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On conformally symmetric warped products

Dedicated to Professor Dr. Andrzej Zajtz on his seventieth birthday

Abstract. We prove the necessary and sufficient conditions for a warped product manifold to be conformally symmetric. Basing on these results we give two examples of such warped products.

1. Introduction

Let (M, g) and (M', g') be two Riemannian manifolds whose metrics need not be positive definite and let $f > 0$ be a smooth function on M. The warped product ([1], [11]) $\overline{M} = M \times_f M'$ is the product manifold $M \times M'$ furnished with the metric

$$\bar{g} = \pi^* g + (f \circ \pi)\sigma^* g',$$

where π and σ are the projections of $M \times M'$ onto M and M', respectively. M is called the base of $\overline{M} = M \times_f M'$, M' the fiber, and f the warping function. An n-dimensional ($n > 3$) Riemannian manifold is said to be conformally symmetric ([2]) if its Weyl conformal curvature tensor

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}\,(g_{ij}S_{hk} - g_{ik}S_{hj} + g_{hk}S_{ij} - g_{hj}S_{ik}) + \frac{\kappa}{(n-1)(n-2)}\,(g_{hk}g_{ij} - g_{hj}g_{ik})$$

is parallel, i.e. $\nabla C = 0$. Such a manifold is said to be essentially conformally symmetric (shortly, e.c.s.) if it is neither conformally flat ($C = 0$) nor locally symmetric ($\nabla R = 0$). Properties and examples of e.c.s. manifolds can be found in [3]-[6] and [12].

In this paper we are concerned with e.c.s. warped products. Some necessary conditions for a warped product to be conformally symmetric can be found in [7] and [10].

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In Section 3 we prove many necessary conditions for a warped product to be an e.c.s. manifold. In Section 4 we state conditions which are also sufficient. Basing on these results we give two examples of e.c.s. warped products.

Throughout this paper, by a manifold we mean a connected paracompact manifold of class C^∞ or analytic. By abuse of notation concerning Riemannian manifolds we often write M instead of (M, g).

2. Preliminaries

Let $(\overline{M}, \bar{g})$ be an n-dimensional $(n > 3)$ warped product $M \times_f M'$ $(\dim M = q,\ 1 \leqslant q < n,\ \dim M' = n - q = s)$. In a suitable product chart $x^1, \dots, x^n$ for $\overline{M}$ we have

$$\bar{g}_{ij} dx^i dx^j = g_{ab} dx^a dx^b + f \cdot g'_{\alpha\beta} dx^\alpha dx^\beta,$$

where $i, j = 1, \dots, n$, $a, b = 1, \dots, q$, $\alpha, \beta = q+1, \dots, n$, g_{ab} and f are functions of (x^a) only, and $g'_{\alpha\beta}$ are functions of (x^α) only.

We denote by Γ^a_{bc},R_{abcd}, S_{ab} and κ, the components of the Levi-Civita connection ∇, the Riemann-Christoffel curvature tensor R, the Ricci tensor S and the scalar curvature of (M, g), respectively. Moreover, when $\overline{\Omega}$ is a quantity formed with respect to $\bar{g}$, we denote by Ω' the similar quantity formed with respect to g'.

It is easy to show that the following relations hold (cf. [10])

$$\bar{\Gamma}^c_{ab} = \Gamma^c_{ab}\,, \qquad \bar{\Gamma}^\alpha_{b\gamma} = \frac{1}{2f} \delta^\alpha_\gamma f_b\,, \qquad \bar{\Gamma}^\alpha_{\beta\gamma} = \Gamma'^\alpha_{\beta\gamma}\,,$$

$$\bar{\Gamma}^a_{\beta\gamma} = -\frac{1}{2} g^{ad} f_d g'_{\beta\gamma}\,, \qquad \bar{\Gamma}^a_{b\gamma} = \bar{\Gamma}^\alpha_{bc} = 0,$$

where $f_b = \partial_b f$, $\partial_b = \frac{\partial}{\partial x^b}$.

