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Locally conformal symplectic structures and their generalizations from the point of view of Lie algebroids

Dedicated to Andrzej Zajtz, with admiration, on his 70th birthday

Abstract. We study locally conformal symplectic structures and their generalizations from the point of view of transitive Lie algebroids. To consider l.c.s. structures and their generalizations we use Lie algebroids with trivial adjoint Lie algebra bundle $M \times \mathbb{R}$ and $M \times \mathfrak{g}$. We observe that important l.c.s's notions can be translated on the Lie algebroid's language. We generalize l.c.s. structures to $\mathfrak{g}$-l.c.s. structures in which we can consider an arbitrary finite dimensional Lie algebra $\mathfrak{g}$ instead of the commutative Lie algebra $\mathbb{R}$.

1. l.c.s. structures from the point of view of Lie algebroids

We study locally conformal symplectic structures and their generalizations from the point of view of transitive Lie algebroids. We recall that an l.c.s. structure on a manifold M is a pair (ω, Ω) of differentiable forms on M such that

(1) ω is a real closed 1-form on M,

(2) Ω is a real non-degenerated 2-form fulfilling the property

$$d\Omega = -\omega \wedge \Omega.$$

From the non-degeneracy of Ω it follows that M has even dimension.

By a *transitive Lie algebroid on a manifold* M ([16]) we mean a system $(A, [\![\cdot,\cdot]\!], \#_A)$ consisting of a vector bundle A over M and mappings

$$[\![\cdot,\cdot]\!] \colon \operatorname{Sec} A \times \operatorname{Sec} A \longrightarrow \operatorname{Sec} A, \qquad \#_A \colon A \longrightarrow TM,$$

such that

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(a) $(\mathrm{Sec}\, A, [\![\cdot,\cdot]\!])$ is a real Lie algebra,

(b) $\#_A$, called an *anchor*, is an epimorphism of vector bundles,

(c) $\mathrm{Sec}\,\#_A\colon \mathrm{Sec}\, A \longrightarrow \mathfrak{X}(M),\ \xi \longmapsto \#_A \circ \xi$, is a homomorphism of Lie algebras,

(d) $[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + (\#_A\circ\xi)(f)\cdot\eta,\ \ \xi,\eta\in \mathrm{Sec}\, A,\ f\in\Omega^0(M) = C^\infty(M)$.

The axiom (c) follows from the remaining ones, see [9], [1].

It follows that $\boldsymbol{g} := \ker \#_A$ is a LAB (Lie algebra bundle), called the *adjoint of* A. The Lie algebra $\boldsymbol{g}_x$ is called the structure Lie algebra at x. The exact sequence

$$0 \longrightarrow \boldsymbol{g} \longrightarrow A \xrightarrow{\#_A} TM \longrightarrow 0$$

is called the *Atiyah sequence* of A, while any splitting $\lambda\colon TM \longrightarrow A$, $\#_A\circ\lambda = \mathrm{id}_{TM}$, is a *connection* in A. The following geometric objects give rise to transitive Lie algebroids:

— Lie groupoids,
— principal fibre bundles,
— vector bundles,
— transversely complete foliations,
— nonclosed Lie subgroups.

Let us remark that differential groupoids (non-transitive, in general), Poisson and Jacobi manifolds as well as any infinitesimal action of a Lie algebra on a manifold produce nontransitive Lie algebroids. The image of the anchor is always an integrable singular (or regular) foliation ([17], [6]) and the restriction of the Lie algebroid to any leaf of this foliation is a transitive Lie algebroid.

To consider l.c.s. structures and their generalizations we use Lie algebroids with trivial adjoint Lie algebra bundle $\boldsymbol{g} = M\times\mathfrak{g}$.

From the general theorem concerning the form of any transitive Lie algebroids (Mackenzie [15], Kubarski [11]) we have:

Each transitive Lie algebroid on M *with a trivial adjoint bundle* $\boldsymbol{g}\cong M\times\mathbb{R}$ *is isomorphic to*

$$A = TM\times\mathbb{R}$$

with $\#_A = \mathrm{pr}_1\colon TM\times\mathbb{R}\longrightarrow TM$ *as the anchor and the bracket* $[\![\cdot,\cdot]\!]$ *in* $\mathrm{Sec}\, A$ *is defined via some flat covariant derivative* ∇ *in* $M\times\mathbb{R}$ *and a 2-form* $\Omega\in\Omega^2(M)$ *fulfilling the Bianchi identity* $\nabla\Omega = 0$ *in the following way*

$$[\![(X,f),(Y,g)]\!] = ([X,Y], \nabla_X g - \nabla_Y f - \Omega(X,Y)).$$

We recall that a covariant derivative ∇ in a vector bundle ξ determines a standard operator $d_\nabla\colon \Omega^*(M;\xi) \longrightarrow \Omega^*(M;\xi)$ and $d_\nabla\theta$ is sometimes denoted by $\nabla\theta$. If ∇ is flat then $(d_\nabla)^2 = 0$ and it determines the cohomology space $H_\nabla(M;\xi)$ in the obvious way.

Each flat covariant derivative in $\boldsymbol{g} = M \times \mathbb{R}$ is of the form

$$\nabla_X f = \partial_X f + \omega(X) \cdot f$$

where ω is a closed differentiable 1-form on M. Then the differential operator d_∇ is denoted rather by d_ω ([7], [8]). We have

$$d_\omega(\theta) = d\theta + \omega \wedge \theta$$

and write $H_\omega(M) := H_{d_\omega}(M)$.

The condition $\nabla\Omega = 0$ is then equivalent to $d\Omega = -\omega \wedge \Omega$.

