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On the rate of convergence theorem for the alternate Poisson integrals for Hermite and Laguerre expansions

Dedicated to Professor Andrzej Zajtz on his 70th birthday

Abstract. The aim of this paper is the study of a rate of convergence of alternate Poisson integrals for Hermite and Laguerre expansions. We state some estimates of the rate of convergence of these integrals using the classical moduli of continuity.

1. Introduction

Let $L^p(\exp(-z^2))$, $p \geq 1$ denote the set of functions f defined on $\mathbb{R} = (-\infty, \infty)$ such that

$$\int_{-\infty}^{\infty} |f(t)|^p \exp\left(-t^2\right) dt < \infty \qquad \text{if } 1 \leq p < \infty$$

and f is bounded a.e. on $\mathbb{R}$ if $p = \infty$.

Muckenhoupt in [2] studied Poisson integrals and alternate Poisson integrals for Hermite polynomial expansions. He considered the Poisson integral $A(f)(r,x)$ of a function $f \in L^p(\exp(-z^2))$ for Hermite expansions defined by

$$A(f)(r,x) = A(f;r,x) = \int_{-\infty}^{\infty} P(r,x,z) f(z) \exp\left(-z^2\right) dz, \quad 0 < r < 1,\ x > 0,$$

where

$$\begin{aligned} P(r,x,z) &= \sum_{n=0}^{\infty} \frac{r^n H_n(x) H_n(z)}{\sqrt{\pi}\, 2^n n!} \\ &= \frac{1}{\sqrt{\pi(1-r^2)}} \exp\left(\frac{-r^2x^2 + 2rxz - r^2z^2}{1-r^2}\right) \end{aligned}$$

and H_n is the nth Hermite polynomial, $n = 0, 1, \dots$.

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The alternate Poisson integral is defined by

$$\begin{aligned} F(f)(x,y) = F(f;x,y) &= \int_0^1 T(x,r)A(f;r,y)\,dr \\ &= \int_{-\infty}^{\infty}\left(\int_0^1 T(x,r)P(r,y,z)\,dr\right) f(z)\exp(-z^2)\,dz, \\ &\qquad\qquad x>0,\ y\in\mathbb{R}, \end{aligned} \tag{1}$$

where

$$T(x,r) = \frac{x\exp\left(\frac{x^2}{2\ln r}\right)}{(2\pi)^{\frac{1}{2}} r(-\ln r)^{\frac{3}{2}}}.$$

Muckenhoupt obtained the following result.

If $f \in L^p(\exp(-z^2))$, then $F(f;x,\cdot) \in L^p(\exp(-z^2))$ for $x>0$ and:

(a) $\|F(f;x,\cdot)\|_p \le \|f(\cdot)\|_p,\ \ 1\le p\le\infty$,

(b) $\|F(f;x,\cdot) - f(\cdot)\|_p \to 0$ as $x\to 0^+$ for $1\le p<\infty$,

(c) $\lim_{x\to 0^+} F(f;x,y) = f(y)$ almost everywhere, $1\le p\le\infty$,

(d) $\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} - 2y\frac{\partial F}{\partial y} = 0$ in $\Omega = \{(x,y):\ x>0,\ y\in\mathbb{R}\}$,

where $\|\ \|_p$ denotes the norm in $L^p\left(\exp(-z^2)\right)$.

By $L^p(z^\alpha\exp(-z))$, $p\ge 1$, $\alpha>-1$ we denote the set of functions f defined on $\mathbb{R}_+ = [0,\infty)$ such that

$$\int_0^\infty |f(t)|^p t^\alpha \exp(-t)\ dt < \infty \qquad \text{if } 1\le p<\infty$$

and f is bounded a.e. on $\mathbb{R}_+$ if $p=\infty$. In [2] the Poisson integral of a function $f\in L^p(z^\alpha\exp(-z))$ is also considered. This integral is defined by

$$B(f)(r,x) = B(f;r,x) = \int_0^\infty K(r,x,z)f(z)z^\alpha\exp(-z)\,dz, \quad 0<r<1,\ x\ge 0$$

with the Poisson kernel

$$\begin{aligned} K(r,x,z) &= \sum_{n=0}^{\infty}\frac{r^n n!}{\Gamma(n+\alpha+1)}L_n^\alpha(x)L_n^\alpha(z) \\ &= \frac{(rxz)^{-\frac{\alpha}{2}}}{1-r}\exp\left(\frac{-r(x+z)}{1-r}\right) I_\alpha\left(\frac{2(rxz)^{\frac{1}{2}}}{1-r}\right), \end{aligned}$$

where L_n^α is the nth Laguerre polynomial, $n = 0, 1, \dots$ and I_α is the modified Bessel function ([1]),

$$I_\alpha(s) = \sum_{n=0}^{\infty} \frac{s^{\alpha+2n}}{2^{\alpha+2n} n! \Gamma(\alpha+n+1)}.$$