In the sequel we will use the following notations

$$G_{abcd} = g_{ad} g_{bc} - g_{ac} g_{bd}\,,$$

$$T_{ab} = -\tfrac{1}{2f}(\nabla_b f_a - \tfrac{1}{2f} f_a f_b), \qquad tr(T) = g^{ab} T_{ab}\,,$$

$$Q = f((s-1)P - tr(T)), \qquad P = \tfrac{1}{4f^2} g^{ab} f_a f_b\,. \tag{1}$$

By an elementary calculation we can show that the only non-zero components of $\bar{R}$, $\bar{S}$ and $\bar{C}$ are those related to:

$$\bar{R}_{abcd} = R_{abcd}\,, \qquad \bar{R}_{a\beta c\delta} = f T_{ac} g'_{\beta\delta}\,, \qquad \bar{R}_{\alpha\beta\gamma\delta} = f R'_{\alpha\beta\gamma\delta} + f^2 P G'_{\alpha\beta\gamma\delta}\,,$$

(2)
$$a) \quad \bar{S}_{ab} = S_{ab} - s T_{ab}\,,$$

$$b) \quad \bar{S}_{\alpha\beta} = S'_{\alpha\beta} + Q g'_{\alpha\beta}\,,$$

(3) $a)$ $\bar{C}_{abcd}$

$$= R_{abcd} - \frac{1}{n-2}(g_{ad}S_{bc} + g_{bc}S_{ad} - g_{ac}S_{bd} - g_{bd}S_{ac}) + \frac{s}{n-2}(g_{ad}T_{bc} + g_{bc}T_{ad} - g_{ac}T_{bd} - g_{bd}T_{ac}) + \frac{\bar{\kappa}}{(n-1)(n-2)}\, G_{abcd}\,,$$

$b)$ $\bar{C}_{\alpha\beta\gamma\delta}$

$$= f\left(R'_{\alpha\beta\gamma\delta} - \frac{1}{n-2}(g'_{\alpha\delta}S'_{\beta\gamma} + g'_{\beta\gamma}S'_{\alpha\delta} - g'_{\alpha\gamma}S'_{\beta\delta} - g'_{\beta\delta}S'_{\alpha\gamma})\right) + (f^2P - \frac{2fQ}{n-2} + \frac{f^2\bar{\kappa}}{(n-1)(n-2)}\, G'_{\alpha\beta\gamma\delta}\,,$$

$c)$ $\bar{C}_{a\beta c\delta}$

$$= \frac{1}{n-2}\left(fg'_{\beta\delta}(S_{ac} + (q-2)T_{ac}) + g_{ac}S'_{\beta\delta} + \left(Q - \frac{f\bar{\kappa}}{n-1}\right) g_{ac}g'_{\beta\delta}\right).$$

Moreover,

$$\bar{\kappa} = \kappa + \frac{\kappa'}{f} + s((s-1)P - 2tr(T)). \tag{4}$$

Similarly, by an elementary but lenghty calculation we can easily show that the only non-zero components of $\overline{\nabla}\bar{R}$ and $\overline{\nabla}\bar{S}$ are those related to:

(5)

$$a)\quad \overline{\nabla}_e\bar{R}_{abcd} = \nabla_e R_{abcd}\,,$$

$$b)\quad \overline{\nabla}_\varepsilon\bar{R}_{\alpha\beta\gamma\delta} = f\nabla'_\varepsilon R'_{\alpha\beta\gamma\delta}\,,$$

$$c)\quad \overline{\nabla}_e\bar{R}_{\alpha\beta\gamma\delta} = -f_e R'_{\alpha\beta\gamma\delta} + f^2(\partial_e P)G'_{\alpha\beta\gamma\delta}\,,$$

$$d)\quad \overline{\nabla}_\varepsilon\bar{R}_{\alpha\beta\gamma d} = -\frac{f_d}{2}R'_{\alpha\beta\gamma\varepsilon} + \frac{f^2}{2}(\partial_d P)G'_{\alpha\beta\gamma\varepsilon}\,,$$

$$e)\quad \overline{\nabla}_e\bar{R}_{a\beta c\delta} = f\nabla_e T_{ac}g'_{\beta\delta}\,,$$

$$f)\quad \overline{\nabla}_\varepsilon\bar{R}_{abc\delta} = \frac{1}{2}g'_{\varepsilon\delta}(f_aT_{bc} - f_bT_{ac}) + \frac{1}{2}f^dR_{abcd}g'_{\varepsilon\delta}\,,$$