Hence any transitive Lie algebroid with the trivial adjoint bundle $\boldsymbol{g} = M \times \mathbb{R}$ is determined by the following data:

$$\text{a closed 1-form } \omega \text{ and a 2-form } \Omega \text{ such that } d\Omega = -\omega \wedge \Omega. \qquad (\star)$$

The Lie algebroid obtained in this way will be denoted by

$$(TM \times \mathbb{R}, \omega, \Omega).$$

Lemma 1.1

A connection $\lambda\colon TM \longrightarrow TM \times \mathbb{R}$ in the Lie algebroid $A = (TM \times \mathbb{R}, \omega, \Omega)$ is of the form $\lambda(X) = (X, \eta(X))$ for a 1-form $\eta \in \Omega^1(M)$. The curvature form $\Omega^\lambda(X,Y) = [\![\lambda X, \lambda Y]\!] - \lambda[X,Y]$ of the connection λ is equal to

$$\Omega^\lambda = d_\omega(\eta) - \Omega = d\eta + \omega \wedge \eta - \Omega. \tag{1.1}$$

According to $(\star)$, the pair (ω, Ω) determining the above Lie algebroid is precisely a locally conformal symplectic structure (l.c.s. structure, for short) on the manifold M provided that the 2-form Ω is non-degenerate. Therefore our transitive Lie algebroids $TM \times \mathbb{R}$ determined by (ω, Ω) are natural generalizations of the locally conformal symplectic structures. For an l.c.s. structure (ω, Ω), following $(\star)$, the form Ω represents the cohomology class $[\Omega] \in H^2_\omega(M)$ which is called the *Lichnerowicz class* of the l.c.s. structure (ω, Ω) ([3]). If the 1-form ω is exact the l.c.s. structure is called *globally* conformal symplectic structure. The property that an l.c.s. structure is global can be equivalently expressed in the language of Lie algebroids ([10], [14]). For this purpose we recall that a transitive Lie algebroid $(A, [\![\cdot,\cdot]\!], \#_A)$ is called *invariantly oriented* ([13]) if there is specified a non-singular cross-section ε of the bundle $\bigwedge^n \boldsymbol{g}$, $\boldsymbol{g} := \ker \#_A$ and $n = \operatorname{rank} \boldsymbol{g}$, which is invariant with respect to the adjoint representation of A in $\bigwedge^n \boldsymbol{g}$, equivalently, if $\boldsymbol{g}$ is orientable and the modular class of the Lie algebroid is zero ([5], [14]). Let us remark that for a transitive Lie algebroid the modular class is equal to the characteristic class of the top-power of the adjoint representation ad_A. The structure Lie algebras $\boldsymbol{g}_x$ of the invariantly oriented Lie algebroid are unimodular.

A cross-section ε of the bundle $\bigwedge^n \boldsymbol{g}$ is invariant if and only if, in any open subset $U \subset M$ on which ε is of the form $\varepsilon_{|U} = (h_1 \wedge \ldots \wedge h_n)_{|U}$, $h_i \in \mathrm{Sec}\,\boldsymbol{g}$, we have, for all $\xi \in \mathrm{Sec}\,A$,

$$\sum_{i=1}^{n} (h_1 \wedge \ldots \wedge [\![\xi, h_i]\!] \wedge \ldots \wedge h_n)_{|U} = 0.$$

In the case $A = (TM \times \mathbb{R}, \omega, \Omega)$ we have $n = 1$ and $\boldsymbol{g} = M \times \mathbb{R}$ and a positive function $\varepsilon \in C^\infty(M) = \mathrm{Sec}(M \times \mathbb{R})$ is invariant if and only if ε is ∇-constant, $\nabla\varepsilon = 0$ ([13, Lemma 6.2.1]). The condition $\nabla\varepsilon = 0$ is equivalent to $\omega = d(-\ln(\varepsilon))$.

Theorem 1.1
Let (ω, Ω) be an l.c.s. structure on an arbitrary m-dimensional connected manifold (oriented or not). The following conditions are equivalent:

(a) *the l.c.s. structure (ω, Ω) is globally conformal symplectic structure (i.e., $[\omega] = 0$),*

(b) *the associated Lie algebroid $A = (TM \times \mathbb{R}, \omega, \Omega)$ is invariantly oriented,*

(c) $H^{m+1}_{\partial^{or}_A, c}(A; or(M)) \neq 0$,

(d) $H^{m+1}_{\partial^{or}_A, c}(A; or(M)) = \mathbb{R}$, *and the pairing*

$$H^j(A) \times H^{m+1-j}_{\partial^{or}_A, c}(A; or(M)) \longrightarrow H^{m+1}_{\partial^{or}_A, c}(A; or(M)) \cong \mathbb{R}$$

is non-degenerate, i.e., $H^j(A) \cong \left(H^{m+1-j}_{\partial^{or}_A, c}(A; or(M))\right)^*$.

Proof. (a) $\Longleftrightarrow$ (b) see [10], (b) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (d) see [14].

Remark 1.1
(1) For an orientable manifold M the conditions (c) and (d) are equal to:

(c′) $H^{m+1}_c(A) \neq 0$,

(d′) $H^{m+1}_c(A) = \mathbb{R}$, and the pairing

$$H^j(A) \times H^{m+1-j}_c(A) \longrightarrow H^{m+1}_c(A) \cong \mathbb{R}$$

is non-degenerate, i.e., $H^j(A) \cong (H^{m+1-j}_c(A))$.

(2) ∂_A^{or} is the canonical representation of A in the orientation bundle $or(M)$, $(\partial_A^{or})_\gamma(\sigma) = (\partial^{or})_{\#_A(\gamma)}(\sigma)$, $\gamma \in A$, $\sigma \in \Gamma(or(M))$. ∂^{or} is the canonical flat structure of the orientation bundle $or(M)$ ([4]).

