

Vasyl M. Fedorchuk, Volodymyr I. Fedorchuk

On the differential first-order invariants for the non-splitting subgroups of the generalized Poincaré group $P(1,4)$

Dedicated to Professor Andrzej Zajtz on his seventieth birthday

Abstract. The functional bases of the differential first-order invariants for all non-splitting subgroups of the generalized Poincaré group $P(1,4)$ are constructed. Some applications of the results obtained are considered.

Introduction

The development of the theoretical physics has required various extensions of the four-dimensional Minkowski space and, correspondingly, various extensions of the Poincaré group $P(1,3)$. The natural extension of this group is the generalized Poincaré group $P(1,4)$. The group $P(1,4)$ is the group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. This group has many applications in theoretical and mathematical physics [1-3]. The group $P(1,4)$ has many subgroups used in theoretical physics [4-8]. Among these subgroups there are the Poincaré group $P(1,3)$ and the extended Galilei group $\tilde{G}(1,3)$ (see also [9]). Thus, the results obtained with the help of the subgroup structure of the group $P(1,4)$ will be useful in relativistic and non-relativistic physics.

The papers [10-12] are devoted to the construction of the first-order differential invariants for the splitting subgroups [4, 5, 7] of the generalized Poincaré group $P(1,4)$.

The present paper is devoted to the construction of functional bases of the differential first-order invariants for the non-splitting subgroups [4, 6-8] of the group $P(1,4)$.

Our paper is organized as follows. In the first section we introduce some notation and results concerning the Lie algebra of the group $P(1,4)$ which we use in the following. Sections 2 and 3 present our main results.

1. The Lie algebra of the group $P(1,4)$ and its non-conjugate subalgebras

The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ ($\mu, \nu = 0, 1, 2, 3, 4$) and P'_μ ($\mu = 0, 1, 2, 3, 4$), satisfying the commutation relations

$$\begin{aligned} [P'_\mu, P'_\nu] &= 0, \\ [M'_{\mu\nu}, P'_\sigma] &= g_{\mu\sigma}P'_\nu - g_{\nu\sigma}P'_\mu, \\ [M'_{\mu\nu}, M'_{\rho\sigma}] &= g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\sigma}M'_{\mu\rho} - g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho}, \end{aligned}$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$.

In order to study the subgroup structure of the group $P(1,4)$ we used the method proposed in [13]. Continuous subgroups of the group $P(1,4)$ have been described in [4–8].

Further we will use the following basis elements:

$$\begin{aligned} G &= M'_{40}, & L_1 &= M'_{32}, & L_2 &= -M'_{31}, & L_3 &= M'_{21}, \\ P_a &= M'_{4a} - M'_{a0}, & C_a &= M'_{4a} + M'_{a0}, & & & & (a = 1, 2, 3), \\ X_0 &= \frac{1}{2}(P'_0 - P'_4), & X_k &= P'_k & & & & (k = 1, 2, 3), \\ X_4 &= \frac{1}{2}(P'_0 + P'_4). \end{aligned}$$

2. The differential first-order invariants of the non-splitting subgroups of the group $P(1,4)$

The group $P(1,4)$ acts on $M(1,3) \times R(u)$ (i.e., on the Cartesian product of the four-dimensional Minkowski space (of the independent variables x_0, x_1, x_2, x_3) and the number axis of the dependent variable u). The group $P(1,4)$ usually acts on $M(1,3) \times R(u)$ as a group generated by translations and rotations of this space.