(6)

$$a)\quad \overline{\nabla}_c\bar{S}_{ab} = \nabla_c S_{ab} - s\nabla_c T_{ab}\,,$$

$$b)\quad \overline{\nabla}_\gamma\bar{S}_{\alpha\beta} = \nabla'_\gamma S'_{\alpha\beta}\,,$$

$$c)\quad \overline{\nabla}_\gamma\bar{S}_{\alpha b} = -\frac{1}{2f}S'_{\alpha\gamma}f_b + \frac{1}{2}g'_{\alpha\gamma}(f^cS_{cb} - sf^cT_{cb} - \frac{Q}{f}f_b),$$

$$d)\quad \overline{\nabla}_c \bar{S}_{\alpha\beta} = (\partial_c Q) g'_{\alpha\beta} - \frac{f_c}{f}(S'_{\alpha\beta} + Q g'_{\alpha\beta}).$$

From the above formulas we immediately obtain

Lemma 2.1
Let $\overline{M}$ be a warped product $M \times_f M'$ with vanishing scalar curvature $\bar{\kappa}$. Then $\overline{M}$ is conformally symmetric if and only if

$$(7)\quad a)\quad \overline{\nabla}_e \bar{R}_{abcd} = \frac{1}{n-2}(\bar{g}_{ad}\overline{\nabla}_e \bar{S}_{bc} + \bar{g}_{bc}\overline{\nabla}_c \bar{S}_{ad} - \bar{g}_{ac}\overline{\nabla}_e \bar{S}_{bd} - \bar{g}_{bd}\overline{\nabla}_e \bar{S}_{ac}),$$

$$b)\quad \overline{\nabla}_\varepsilon \bar{R}_{\alpha\beta\gamma\delta} = \frac{1}{n-2}(\bar{g}_{\alpha\delta}\overline{\nabla}_\varepsilon \bar{S}_{\beta\gamma} + \bar{g}_{\beta\gamma}\overline{\nabla}_\varepsilon \bar{S}_{\alpha\delta} - \bar{g}_{\alpha\gamma}\overline{\nabla}_\varepsilon \bar{S}_{\beta\delta} - \bar{g}_{\beta\delta}\overline{\nabla}_\varepsilon \bar{S}_{\alpha\gamma}),$$

$$c)\quad \overline{\nabla}_e \bar{R}_{\alpha\beta\gamma\delta} = \frac{1}{n-2}(\bar{g}_{\alpha\delta}\overline{\nabla}_e \bar{S}_{\beta\gamma} + \bar{g}_{\beta\gamma}\overline{\nabla}_e \bar{S}_{\alpha\delta} - \bar{g}_{\alpha\gamma}\overline{\nabla}_e \bar{S}_{\beta\delta} - \bar{g}_{\beta\delta}\overline{\nabla}_e \bar{S}_{\alpha\gamma}),$$

$$d)\quad \overline{\nabla}_\varepsilon \bar{R}_{\alpha\beta\gamma d} = \frac{1}{n-2}(\bar{g}_{\beta\gamma}\overline{\nabla}_\varepsilon \bar{S}_{\alpha d} - \bar{g}_{\alpha\gamma}\overline{\nabla}_\varepsilon \bar{S}_{\beta d}),$$