(3) Each representation ∇ of a Lie algebroid A in a vector bundle ξ (i.e., a homomorphism of a Lie algebroid A in the Lie algebroid $A(\xi)$ of the vector bundle ξ ([12], [15])) determines a standard differential operator $d_\nabla \colon \Omega(A;\xi) \longrightarrow \Omega(A;\xi)$ and $H_\nabla(A;\xi)$ is the space of cohomology of the complex $(\Omega(A;\xi), d_\nabla)$. Local trivializations of $A(\mathfrak{f})$ are constucted in the following way: Let $\psi\colon U \times V \longrightarrow p^{-1}[U] = \mathfrak{f}_{|U}$ be a local trivialization of a vector bundle $\mathfrak{f}$; V is the typical fibre. Consider the trivial Lie algebroid $TU \times \mathrm{End}(V)$. For a cross-section $\sigma \in \mathrm{Sec}\,\mathfrak{f}$, denote by σ_ψ the V-valued function $U \ni x \longmapsto \psi_x^{-1}(\sigma(x)) \in V$. The mapping

$$\bar{\psi}\colon TU \times \mathrm{End}(V) \longrightarrow A(\mathfrak{f})_{|U}$$
$$\bar{\psi}(v,a)(\sigma) = \psi_x(v(\sigma_\psi) + a(\sigma_\psi(x))),$$

$(v \in T_xU,\ x \in U,\ a \in \mathrm{End}(V),\ \sigma \in \mathrm{Sec}\,\mathfrak{f})$ is an isomorphism of Lie algebroids ([12]).

(4) The associated Lie algebra bundle of the considered Lie algebroid $A = (TM \times \mathbb{R}, \omega, \Omega)$ is the trivial line bundle $\boldsymbol{g} = M \times \mathbb{R}$. Therefore, the top group of cohomology $H^{m+1}_{\partial_A^{or},c}(A; or(M))$ can be written (analogously to real coefficients, see [10]) as follows

$$H^{m+1}_{\partial_A^{or},c}(A; or(M)) = H^m_{d_{\partial^{-\omega}\otimes\partial^{or}}}(M; or(M))$$
$$= H^m_{(\partial^{or})^{-\omega}}(M; or(M)).$$

Then the equivalence (a) $\Longleftrightarrow$ (c) follows trivially, since

$$H^m_{(\partial^{or})^{-\omega}}(M; or(M)) \neq 0 \iff [-\omega] = 0,$$

see [14].

Two l.c.s. structures (ω,Ω) and (ω',Ω') on a manifold M are called *conformally equivalent* if

$$\Omega' = \frac{1}{a}\Omega, \qquad \omega' = \omega + \frac{da}{a},$$

for a nowhere vanishing function a on M (non-singular for short).

If two l.c.s. structures (ω',Ω') and (ω,Ω) on a manifold M are conformally equivalent then the associated Lie algebroids $A' = (TM \times \mathbb{R}, \omega', \Omega')$ and $(TM \times \mathbb{R}, \omega, \Omega)$ are isomorphic via the mapping

$$H\colon (TM \times \mathbb{R}, \omega', \Omega') \longrightarrow (TM \times \mathbb{R}, \omega, \Omega)$$
$$H(X, f) = (X, a \cdot f)$$

where $a \in C^\infty(M)$ is a non-singular smooth function. The isomorphism $H\colon A' \longrightarrow A$ of the above form will be called a *conformal isomorphism.*

We must add that the general form of a homomorphism $H\colon TM \times \mathbb{R} \longrightarrow TM \times \mathbb{R}$ of vector bundles commuting with anchors $\#_A = pr_1$ is as follows

$$H(X,f) = H_{\eta,a}(X,f) := (X, \eta(X) + a \cdot f), \tag{$\star\star$}$$

for $\eta \in \Omega^1(M)$ and $a \in C^\infty(M)$.

Proposition 1.1

(A) *The following conditions are equivalent:*

(1) $H_{\eta,a}$ *is a homomorphism of Lie algebroids,*

(2) (a) $\nabla\eta = \Omega - a \cdot \Omega'$,

(b) $\nabla_X(a \cdot f) = a \cdot \nabla'_X f$,

(3) (a) $d_\omega(\eta) = d\eta + \omega \wedge \eta = \Omega - a \cdot \Omega'$,

(b) $a \cdot (\omega' - \omega) = da$.

The homomorphism $H_{\eta,a}$ is an isomorphism of Lie algebroids if and only if a is non-singular. Conditions (1), (2), (3) are then equivalent to

(4) (a) $\Omega' = \frac{1}{a} \cdot (\Omega - d_\omega(\eta))$,

(b) $\omega' = \omega + d(\ln|a|)$.

(B) *For an arbitrary Lie algebroid $A' = (TM \times \mathbb{R}, \omega', \Omega')$ and data (η, a) where $\eta \in \Omega^1(M)$ and a is a non-singular function, the differential forms $\omega = \omega' - d(\ln|a|)$, $\Omega = a \cdot \Omega' + d_\omega(\eta)$ fulfil the condition $d\Omega = -\omega \wedge \Omega$, i.e., the data (ω, Ω) determines a Lie algebroid $A = (TM \times \mathbb{R}, \omega, \Omega)$ and $H_{\eta,a}\colon A' \longrightarrow A$ given by $(\star\star)$ is an isomorphism of Lie algebroids.*

Proof. Easy calculation.