Let

$$X = \sum_{i=0}^3 \xi_i(x, u) \frac{\partial}{\partial x_i} + \eta(x, u) \frac{\partial}{\partial u}$$

be one of the basis infinitesimal operators. The first prolongation of X has the form

$$X^{(1)} = X + \sum_{i=0}^3 \left(\frac{\partial \eta}{\partial x_i} + \frac{\partial \eta}{\partial u} u_i - \sum_{j=0}^3 u_j \frac{\partial \xi_j}{\partial x_i} - \sum_{j=0}^3 u_i u_j \frac{\partial \xi_j}{\partial u} \right) \frac{\partial}{\partial u_i}.$$

Now, a function $J(x, u^{(1)})$ is a first-order differential invariant if

$$X^{(1)} \cdot J(x, u^{(1)}) = 0.$$

Here $u^{(1)} = (u, u_0, u_1, u_2, u_3)$ is an element of the first prolongation $R(u)^{(1)}$.

Let us consider the following representation of the Lie algebra of the group $P(1, 4)$:

$$\begin{aligned} P'_0 &= \frac{\partial}{\partial x_0}, & P'_1 &= -\frac{\partial}{\partial x_1}, & P'_2 &= -\frac{\partial}{\partial x_2}, & P'_3 &= -\frac{\partial}{\partial x_3}, \\ P'_4 &= -\frac{\partial}{\partial u}, & M'_{\mu\nu} &= -(x_\mu P'_\nu - x_\nu P'_\mu), & x_4 &\equiv u. \end{aligned}$$

More details about this representation can be found in [14-16].

In the construction of the differential invariants it has turned out that different non-splitting subalgebras of the Lie algebra of the group $P(1, 4)$ can have the same functional basis of the first-order differential invariants. Consequently, there is no one-to-one correspondence between non-conjugate subalgebras of the Lie algebra of the group $P(1, 4)$ and corresponding differential invariants.

DEFINITION 1

We call two subalgebras L^1 and L^2 of the Lie algebra of the group $P(1, 4)$ equivalent if they have the same functional basis of the first-order differential invariants.

It is possible to prove that the relation of equivalence of subalgebras L^1 and L^2 given by Definition 1 is the set-theoretical equivalence relation. With respect to this equivalence relation, all non-splitting subalgebras of the Lie algebra of the group $P(1, 4)$ split into classes of equivalent subalgebras. Each two different classes have different functional bases of the first-order differential invariants.

DEFINITION 2

We call two functional bases of the first-order differential invariants of the non-splitting subalgebras of the Lie algebra of the group $P(1, 4)$ equivalent if they belong to the equivalent subalgebras.

One of the results in this section can be formulated as follows.

PROPOSITION

The non-splitting subgroups of the group $P(1, 4)$ have 264 non-equivalent functional bases of the first-order differential invariants.

Proof. Here, we only give a sketch of the proof. Following the sketch, for the purpose of proving the Proposition, we have to use:

- the list of the non-splitting subalgebras of the Lie algebra of the group $P(1,4)$ [17];
- the general ranks of the matrices which contain coordinates of the one time prolonged basis elements of the subalgebras of the considered Lie algebra;
- theorem on number of invariants of the Lie group of the point transformations (see, for example, [18, 19]);
- Definition 1 and Definition 2.

For all non-splitting subgroups of the group $P(1,4)$ the functional bases of the first-order differential invariants have been constructed.

Below, for some of the non-splitting subalgebras of the Lie algebra of the group $P(1,4)$ we give their respective basis elements and corresponding functional basis of differential invariants.

One-dimensional subalgebras

1. $\langle L_3 + eG + \kappa_3 X_3, e > 0, \kappa_3 < 0 \rangle$,

$$\begin{aligned}
 J_1 &= (x_0^2 - u^2)^{\frac{1}{2}}, & J_2 &= (x_1^2 + x_2^2)^{\frac{1}{2}}, \\
 J_3 &= \kappa_3 \ln(x_0 + u) - ex_3, & J_4 &= x_3 + \kappa_3 \arctan \frac{x_1}{x_2}, \\
 J_5 &= \frac{x_1 u_2 - x_2 u_1}{x_1 u_1 + x_2 u_2}, & J_6 &= u_3 \frac{x_0 + u}{u_0 + 1}, \\
 J_7 &= \frac{u_3^2}{u_0^2 - 1}, & J_8 &= \frac{u_1^2 + u_2^2}{u_3^2}, \\
 u_\mu &\equiv \frac{\partial u}{\partial x_\mu}, \quad \mu = 0, 1, 2, 3;
 \end{aligned}$$