$$e)\quad \overline{\nabla}_e \bar{R}_{a\beta c\delta} = \frac{1}{n-2}(-\bar{g}_{ac}\overline{\nabla}_e \bar{S}_{\beta\delta} - \bar{g}_{\beta\delta}\overline{\nabla}_e \bar{S}_{ac}),$$

$$f)\quad \overline{\nabla}_\varepsilon \bar{R}_{abc\delta} = \frac{1}{n-2}(\bar{g}_{bc}\overline{\nabla}_\varepsilon \bar{S}_{a\delta} - \bar{g}_{ac}\overline{\nabla}_\varepsilon \bar{S}_{b\delta}),$$

$$g)\quad \overline{\nabla}_\varepsilon \bar{S}_{\beta\delta} = 0.$$

In the sequel we shall need the following properties of e.c.s. manifolds:

Lemma 2.2 ([5], [6])
Every e.c.s. manifold $(\overline{M}, \bar{g})$ satisfies the relations:

$$(8)\qquad \bar{\kappa} = 0,$$

$$(9)\qquad \overline{\nabla}_k \bar{S}_{ij} = \overline{\nabla}_j \bar{S}_{ik}\,,$$

$$(10)\qquad \bar{S}_{il}\bar{C}_{hmjk} + \bar{S}_{ij}\bar{C}_{hmkl} + \bar{S}_{ik}\bar{C}_{hmlj} = 0.$$

Lemma 2.3 ([6])
Let $(\overline{M}, \bar{g})$ be an e.c.s. manifold. Then $\overline{M}$ admits a unique function $\bar{F}$ such that

$$(11)\qquad \bar{F}\bar{C}_{hijk} = \bar{S}_{hk}\bar{S}_{ij} - \bar{S}_{hj}\bar{S}_{ik}.$$

$\bar{F}$ is said to be the fundamental function of $\overline{M}$. It is clear that $\bar{F}(x) = 0$ if and only if rank $\bar{S}(x) \leqslant 1$.

Moreover we shall use the following fact

Lemma 2.4 ([9], Theorem 1)
Let M be an n-dimensional Riemannian manifold. If B is a generalized curvature tensor satisfying $\nabla_m \nabla_l B_{hijk} = \nabla_l \nabla_m B_{hijk}$ and P is a vector field such that $w^r R_{rijk} = P_k g_{ij} - P_j g_{ik}$ for some vector field w, then

$$P_h \left(B_{lijk} - \frac{\kappa(B)}{n(n-1)} (g_{ij} g_{lk} - g_{ik} g_{lj}) \right) = 0,$$

where $\kappa(B) = B_{rijs} g^{rs} g^{ij}$.

3. Necessary conditions

Lemma 3.1
If $\overline{M} = M \times_f M'$ is an e.c.s. manifold then $\dim M = q > 1$.

Proof. We shall use the following fact due to Kručkovič

Theorem 3.1 ([11], p. 116)
A Riemannian space $\overline{V}^n$ admits a solution $k \neq constant$ of the equation

(12) $$\overline{\nabla}_j \overline{\nabla}_i k = \phi \bar{g}_{ij}$$

such that $\operatorname{grad} k$ *is non-null vector field if and only if $\overline{V}^n$ is a warped product with one-dimensional base.*

Thus supposing that $q = 1$, we have (12). Differentiating (12) covariantly and alternating the resulting equation, by Ricci identity, we easily obtain

$$k_r \bar{R}^r_{ijk} = \overline{\nabla}_j \phi \bar{g}_{ik} - \overline{\nabla}_k \phi \bar{g}_{ij}\,.$$