Clearly $H_{\eta',a'} \circ H_{\eta,a} = H_{\eta' + a'\cdot\eta, a'\cdot a}$, $(H_{\eta,a})^{-1} = H_{-\frac{\eta}{a}, \frac{1}{a}}$. In particular,

$$H_{\eta,a} = H_{\eta,1} \circ H_{0,a},$$

see the diagram

$$\begin{array}{ccc} A' = (TM \times \mathbb{R}, \omega', \Omega') & \xrightarrow{\;H_{\eta,a}\;} & (TM \times \mathbb{R}, \omega, \Omega) = A \\ & {\scriptstyle H_{0,a}} \searrow \quad \swarrow {\scriptstyle H_{\eta,1}} & \\ & (TM \times \mathbb{R}, \omega, a \cdot \Omega') & \end{array}$$

It means that if A' is isomorphic to A then there exists a Lie algebroid $A'' = (TM \times \mathbb{R}, \omega, \Omega'')$, $\Omega'' = a \cdot \Omega'$, conformally isomorphic to A', i.e., such that $[A], [A''] \in Opext(TM, \nabla, M \times \mathbb{R})$ – the set of isomorphic classes of Lie algebroids having the same representation ∇ (a flat covariant derivative ∇).

Let (ω', Ω') and (ω, Ω) be l.c.s. structures. We observe that the isomorphism $H_{\eta,a} \colon A' \longrightarrow A$ given by $(\star\star)$ is equivalent to conformal equivalence of the associated l.c.s. structures if and only if $\eta = 0$.

How can we formulate the problem of existence of l.c.s. structures? We have the simple

Proposition 1.2
Any Lie algebroid $A' = (TM \times \mathbb{R}, \omega', \Omega')$ is isomorphic to $A = (TM \times \mathbb{R}, \omega, \Omega)$ with Ω non-degenerate (i.e. (ω, Ω) is an l.c.s. structure) if and only if there exists in A' a connection for which the curvature tensor is non-degenerate.

Proof. Let $H_{\eta,a} \colon A' \longrightarrow A$ be an isomorphism of Lie algebroids

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M \times \mathbb{R} & \longrightarrow & (TM \times \mathbb{R}, \omega', \Omega') & \underset{\lambda'}{\overset{\longrightarrow}{\longleftarrow}} & TM & \longrightarrow & 0 \\
 & & \Big\downarrow H^{+}_{\eta,a} & & \Big\downarrow H_{\eta,a} & & \Big\downarrow & & \\
0 & \longrightarrow & M \times \mathbb{R} & \longrightarrow & (TM \times \mathbb{R}, \omega, \Omega) & \underset{\lambda}{\overset{\longrightarrow}{\longleftarrow}} & TM & \longrightarrow & 0
\end{array} \tag{1.2}$$

$H^{+}_{\eta,a}(f) = a \cdot f$. For arbitrary connections λ' and λ in A' and A, respectively, such that $H_{\eta,a} \circ \lambda' = \lambda$ we have the following equality for curvature tensors

$$\Omega^{\lambda} = H^{+}_{\eta,a} \circ \Omega^{\lambda'}.$$

Therefore, if Ω is nondegenerate and λ' is a connection such that $H_{\eta,a} \circ \lambda' = \lambda$ where $\lambda(v) = (v, 0)$, then $\Omega^{\lambda} = -\Omega$ (see Lemma 1.1) and, clearly, $\Omega^{\lambda'}$ is non-degenerate.

Conversely, if $\lambda'(X) = (X, \eta(X))$ is any connection in A' such that $\Omega^{\lambda'}$ is non-degenerate, then $H_{-\eta,1}$ is an isomorphism of A' on $A := \big(TM \times \mathbb{R}, \omega', -\Omega^{\lambda'}\big)$ (see (1.1)) and $\big(\omega', -\Omega^{\lambda'}\big)$ is an l.c.s. structure.

So, the problem of existing of l.c.s. structures can be precisely formulated as follows:

Problem 1.1
We introduce into the class of pairs (ω, Ω) fulfilling $(\star)$, i.e., $d\Omega = -\omega \wedge \Omega$, the equivalence relation

r) $(\omega', \Omega') \approx (\omega, \Omega) \equiv$ the Lie algebroids $A' = (TM \times \mathbb{R}, \omega', \Omega')$ and $A = (TM \times \mathbb{R}, \omega, \Omega)$ are isomorphic, i.e., there exists $\eta \in \Omega^1(M)$ and $a \in C^{\infty}(M)$, $a(x) \neq 0$ for all $x \in M$, such that (4a), (4b) hold: (4a) $\Omega' = \frac{1}{a}(\Omega - d\eta - \omega \wedge \eta)$, (4b) $\omega' = \omega + \frac{da}{a}$.

Let $\dim M$ be even. We can ask: Does there in every (in given) equivalence class $[(\omega', \Omega')]$ exist (ω, Ω) being an l.c.s. structure; equivalently, does there in the Lie algebroid $A' = (TM \times \mathbb{R}, \omega', \Omega')$ exist a connection with non-degenerate curvature tensor, i.e., equivalently, does there exist a 1-form $\eta \in \Omega^1(M)$ such that $d\eta + \omega \wedge \eta - \Omega$ is non-degenerate.

This problem has a local solution, see Proposition 2.5 below for more general situations.

We must add that for a fixed closed form ω, i.e., a flat covariant derivative $\nabla_X f = \partial_X f + \omega(X) \cdot f$ in the trivial bundle $M \times \mathbb{R}$, the classification of Lie algebroids of the form $(TM \times \mathbb{R}, \omega, \cdot)$ up to isomorphism is as follows: for the class of isomorphic Lie algebroids $Opext(TM, \nabla, M \times \mathbb{R})$ we have ([15])

$$Opext(TM, \nabla, M \times \mathbb{R}) \cong H^2_\nabla(M; \mathbb{R}), \qquad [(TM \times \mathbb{R}, \omega, \Omega)] \longmapsto [\Omega].$$

A. Banyaga ([3]) gives examples of l.c.s. structures (ω, Ω) such that the Lichnerowicz class $[\Omega]$ is not trivial, $[\Omega] \neq 0$. For deformations and equivalence of l.c.s. structures see [2].