2. $\langle P_3 + X_0 \rangle$,

$$\begin{aligned}
 J_1 &= x_1, & J_2 &= x_2, \\
 J_3 &= (x_0 + u)^2 - 2x_3, & J_4 &= x_0 - u + \frac{2}{3}(x_0 + u)^3 - 2x_3(x_0 + u), \\
 J_5 &= x_0 + u + \frac{u_3}{u_0 + 1}, & J_6 &= \frac{u_1}{u_2}, \\
 J_7 &= \frac{u_1}{u_0 + 1}, & J_8 &= \frac{u_3^2}{(u_0 + 1)^2} + \frac{2}{u_0 + 1}.
 \end{aligned}$$

Two-dimensional subalgebras

1. $\langle G, L_3 + dX_3, d < 0 \rangle$,

$$\begin{aligned} J_1 &= x_3 + d \arctan \frac{x_1}{x_2}, & J_2 &= (x_1^2 + x_2^2)^{\frac{1}{2}}, \\ J_3 &= (x_0^2 - u^2)^{\frac{1}{2}}, & J_4 &= (x_0 + u)^2 \frac{1 - u_0}{u_0 + 1}, \\ J_5 &= \frac{x_1 u_2 - x_2 u_1}{x_1 u_1 + x_2 u_2}, & J_6 &= \frac{u_3^2}{u_0^2 - 1}, \\ J_7 &= \frac{u_1^2 + u_2^2}{u_3^2}; \end{aligned}$$

2. $\langle L_3 - X_4, P_3 \rangle$,

$$\begin{aligned} J_1 &= x_0 + u, & J_2 &= (x_1^2 + x_2^2)^{\frac{1}{2}}, \\ J_3 &= x_0^2 - x_3^2 - u^2 + (x_0 + u) \arctan \frac{x_1}{x_2}, & J_4 &= \frac{x_3}{x_0 + u} + \frac{u_3}{u_0 + 1}, \\ J_5 &= \frac{x_1 u_2 - x_2 u_1}{x_1 u_1 + x_2 u_2}, & J_6 &= \frac{u_1^2 + u_2^2}{(u_0 + 1)^2}, \\ J_7 &= \frac{u_3^2}{(u_0 + 1)^2} + \frac{2}{u_0 + 1}. \end{aligned}$$

Three-dimensional subalgebras

1. $\langle G + a_1 X_1 + a_3 X_3, P_3, X_4, a_1 < 0, a_3 < 0 \rangle$,

$$\begin{aligned} J_1 &= x_2, & J_2 &= x_1 - a_1 \ln(x_0 + u), \\ J_3 &= x_3 - a_3 \ln(x_0 + u) + u_3 \frac{x_0 + u}{u_0 + 1}, & J_4 &= (x_0 + u) \frac{u_1}{u_0 + 1}, \\ J_5 &= \frac{u_1}{u_2}, & J_6 &= \frac{u_0^2 - u_3^2 - 1}{u_1^2}; \end{aligned}$$

2. $\langle L_3 - P_3 + \alpha_0 X_0, X_3, X_4, \alpha_0 < 0 \rangle$,

$$\begin{aligned} J_1 &= (x_1^2 + x_2^2)^{\frac{1}{2}}, & J_2 &= \alpha_0 \arctan \frac{x_1}{x_2} - x_0 - u, \\ J_3 &= \arctan \frac{u_1}{u_2} - \frac{u_3}{u_0 + 1}, & J_4 &= x_0 + u - \alpha_0 \frac{u_3}{u_0 + 1}, \\ J_5 &= \frac{u_1^2 + u_2^2}{(u_0 + 1)^2}, & J_6 &= \frac{u_3^2 + 2(u_0 + 1)}{(u_0 + 1)^2}. \end{aligned}$$