Now, using Lemma 2.4 for $M = \overline{M}$, $B = C$ and $P = \operatorname{grad} \phi$, we have $\overline{\nabla}_j \phi = 0$. Thus $\overline{\nabla}_j \overline{\nabla}_i k = c \cdot \bar{g}_{ij}$, $c = constant$. We assert that $c = 0$. Suppose that $c \neq 0$. Then the manifold $\overline{M}$ admits a vector field v such that $\overline{\nabla}_j v_i = \bar{g}_{ij}$. This equation immediately implies $v_r \bar{R}^r_{hij} = 0$ and $v_r \bar{S}^r_j = 0$ and next $v_r \overline{\nabla}_l \bar{R}^r_{hij} = -\bar{R}_{lhij}$, $v_r \overline{\nabla}_l \bar{S}^r_h = -\bar{S}_{lh}$. Using now the second Bianchi identity, we have $v^r \overline{\nabla}_r \bar{R}_{ijhl} = -2\bar{R}_{ijhl}$ and $v^r \overline{\nabla}_r \bar{S}_{il} = -2\bar{S}_{il}$. Transvection of (9) with v^j leads to $\bar{S} = 0$, a contradiction. Thus the gradient of k is non-null parallel vector field. But in any e.c.s. manifold every parallel vector field must be isotropic ([5], Theorem 11 and [12]). This completes the proof.

Proposition 3.1
Let $\overline{M} = M \times_f M'$ be an e.c.s. manifold. Then M' is of constant curvature. Moreover, if $\dim M' = s > 1$ *then*

(13) $$s(s-1)((n-2)f^2 \partial_e P - f \partial_e Q + f_e Q) = f_e (q-s) \kappa'.$$

Proof. We can assume that $s > 1$. Using (7)c), (5)c) and (6)d), we have

$$\begin{aligned}f_e((n-2)R'_{\alpha\beta\gamma\delta} - (g'_{\alpha\delta}S'_{\beta\gamma} + g'_{\beta\gamma}S'_{\alpha\delta} - g'_{\alpha\gamma}S'_{\beta\delta} - g'_{\beta\delta}S'_{\alpha\gamma})) \\ = ((n-2)f^2\partial_e P - f\partial_e Q + f_e Q)G'_{\alpha\beta\gamma\delta}\,.\end{aligned} \tag{14}$$

Contracting (14) with $g'^{\beta\gamma}$, we obtain

$$f_e(qS'_{\alpha\delta} - \kappa' g'_{\alpha\delta}) = ((n-2)f^2\partial_e P - f\partial_e Q + f_e Q)s(s-1). \tag{15}$$

Further contraction with $g'^{\alpha\delta}$ leads to (13). Substituting (13) into (15), we get $S'_{\alpha\delta} = \frac{\kappa'}{s}g'_{\alpha\delta}$ which together with (13) turns (14) into $R' = \frac{\kappa'}{s(s-1)}G'$. This completes the proof.

LEMMA 3.2
Let $\overline{M} = M \times_f M'$ be an e.c.s. manifold. Then $\bar{C}_{abcd} \neq 0$ at every point x of $\overline{M}$.

Proof. Suppose that there exists point $x \in \overline{M}$ at which $\bar{C}_{abcd} = 0$. Using (3)a), we have ($q > 1$ in virtue of Lemma 3.1)

$$(n-2)R_{abcd} = g_{ad}(S_{bc} - sT_{bc}) + g_{bc}(S_{ad} - sT_{ad}) - g_{ac}(S_{bd} - sT_{bd}) - g_{bd}(S_{ac} - sT_{ac}).$$

Contracting this equation with g^{bc}, we get

$$s(S_{ad} + (q-2)T_{ad}) = \kappa - s \cdot tr(T) \tag{16}$$

and, after contraction with g^{ad},

$$(q-s)\kappa = 2s(q-1)tr(T). \tag{17}$$

Now substituting $S'_{\beta\delta} = \frac{\kappa'}{s}g'_{\beta\delta}$, which is an obvious consequence of Proposition 3.1, and (16) into (3)c) and using (17), we have $\bar{C}_{a\beta c\delta} = 0$. Finally, in the same way, using (3)a) and

$$\kappa f + \kappa' + sf((s-1)P - 2tr(T)) = 0 \tag{18}$$

which follows from (4) and (8), we easily obtain $\bar{C}_{\alpha\beta\gamma\delta} = 0$. Thus $\bar{C} = 0$ at x, a contradiction. This completes the proof.