To sum up we see that important l.c.s's notions can be translated into the Lie algebroid's language. We have the following table:

l.c.s.	**Lie algebroid**
$(M, \omega, \Omega) \equiv$ ω is closed, $d\Omega = -\omega \wedge \Omega$.	$A = TM \times \mathbb{R}$ with anchor $\#_A = pr_1 \colon TM \times \mathbb{R} \longrightarrow TM$, with bracket $[\![(X, f), (Y, g)]\!] = ([X, Y], \nabla_X g - \nabla_Y f - \Omega(X, Y))$ where $\nabla_X g = \partial_X g + \omega(X) \cdot g$ ∇ is flat and $\nabla\Omega = 0$.
Globally c.s. $\equiv$ ω is exact.	A is invariantly oriented.
Two l.c.s. structures (ω', Ω') and (ω, Ω) on M are conformally equivalent $\equiv$ $\omega' = \omega + \frac{da}{a}$, $\Omega' = \frac{1}{a}\Omega$.	The corresponding Lie algebroids are isomorphic via $H_{0,a} \colon TM \times \mathbb{R} \longrightarrow TM \times \mathbb{R}$, $H(X, f) = (X, a \cdot f)$, $a \in C^\infty(M)$, $a(x) \neq 0$ for all x.

2. Generalizations: $\mathfrak{g}$-l.c.s. structures and Lie algebroids

We generalize l.c.s. structures to $\mathfrak{g}$-l.c.s. structures in which we can consider an arbitrary finite dimensional Lie algebra $\mathfrak{g}$ instead of the commutative Lie algebra $\mathbb{R}$. From the general theorem on the form of Lie algebroids, mentioned above, we have ([15], [11]):

THEOREM 2.1
Each transitive Lie algebroid with a trivial adjoint bundle of Lie algebras $M\times\mathfrak{g}$ *is isomorphic to* $TM\times\mathfrak{g}$ *with* $\#_A = \mathrm{pr}_1 : TM\times\mathfrak{g}\longrightarrow TM$ *as the anchor and the bracket*

$$[\![(X,\sigma),(Y,\eta)]\!] = ([X,Y],\nabla_X\eta-\nabla_Y\sigma+[\sigma,\eta]-\Omega(X,Y))$$

in $\operatorname{Sec} A$ *is defined via the following data* (∇,Ω)*: a covariant derivative* ∇ *in the trivial vector bundle* $M\times\mathfrak{g}$ *and a 2-form* $\Omega\in\Omega^2(M;\mathfrak{g})$ *fulfilling the conditions:*

(1) $R^{\nabla}_{X,Y}\sigma = -[\Omega(X,Y),\sigma]$, R^{∇} *being the curvature tensor of* ∇,

(2) $\nabla_X[\sigma,\eta] = [\nabla_X\sigma,\eta]+[\sigma,\nabla_X\eta]$, $\sigma,\eta\in C^\infty(M;\mathfrak{g})$,

(3) $\nabla\Omega = 0$.

The Lie algebroid obtained in the above way via the data (∇,Ω) fulfilling (1)-(3) above will be denoted here by

$$(TM\times\mathfrak{g},\nabla,\Omega). \tag{2.3}$$

The form $-\Omega$ is the curvature form of the connection $\lambda\colon TM\longrightarrow TM\times\mathfrak{g}$, $\lambda(v)=(v,0)$, in this Lie algebroid $(TM\times\mathfrak{g},\nabla,\Omega)$.

$$0\longrightarrow M\times\mathfrak{g}\longrightarrow TM\times\mathfrak{g}\underset{\overleftarrow{\lambda}}{\longrightarrow} TM\longrightarrow 0.$$

More generally, the curvature form of an arbitrary connection $\lambda(X)=(X,\eta(X))$, $\eta\in\Omega^1(M;\mathfrak{g})$, is given by

$$\Omega^\lambda(X,Y) = (\nabla\eta)(X,Y)+[\eta X,\eta Y]-\Omega(X,Y). \tag{2.4}$$

We write the covariant derivative ∇ in the trivial bundle $M\times\mathfrak{g}$ in the form

$$\nabla_X\sigma = \partial_X\sigma+\omega(X)(\sigma)$$

for a 1-form $\omega\in\Omega^1(M;\operatorname{End}\mathfrak{g})$. Then $\nabla\theta = d_\nabla\theta = d_{dR}\theta+\omega\wedge\theta$. The curvature tensor R^∇ of ∇ is equal to

$$R^{\nabla}_{X,Y}\sigma = d\omega(X,Y)(\sigma)+[\omega(X),\omega(Y)](\sigma).$$

Theorem 3.31, Chapter IV from [15] classifies all transitive Lie algebroids having a given coupling Ξ. For the Lie algebroid (2.3) we have,

$$\Xi\colon TM\longrightarrow \operatorname{OutDo}[(M\times\mathfrak{g})] = TM\times\operatorname{Der}(\mathfrak{g})/\operatorname{ad}(\mathfrak{g}),$$
$$\Xi(v) = (v,[a_v]),$$

where $a_v(\sigma) = \nabla_v\tilde{\sigma} - v(\tilde{\sigma})$, $\tilde{\sigma}\colon M \longrightarrow \mathfrak{g}$, $\tilde{\sigma}(x) \equiv \sigma \in \mathfrak{g}$,

$$Opext(TM, \Xi, M \times \mathfrak{g}) \cong H^2_{\rho^\Xi}(M, Z\mathfrak{g}) \tag{2.5}$$

where $Z\mathfrak{g}$ is the center of $\mathfrak{g}$ and $\rho^\Xi\colon TM \longrightarrow TM \times \mathrm{End}(Z\mathfrak{g})$ is the central representation $\rho^\Xi(v) = (v, a_v)$ for Ξ.