Four-dimensional subalgebras

- 1.
- $\langle G + a_3 X_3, L_3, P_3, X_4, a_3 < 0 \rangle$
- ,

$$J_1 = (x_1^2 + x_2^2)^{\frac{1}{2}},$$

$$J_2 = \frac{x_1 u_2 - x_2 u_1}{x_1 u_1 + x_2 u_2},$$

$$J_3 = x_3 - a_3 \ln(x_0 + u) + \frac{x_0 + u}{u_0 + 1} u_3,$$

$$J_4 = (u_1^2 + u_2^2) \frac{(x_0 + u)^2}{(u_0 + 1)^2},$$

$$J_5 = \frac{u_0^2 - u_3^2 - 1}{u_1^2 + u_2^2};$$

- 2.
- $\langle L_3, P_1, P_2, P_3 + X_3 \rangle$
- ,

$$J_1 = x_0 + u,$$

$$J_2 = x_0^2 - x_1^2 - x_2^2 - u^2 - \frac{x_0 + u}{x_0 + u - 1} x_3^2,$$

$$J_3 = \frac{x_3}{x_0 + u - 1} + \frac{u_3}{u_0 + 1},$$

$$J_4 = \left(\frac{x_1}{x_0 + u} + \frac{u_1}{u_0 + 1} \right)^2 + \left(\frac{x_2}{x_0 + u} + \frac{u_2}{u_0 + 1} \right)^2,$$

$$J_5 = \frac{u_1^2 + u_2^2 + u_3^2 + 2(u_0 + 1)}{(u_0 + 1)^2}.$$

Five-dimensional subalgebras

- 1.
- $\langle G + a_2 X_1, P_1, P_2, P_3, X_4, a_2 < 0 \rangle$
- ,

$$J_1 = x_1 + \frac{x_0 + u}{u_0 + 1} u_1 - a_2 \ln(x_0 + u),$$

$$J_2 = x_2 + \frac{x_0 + u}{u_0 + 1} u_2,$$

$$J_3 = x_3 + (x_0 + u) \frac{u_3}{u_0 + 1},$$

$$J_4 = (u_0^2 - u_1^2 - u_2^2 - u_3^2 - 1) \frac{(x_0 + u)^2}{(u_0 + 1)^2};$$

- 2.
- $\langle L_3, P_1 + X_2, P_2 - X_1, X_3, X_4 \rangle$
- ,

$$J_1 = x_0 + u,$$

$$\begin{aligned}
 J_2 &= \left(\frac{x_1}{x_0 + u} + \frac{u_2}{(x_0 + u)(u_0 + 1)} + \frac{u_1}{u_0 + 1} \right)^2 \\
 &\quad + \left(\frac{x_2}{x_0 + u} - \frac{u_1}{(x_0 + u)(u_0 + 1)} + \frac{u_2}{u_0 + 1} \right)^2, \\
 J_3 &= \frac{u_3}{u_0 + 1}, \\
 J_4 &= \frac{u_1^2 + u_2^2}{(u_0 + 1)^2} + \frac{2}{u_0 + 1}.
 \end{aligned}$$

Six-dimensional subalgebras

1. $\langle G + aX_3, L_3 + dX_3, P_1, P_2, P_3, X_4, a < 0, d < 0 \rangle$,

$$\begin{aligned}
 J_1 &= (x_0 + u)^2 \left(\frac{u_1^2 + u_2^2 + u_3^2 + 2u_0 + 2}{(u_0 + 1)^2} - 1 \right), \\
 J_2 &= \left(x_1 + \frac{x_0 + u}{u_0 + 1} u_1 \right)^2 + \left(x_2 + \frac{x_0 + u}{u_0 + 1} u_2 \right)^2, \\
 J_3 &= x_3 - a \ln(x_0 + u) + (x_0 + u) \frac{u_3}{u_0 + 1} \\
 &\quad + d \arctan \left(\frac{x_1(u_0 + 1) + u_1(x_0 + u)}{x_2(u_0 + 1) + u_2(x_0 + u)} \right);
 \end{aligned}$$