LEMMA 3.3
Let $\overline{M} = M \times_f M'$ be an e.c.s. manifold. Then $\kappa = 0$, $\kappa' = 0$, $tr(T) = 0$, $Q = 0$ and if $s > 1$ also $P = 0$.

Proof. Using (10), (2) and (3), we have

$$\bar{S}_{\alpha\beta}\bar{C}_{abcd} = 0, \qquad \bar{S}_{ab}\bar{C}_{\alpha\beta\gamma\delta} = 0.$$

Thus, in virtue of Lemma 3.2, we get

$$\bar{C}_{\alpha\beta\gamma\delta} = 0, \tag{19}$$

$$\bar{S}_{\alpha\beta} = 0 \tag{20}$$

which, in view of (2)b) and Proposition 3.1, is equivalent to

$$\frac{\kappa'}{s} + Q = 0. \tag{21}$$

If $s = 1$, then $\kappa' = 0$ and $Q = 0$. Using now (1), we get $tr(T) = 0$. Applying now (18), we have $\kappa = 0$.

Consider now the case $s > 1$. Substituting (21) into (18), we obtain

$$\kappa = s \cdot tr(T). \tag{22}$$

Using (19), (21) and Proposition 3.1, we have

$$\frac{\kappa'}{s(s-1)} + fP = 0 \tag{23}$$

which implies $fP = constant$ and further $fP_e = -f_e P$. Substituting the last equality and (21) into (13) we get $(s-1)\kappa' = 0$. Thus $\kappa' = 0$, $P = 0$ (by (23)), $Q = 0$ (by (21)), $tr(T) = 0$ (by (1)) and $\kappa = 0$ (by (22)). This completes the proof.

Proposition 3.2
In every e.c.s. warped product $M \times_f M'$ the tensor $S + (q-2)T$ is parallel and the tensor T is a Codazzi tensor.

Proof. The first assertion is an immediate consequence of (7)e), (5)e), (6)a), (6)d) and

$$\overline{\nabla}_e \bar{S}_{\beta\delta} = 0 \tag{24}$$

which simply follows from $Q = 0$ and $\kappa' = 0$. Using (9) and (24), we have $\overline{\nabla}_\varepsilon \bar{S}_{a\delta} = 0$. Thus (7)f), in view of (5)f) takes the form

$$f^d R_{abcd} = f_b T_{ac} - f_a T_{bc}\,. \tag{25}$$

But this equation, via Ricci identity, is equivalent to $\nabla_c T_{ab} = \nabla_b T_{ac}$. This completes the proof.

Lemma 3.4
Let $\overline{M} = M \times_f M'$ be an e.c.s. manifold. Then $q = \dim M > 2$.

Proof. Suppose that $q = 2$. From $\kappa = 0$ and Proposition 3.2 we obtain $S = 0$, which yields, in view of Lemma 3.3 and Proposition 3.1, $\bar{C}_{a\beta c\delta} = 0$. Now the formula (3)a) reduces to

$$\bar{C}_{1212} = -g_{11}T_{22} - g_{22}T_{11}\,.$$

But $tr(T) = 0$, so $\bar{C}_{1212} = 0$. Taking (19) into account we see that $\bar{C} = 0$, a contradiction.

Lemma 3.5
Let $\overline{M} = M \times_f M'$ be an e.c.s. manifold. If $S + (q-2)T = 0$ then $q > 3$.

Proof. In virtue of (19), (3)c) and Lemma 3.3, we have $\bar{C}_{\alpha\beta\gamma\delta} = 0 = \bar{C}_{a\beta c\delta}$ and it sufficies to show that if $q = 3$ then $\bar{C}_{abcd} = 0$. But substituting assumed equation into (3)a), in view of (8), we get $\bar{C}_{abcd} = C_{abcd} = 0$ $(q = 3)$.

Proposition 3.3 (cf. [7])
If $\overline{M} = M \times_f M'$ is an e.c.s. manifold then M is conformally symmetric.