Proposition 2.1
The conditions (1)-(3) *characterizing the data* (∇, Ω) *determining the Lie algebroid* $(TM \times \mathfrak{g}, \nabla, \Omega)$ *can be expressed as follows*

— *the condition* (1) *is equivalent to*

$$d\omega(X,Y)(\sigma) + [\omega(X), \omega(Y)](\sigma) = -[\Omega(X,Y), \sigma],$$

— *the condition* (2) *is equivalent to* $\omega_x \in \mathrm{Der}(\mathfrak{g})$, *i.e.,* ω_x *is a differentiation of the Lie algebra* $\mathfrak{g}$,

— *the condition* (3) *is equivalent to*

$$d\Omega = -\omega \wedge \Omega$$

(the values of forms ω *and* Ω *are multiplied with respect to the 2-linear homomorphism* $\mathrm{End}\,\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, $(a, \sigma) \longmapsto a(\sigma)$.

Definition 2.1
The pair (∇, Ω) determining the above Lie algebroid $(TM \times \mathfrak{g}, \nabla, \Omega)$ will be called $\mathfrak{g}$-*locally conformal symplectic structure* ($\mathfrak{g}$-l.c.s. structure, for short) on the manifold provided that the 2-form Ω is *non-degenerate* in the following sense: for each point $x \in M$ the mapping

$$T_xM \longrightarrow L(T_xM, \mathfrak{g}), \qquad v \longmapsto \Omega_x(v, \cdot), \tag{2.6}$$

is a monomorphism.

It is easy to see that if the mapping (2.6) is a monomorphism at a point x then it is a monomorphism at every point near x.

We notice that if $\dim\mathfrak{g} \geq 2$ there is no dimensional obstructions to the existence of an non-degenerate tensors:

Lemma 2.1
For arbitrary vector spaces V *and* $\mathfrak{g}$ *such that* $\dim\mathfrak{g} \geq 2$ *there exists a 2-linear skew-symmetric non-degenerate tensor* $\Omega \in \Omega^2(V; \mathfrak{g})$.

Proof. Let $(e_1, \ldots, e_n)$ be a basis of $\mathfrak{g}$. If $\dim V$ is even, then there exists a real 2-linear skew-symmetric non-degenerate tensor, say Ω_0. The form $\Omega := \Omega_0 \cdot e_1 \in \Omega^2(V; \mathfrak{g})$ is non-degenerate. If $\dim V = 2k + 1$ and $(v_1, \ldots, v_{2k+1})$ is a basis of V and $u^1, \ldots, u^{2k+1}$ is a dual basis, then put

$$\Omega_0 = u^1 \wedge u^2 + \cdots + u^{2k-1} \wedge u^{2k},$$
$$\Omega_1 = u^{2k} \wedge u^{2k+1}.$$

The form $\Omega := \Omega_0 \cdot e_1 + \Omega_1 \cdot e_2$ is non-degenerate.

Definition 2.2
A $\mathfrak{g}$-l.c.s. structure is called *globally* conformal symplectic structure if the associated Lie algebroid $(TM \times \mathfrak{g}, \nabla, \Omega)$ is invariantly oriented.

Theorem 2.2
Let (∇, Ω) *be a* $\mathfrak{g}$*-l.c.s. structure on an arbitrary* m*-dimensional connected manifold (oriented or not),* $\dim \mathfrak{g} = n$. *Write* $\nabla_X \sigma = \partial_X \sigma + \omega(X)(\sigma)$ *for* $\omega \in \Omega^1(M; \operatorname{End} \mathfrak{g})$. *The following conditions are equivalent:*

(a) *The Lie algebroid* $(TM \times \mathfrak{g}, \nabla, \Omega)$ *is invariantly oriented (i.e.,* (∇, Ω) *is a globally conformal symplectic structure),*

(b) $\mathfrak{g}$ *is unimodular and* $\operatorname{tr} \omega$ *is an exact form. [Let* $e_1, \ldots, e_n$ *be a basis of* $\mathfrak{g}$. *For a non-singular function* $f \in C^\infty(M)$ *the element* $\varepsilon = f \cdot e_1 \wedge \ldots \wedge e_n$ *is an invariant cross-section if and only if* $\operatorname{tr} \omega = d(-\ln |f|)$*],*

(c) *the modular class of* $A = (TM \times \mathfrak{g}, \nabla, \Omega)$ *is zero,* $m_A = 0$*,*

(d) $H^{m+n}_{\partial_A^{or}, c}(A; or(M)) \neq 0$*,*

(e) $H^{m+n}_{\partial_A^{or}, c}(A; or(M)) = \mathbb{R}$*, and the pairing*

$$H^j(A) \times H^{m+n-j}_{\partial_A^{or}, c}(A; or(M)) \longrightarrow H^{m+n}_{\partial_A^{or}, c}(A; or(M)) \cong \mathbb{R}$$

is non-degenerate, i.e., $H^j(A) \cong \left(H^{m+n-j}_{\partial_A^{or}, c}(A; or(M))\right)^*$.

