M *in* $\mathbb{E}_s^{n+k}$. *If at every point* $x \in M$ *the second fundamental tensors* H_z *of* M *satisfy* (39) *and for any* $x \neq y$ *the subspaces* $\mathrm{lin}\,\{U_x, W_x\}$ *and* $\mathrm{lin}\,\{U_y, W_y\}$ *are mutually orthogonal then* M *is a semisymmetric manifold.*

Example 4.1 (cf. [34], Chapter VII, Theorem 1)
First of all we note that the product manifold of k, $k \geq 2$, semisymmetric manifolds is also a semisymmetric manifold. Let now M_a, $\dim M_a = n_a$, be a hypersurface of rank 2 immersed isometrically in a Euclidean space $\mathbb{E}^{n_a}$, $a = 1, \ldots, k$. Such hypersurface is a semisymmetric manifold (cf. Theorem 4.2). By an standard construction, the Cartesian product manifold $M_1 \times \ldots \times M_k$ of the manifolds $M_1, \ldots, M_k$ is a semisymmetric submanifold in a Euclidean space $\mathbb{E}^{n_1+n_2+k}$ such that (39) is satisfied and for any $x \neq y$ $\mathrm{lin}\,\{U_x, W_x\}$ and $\mathrm{lin}\,\{U_y, W_y\}$ are orthogonal.

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