2. $\langle P_1 + X_3, P_2, X_0, X_1, X_2, X_4 \rangle$,

$$\begin{aligned}
 J_1 &= \frac{u_1}{u_0 + 1} - x_3, & J_2 &= \frac{u_3}{u_0 + 1}, \\
 J_3 &= \frac{u_1^2 + u_2^2}{(u_0 + 1)^2} + \frac{2}{u_0 + 1}.
 \end{aligned}$$

Seven-dimensional subalgebras

1. $\langle G + a_3X_3, L_3, P_1, P_2, X_1, X_2, X_4, a_3 < 0 \rangle$,

$$\begin{aligned}
 J_1 &= x_3 - a_3 \ln(x_0 + u), & J_2 &= (x_0 + u) \frac{u_3}{u_0 + 1}, \\
 J_3 &= \frac{u_0^2 - u_1^2 - u_2^2 - 1}{u_3^2};
 \end{aligned}$$

2. $\langle L_3 - P_3 + \alpha_0X_0, P_1, P_2, X_1, X_2, X_3, X_4, \alpha_0 < 0 \rangle$,

$$J_1 = x_0 + u - \alpha_0 \frac{u_3}{u_0 + 1}, \quad J_2 = \frac{u_1^2 + u_2^2 + u_3^2 + 2(u_0 + 1)}{(u_0 + 1)^2}.$$

Eight-dimensional subalgebras

1. $\langle G + a_3 X_3, L_3, P_1, P_2, X_0, X_1, X_2, X_4, a_3 < 0 \rangle,$

$$J_1 = x_3 + a_3 \ln \left(\frac{u_3}{u_0 + 1} \right), \quad J_2 = \frac{u_0^2 - u_1^2 - u_2^2 - 1}{u_3^2};$$

2. $\langle L_3 - X_0, P_1, P_2, P_3, X_1, X_2, X_3, X_4 \rangle,$

$$J_1 = \frac{u_1^2 + u_2^2 + u_3^2 + 2(u_0 + 1)}{(u_0 + 1)^2}.$$

3. On some applications of the results obtained

The differential invariants of the local Lie groups of the point transformations play an important role in the group-analysis of differential equations (see, for example [18-28]). In particule, with the help of these invariants we can construct differential equations with non-trivial symmetry groups.

In our case the considered equations can be written in the following form (see, for example [18-20]):

$$F(J_1, J_2, \dots, J_t) = 0,$$

where F is an arbitrary smooth function of its arguments, $\{J_1, J_2, \dots, J_t\}$ is a functional basis of the first-order differential invariants of the non-splitting subgroups of the group $P(1, 4)$.

Since the Lie algebra of the group $P(1, 4)$ contains, as subalgebras, the Lie algebra of the Poincaré group $P(1, 3)$ and the Lie algebra of the extended Galilei group $\tilde{G}(1, 3)$ (see also [9]), the results obtained can be used in relativistic and non-relativistic physics.

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Institute of Mathematics and
Pedagogical University
ul. Podchorążych 2
PL-30-084 Kraków
Poland

and *Pidstryhach Institute of Applied*
Problems of Mechanics and Mathematics
of the National Ukrainian Academy of Sciences
Naukova Street 3b
Lviv, 79601
Ukraine

E-mail: fedorchuk@ap.krakow.pl

Franko Lviv National University
Universytetska Street 1
Lviv, 79000
Ukraine
E-mail: fedorchukv@ukr.net