Proof. Using Proposition 3.2 and (6)a), we have $\overline{\nabla}_c\bar{S}_{ab} = \frac{n-2}{q-2}\nabla_c S_{ab}$ which together with (5)a) turns (7)a) into

$$\nabla_e R_{abcd} = \frac{1}{q-2}(g_{ad}\nabla_e S_{bc} + g_{bc}\nabla_e S_{ad} - g_{ac}\nabla_e S_{bd} - g_{bd}\nabla_e S_{ac}).$$

But this equation, in virtue of $\kappa = 0$, is equivalent to our assertion.

4. Main results

We are now in a position to prove main results of this paper.

Proposition 4.1
Let $\overline{M} = M \times_f M'$ be an e.c.s. manifold. If $S + (q-2)T \neq 0$ then M is Ricci-recurrent conformally symmetric manifold and $\overline{M}$ is not Ricci-recurrent.

Proof. Using (11), in virtue of (20), we have

$$\bar{F} \cdot \bar{C}_{a\beta c\delta} = 0 \qquad (26)$$

which in view of our assumption gives $\bar{F} = 0$, i.e., rank $\bar{S} \leq 1$. On the other hand, (10) implies

$$\bar{S}_{ab}\bar{C}_{c\beta d\delta} = \bar{S}_{ac}\bar{C}_{b\beta d\delta}$$

which can be written in the form

$$(S_{ab} - sT_{ab})(S_{cd} + (q-2)T_{cd}) = (S_{ac} - sT_{ac})(S_{bd} + (q-2)T_{bd}).$$

This means that rank of tensors $S - sT$ and $S + (q-2)T$ is equal to 1 and since $S + (q-2)T$ is parallel so

$$S + (q-2)T = e \cdot v \otimes v, \qquad |e| = 1$$

for some parallel vector field v. Consequently $S = \phi v \otimes v$, ϕ being a function and M is Ricci-recurrent.

The second part of our assertion follows, in a purely algebraic manner from (2), (6) and the above equalities.

THEOREM 4.1
Let $\overline{M}$ be a warped product $M \times_f M'$. Then the following conditions are equivalent:

(i) *$\overline{M}$ is Ricci-recurrent e.c.s. manifold,*

(ii) *M' is flat, M is Ricci-recurrent e.c.s. manifold, $S + (q-2)T = 0$ and* grad f *is null when* $\dim M' > 1$.

Proof. If $\overline{M}$ is Ricci-recurrent e.c.s. manifold then Proposition 4.1 leads to $S + (q-2)T = 0$. Thus $\bar{S}_{ab} = \frac{n-2}{q-2} S_{ab}$ and because $\bar{S}_{\alpha\beta} = 0$ so $\bar{S}$ and S are simultaneously recurrent and non-parallel. M cannot be conformally flat since $C_{abcd} = \bar{C}_{abcd} \neq 0$. Therefore M is Ricci-recurrent e.c.s. manifold and Proposition 3.1 and Lemma 3.3 imply $R' = 0$ and $P = 0$ if $s > 1$.

Now assume (ii). Since $T = -\frac{1}{q-2} S$, so $\kappa = 0$ implies $tr(T) = 0$ and $Q = 0$. S as well as T are Codazzi tensors. Thus we have (25) which implies $f^c S_{cb} = f^c T_{cb}$ and $f^c S_{cb} - s f^c T_{cb} = 0$. Therefore the only non-zero components of $\overline{\nabla}\bar{S}$ and $\overline{\nabla}\bar{R}$ are those related to $\overline{\nabla}_c \bar{S}_{ab}$, $\overline{\nabla}_e \bar{R}_{abcd}$, $\overline{\nabla}_e \bar{R}_{\alpha\beta c\delta}$ and $\overline{\nabla}_\varepsilon \bar{R}_{abc\delta}$. Using above facts we can easily see that all conditions (7) are satisfied. Thus $\overline{M}$ is conformally symmetric. As in the first part of this proof, we obtain $\bar{C}_{abcd} \neq 0$ and $\overline{\nabla}_c \bar{S}_{ab} \neq 0$ which imply that $\overline{M}$ is e.c.s. manifold. This completes the proof.