Proof. (a) $\Longleftrightarrow$ (b) The very easy proof will be omitted. (a) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (d) $\Longleftrightarrow$ (e) see [14].

Theorem 2.3
If the Lie algebra $\mathfrak{g}$ *is semisimple, then each* $\mathfrak{g}$*-l.c.s. structure is globally c.s. structure.*

Proof. According to Theorem 7.2.3 from [11] (see independently (2.5)) for the trivial LAB $\boldsymbol{g} = M \times \mathfrak{g}$ there exists exactly one, up to isomorphism, a transitive Lie algebroid A with the adjoint LAB $\boldsymbol{g} = M \times \mathfrak{g}$. Therefore, A must be isomorphic to the trivial Lie algebroid $A = TM \times \mathfrak{g}$ with the data $(\partial, 0)$. This Lie algebroid is invariantly oriented: $\varepsilon(x) \equiv \varepsilon_o \in \bigwedge^n \mathfrak{g}$ is an invariant cros-section.

Let $(e_1, \ldots, e_n)$ be a basis of $\mathfrak{g}$ with the structure constants c_{ij}^k. The covariant derivative ∇ determines a matrix of 1-forms $\omega_i^j \in \Omega^1(M)$ by

$$\nabla_X e_i = \sum_j \omega_i^j(X) e_j \,.$$

Analogously we have a collection of 2-forms Ω^j by

$$\Omega_{X,Y} = \sum_j \Omega^j_{X,Y} e_j.$$

We interpret the data (1)-(3) concerning (∇, Ω) in the terms of the matrix ω_i^j and the collection Ω^j and the structure constants c_{ij}^k.

Proposition 2.2

(A) *The conditions* (1)-(3) *characterizing the data* (∇, Ω) *determining the Lie algebroid* $(TM \times \mathfrak{g}, \nabla, \Omega)$ *can be expressed as follows.*

— *The condition* (1) *is equivalent to*

$$-\sum_j \Omega^j_{X,Y} \cdot c^r_{j,i} = d\omega_i^r(X,Y) - \sum_j \left(\omega_i^j(X)\omega_j^r(Y) - \omega_i^j(Y)\omega_j^r(X)\right),$$

— *the condition* (2) *is equivalent to*

$$\sum_k c_{ij}^k \cdot \omega_k^r(X) = \sum_k \left(\omega_i^k(X) c_{kj}^r - \omega_j^k(X) c_{ki}^r\right),$$

— *the condition* (3) *is equivalent to* $d\Omega^j = -\sum_i \Omega^i \wedge \omega_i^j$.

(B) *For an abelian Lie algebra* $\mathfrak{g} = \mathbb{R}^n$ *(i.e.,* $c_{ij}^k = 0$*) the conditions above are equivalent to*

— $d\omega(X,Y) = -\omega(X) \circ \omega(Y) + \omega(Y) \circ \omega(X)$
(equivalently $d\omega_i^r(X,Y) = \sum_j \left(\omega_i^j(X)\omega_j^r(Y) - \omega_i^j(Y)\omega_j^r(X)\right)$ *),*

— $d\Omega^j = -\sum_i \Omega^i \wedge \omega_i^j$.

Two $\mathfrak{g}$-l.c.s. structures (∇', Ω'), (∇, Ω) on a manifold M will be called *$\mathfrak{g}$-conformally equivalent* if the associated Lie algebroids are isomorphic via an isomorphism of the special form (called $\mathfrak{g}$-*conformal*) $H(X, \sigma) = (X, a(\sigma))$ for some mapping $a\colon M \longrightarrow \operatorname{Aut}(\mathfrak{g})$. Then the equivalent relations between the data (∇, Ω) and (∇', Ω') are as follows:

— $\Omega' = a^{-1} \circ \Omega$,

— $a \circ \nabla'_X(\sigma) = \nabla_X(a \circ \sigma)$.

We use the notation $a \circ \sigma$ for the cross-section defined by $(a \circ \sigma)_x = a_x(\sigma_x)$.

Writing ∇' and ∇ with using 1-forms $\omega', \omega \in \Omega^1(M; \operatorname{End}\mathfrak{g})$ (as above) the last condition can be equivalently written in the form

$$\omega(X) \circ a = -\partial_X a + a \circ \omega'(X).$$

In the terms of the matrices $\omega_i'^j$ and ω_i^j this condition is equivalent to

$$\sum_j \omega_i'^j(X) \cdot a_j^k - \sum_j a_i^j \cdot \omega_j^k(X) = \partial_X(a_i^k).$$

The general form of a homomorphism $H\colon TM \times \mathfrak{g} \longrightarrow TM \times \mathfrak{g}$ commuting with anchors pr_1 is as follows

$$H(X, \sigma) = H_{\eta,a}(X, \sigma) = (X, \eta(X) + a \circ \sigma) \tag{2.7}$$

for $\eta \in \Omega^1(M; \mathfrak{g})$, $a \in C^\infty(M, \operatorname{End}\mathfrak{g})$. Consider two Lie algebroids

$$A' = (TM \times \mathfrak{g}, \nabla', \Omega') \qquad \text{and} \qquad A = (TM \times \mathfrak{g}, \nabla, \Omega)$$

Proposition 2.3
The following conditions are equivalent.