THEOREM 4.2
Let $\overline{M}$ be a warped product $M \times_f M'$. Then $\overline{M}$ is an e.c.s. manifold with $\bar{F} \neq 0$ if and only if M' is flat, M is e.c.s. manifold with $F \neq 0$, $S + (q-2)T = 0$ and grad f *is null when* $\dim M' > 1$.

Proof. If $\overline{M}$ is e.c.s. manifold with $\bar{F} \neq 0$ then (26) implies $\bar{C}_{\alpha\beta c\delta} = 0$ which leads to $S + (q-2)T = 0$. To prove remaining parts of this theorem, we use the same procedure as in the proof of Theorem 4.1.

Now we give two examples of e.c.s. warped products. The first warped product is Ricci-recurrent, the second one is not Ricci-recurrent. We shall use the following

THEOREM 4.3 ([12])
Let M denote the Euclidean q-space ($q \geqslant 4$) endowed with the metric g given by

(27) $$g_{ab} dx^a dx^b = \Phi (dx^1)^2 + k_{\lambda\mu} dx^\lambda dx^\mu + 2 dx^1 dx^q,$$

$$\Phi = (Ak_{\lambda\mu} + a_{\lambda\mu})x^{\lambda}x^{\mu},$$

where $a, b = 1, \ldots, q$ *and* $\lambda, \mu = 2, \ldots, q-1$ *and* A *is a non-constant function of* x^1 *only,* $[k_{\lambda\mu}]$ *and* $[a_{\lambda\mu}]$ *are non-zero symmetric matrices such that* $[k_{\lambda\mu}]$ *is non-singular and* $k^{\lambda\mu}a_{\lambda\mu} = 0$*,* $[k^{\lambda\mu}]$ *being the reciprocal of* $[k_{\lambda\mu}]$*. Then* M *is an e.c.s. Ricci-recurrent Riemannian manifold.*

Example 4.1
Let M be the Euclidean q-space endowed with the metric g given by (27) with $A = \frac{2}{(x^1)^2}$.

The only Christoffel symbols not identically zero are

$$\Gamma^{\lambda}_{11} = -\frac{1}{2}k^{\lambda\omega}\partial_{\omega}\Phi, \qquad \Gamma^{q}_{11} = \frac{1}{2}\partial_1\Phi, \qquad \Gamma^{q}_{1\gamma} = \frac{1}{2}\partial_{\gamma}\Phi.$$

$S_{11} = (q-2)A$ and all other components of S are identically zero. Taking $f = f(x^1)$ we can easily show that the equation $S + (q-2)T = 0$ is equivalent to $f'' - \frac{(f')^2}{2f} = 2Af$, where $f' = \partial_1 f$. One can easily verify that the function $f(x^1) = (x^1)^4$ satisfies this equation and $\operatorname{grad} f$ is null ($g^{11} = 0$). Thus taking above described M and f, and a flat manifold M', via Theorem 4.1, we obtain Ricci-recurrent e.c.s. manifold $\overline{M} = M \times_f M'$.

Remark 4.1
The example satisfying assumptions of Theorem 4.1 can be found in [8]. However the authors of that paper were interested in conformally recurrent or birecurrent manifolds, but $k = \frac{1}{2}$ in (4.4) of [8] leads to a conformally symmetric warped product.

Example 4.2
Let M be as above with $A = \frac{2}{(x^1)^2} - c$, $c = constant$. In this metric the null parallel vector field is of the form $v_i = \delta^1_i$. Taking $f(x^1) = (x^1)^4$, we get

$$S + (q-2)T = c(q-2)v \otimes v.$$

By an elementary calculation we can show that if M' is flat then $\overline{M} = M \times_f M'$ is e.c.s. manifold which is not Ricci-recurrent.

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