(1) *H is a homomorphism of Lie algebroids $H\colon A' \longrightarrow A$,*

(2) (a) *a_x is a homomorphism of Lie algebras,*
(b) $(\nabla\eta)(X, Y) + [\eta(X), \eta(Y)] = (\Omega - a\Omega')(X, Y)$,
(c) $a \circ \nabla'_X \sigma = \nabla_X(a \circ \sigma) + [\eta(X), a \circ \sigma]$,

(3) *For the basis $e_1, \ldots, e_n$ and the matrix a_i^j defined by $a(e_i) = \sum_j a_i^j(e_j)$*

(a) *a_x is a homomorphism of Lie algebras,*
(b) $d\eta^k(X, Y) - \left(\sum_i \eta^i \wedge \omega_i^k\right)(X, Y) + \sum_{i,j} \eta^i(X) \cdot \eta^j(Y) \cdot c_{ij}^k$
$= \left(\Omega^k - \sum_i \Omega'^i \cdot a_i^k\right)(X, Y)$,
(c) $\sum_j \omega_i'^j(X) \cdot a_j^k = \sum_j a_i^j \cdot \omega_j^k(X) + \partial_X a_i^k + \sum_{j,s} \eta^j(X) \cdot a_i^s \cdot c_{js}^k$.

The homomorphism $H_{\eta,a}$ is an isomorphism of Lie algebroids if and only if a_x is an isomorphism of Lie algebras.

Proof. Straightforward calculations.

If (∇', Ω') and (∇, Ω) are $\mathfrak{g}$-l.c.s. structures and A' and A are corresponding Lie algebroids, then the isomorphism $H_{\eta,a}$ given by (2.7) is equivalent to conformal equivalence of the associated $\mathfrak{g}$-l.c.s. structures (∇', Ω') and (∇, Ω) if and only if $\eta = 0$.

Analogously, we can put the problem of existence of l.c.s. structures. We have firstly the simple

PROPOSITION 2.4
Any Lie algebroid $A' = (TM \times \mathfrak{g}, \nabla', \Omega')$ is isomorphic to $A = (TM \times \mathfrak{g}, \nabla, \Omega)$ with Ω non-degenerate (i.e., (∇, Ω) is a $\mathfrak{g}$-l.c.s. structure) if and only if there exists in A' a connection for which the curvature tensor is non-degenerate.

PROBLEM 2.1
We introduce into the class of pairs (∇, Ω) fulfilling (1)-(3) from Theorem 2.1, the equivalence relation

r$\mathfrak{g}$) $(\nabla', \Omega') \approx (\nabla, \Omega) \equiv$ the Lie algebroids $A' = (TM \times \mathfrak{g}, \nabla', \Omega')$ and $A = (TM \times \mathfrak{g}, \nabla, \Omega)$ are isomorphic,

i.e., there exist $\eta \in \Omega^1(M; \mathfrak{g})$, $a \in C^\infty(M, \operatorname{Aut}\mathfrak{g})$ such that (2b) and (2c), from Proposition 2.3 holds: $(\nabla\eta)(X,Y) + [\eta(X), \eta(Y)] = (\Omega - a\Omega')(X,Y)$ and $a \circ \nabla'_X \sigma = \nabla_X(a \circ \sigma) + [\eta(X), a \circ \sigma]$.

We can ask: does there in every (in given) equivalence class $[(\nabla', \Omega')]$ exist (∇, Ω) being a $\mathfrak{g}$-l.c.s. structure; equivalently, does there in the Lie algebroid $A' = (TM \times \mathfrak{g}, \nabla', \Omega')$ exist a connection with non-degenerate curvature tensor, i.e., equivalently, does there exists a 1-form $\eta \in \Omega^1(M; \mathfrak{g})$ such that the 2-form $(\nabla\eta)(X,Y) + [\eta X, \eta Y] - \Omega(X,Y)$ is a non-degenerate.

For $\mathfrak{g} = \mathbb{R}$ we obtain Problem 1.1 and we need to assume that $\dim M$ is even.

PROPOSITION 2.5
The above problem has a local solution.

Proof. Let $a\colon T_{x_0}M \times T_{x_0}M \longrightarrow \mathfrak{g}$ be an arbitrary non-degenerate 2-linear skew-symmetric tensor (for $\dim \mathfrak{g} \geq 2$ see Lemma 2.1). We can locally extend $\Omega_{x_0} + a$ to a closed 2-form Φ and find by the Poincaré lemma a 1-form η such that $d\eta = \Phi$; therefore that $(d\eta)_{x_0} = \Omega_{x_0} + a$. Slightly modifying η we can assume that $\eta_{x_0} = 0$, indeed, locally there is a closed 1-form θ such that $\theta_{x_0} = \eta_{x_0}$, so $\eta - \theta$ is zero at x_0 and $d(\eta - \theta)_{x_0} = (d\eta)_{x_0}$. Clearly $(\nabla\eta)_{x_0}(X,Y) + [\eta_{x_0}X, \eta_{x_0}Y] - \Omega_{x_0}(X,Y) = a(X,Y)$ so the curvature tensor Ω^λ of the connection $\lambda(X) = (X, \eta(X))$, see (2.4), is a non-degenerate near x_0.

Problem 2.2
It would be interesting to investigate the group of all compactly supported diffeomorphisms of M that preserve the $\mathfrak{g}$-l.c.s. structure up to $\mathfrak{g}$-conformal equivalence (analogously as it was given for usual l.c.s. structures by Haller and Rybicki in [8]).

Let us remark that two extreme cases: (1) $\mathfrak{g}$ commutative (for example $\mathfrak{g} = \mathbb{R}$) and (2) $\mathfrak{g}$ semisimple, are quite different. In the second case all Lie algebroids of the form $(TM \times \mathfrak{g}, \nabla, \Omega)$ (i.e., with the trivial adjoint Lie algebra $M \times \mathfrak{g}$) are isomorphic, clearly to the trivial one $TM \times \mathfrak{g}$ with the structure given by the data $(\partial, 0)$. Let us remark that not each isomorphism is $\mathfrak{g}$-conformal. This Lie algebroid is invariantly oriented.

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