We define the alternate Poisson integral by

$$\begin{aligned} G(f)(x,y) = G(f;x,y) &= \int_0^1 U(x,r) B(f;r,y)\, dr \\ &= \int_0^\infty \left(\int_0^1 U(x,r) K(r,y,z)\, dr \right) f(z) z^\alpha \exp(-z)\, dz, \\ &\qquad\qquad x > 0,\ y \geq 0, \end{aligned}$$

where

$$U(x,r) = T\left(\frac{x}{\sqrt{2}}, r \right).$$

It was proved in [2] that if $f \in L^p(z^\alpha \exp(-z))$, then $G(f;x,\cdot) \in L^p(z^\alpha \exp(-z))$ for $x > 0$ and:

(a) $\|G(f;x,\cdot)\|_p \leq \|f(\cdot)\|_p,\ 1 \leq p \leq \infty$,

(b) $\|G(f;x,\cdot) - f(\cdot)\|_p \to 0$ as $x \to 0^+$ for $1 \leq p < \infty$,

(c) $\lim_{x \to 0^+} G(f;x,y) = f(y)$ almost everywhere in $[0,\infty)$, $1 \leq p \leq \infty$,

(d) $\frac{\partial^2 G}{\partial x^2} + y \frac{\partial^2 G}{\partial y^2} + (\alpha + 1 - y) \frac{\partial G}{\partial y} = 0$ in $\Omega = \{(x,y):\ x > 0,\ y \geq 0\}$.

The symbol $\|\ \|_p$ is used here to denote the norm in $L^p(z^\alpha \exp(-z))$.

This note contains some estimates of the rate of convergence of the alternate Poisson integrals $F(f)$, $G(f)$. We state these estimates using the moduli of continuity, severally for $F(f)$ and $G(f)$.

2. Auxiliary results

In this section we shall give some properties of the above operators, which we shall apply to the proofs of the main theorems.

First we prove

Lemma 1
Let $x > 0$. For each $y \in \mathbb{R}$ the following equalities hold

$$F(1;x,y) = 1,$$

$$F(z-y;x,y) = y\left(\exp(-\sqrt{2}x) - 1\right),$$
$$F((z-y)^2;x,y) = y^2\left(1 - 2\exp(-\sqrt{2}x) + \exp(-2x)\right) + \frac{1}{2}\left(1 - \exp(-2x)\right).$$

Proof. Using equality (3.9) in [2]

$$\int_0^1 T(x,r)r^n\,dr = \exp\left(-(2n)^{\frac{1}{2}}x\right), \qquad n = 0,1,\dots$$

and [3, Lemma 2.3]

$$A(z-x;r,x) = -x(1-r),$$
$$A((z-x)^2;r,x) = (1-r)\left(x^2(1-r) + \frac{1}{2}(r+1)\right)$$

we obtain from (1) and by elementary calculations the assertion of Lemma 1.

Similarly, using the formula established in [2]

$$\int_0^1 U(x,r)r^n\,dr = \exp(-\sqrt{n}x), \qquad n = 0,1,\dots$$

and [3, Lemma 2.4]

$$B(z-x;r,x) = (1-r)(1+\alpha-x),$$
$$B((z-x)^2;r,x) = (1-r)\big(x^2(1-r) + 2(\alpha+2)rx - 2(\alpha+1)x + (\alpha+2)(\alpha+1)(1-r)\big),$$

where $\alpha > -1$, we can prove

Lemma 2

Let $x > 0$ and $\alpha > -1$. For each $y \in \mathbb{R}_+$

$$G(1;x,y) = 1,$$
$$G(z-y;x,y) = (1+\alpha-y)(1-\exp(-x)),$$
$$G((z-y)^2;x,y) = y^2\left(1 - 2\exp(-x) + \exp(-\sqrt{2}x)\right) + 2(\alpha+2)y\left(\exp(-x) - \exp(-\sqrt{2}x)\right) + 2(\alpha+1)y(\exp(-x) - 1) + (\alpha+2)(\alpha+1)\left(1 - 2\exp(-x) + \exp(-\sqrt{2}x)\right)$$

hold.

3. Rate of convergence

In this part we shall give some estimates of the rate of convergence of the integrals $F(f)$ and $G(f)$. We shall use the classical modulus of continuity defined by

$$\omega(f,\delta) = \sup_{\substack{0\le t\le\delta \\ x\,\in\, Q}} |f(x+t)-f(x)|,$$

where $Q=\mathbb{R}$ or $Q=\mathbb{R}_+$, respectively.

Let $C(Q)$ be the set of all continuous functions on $Q=\mathbb{R}$ or $Q=\mathbb{R}_+$. By $C^1(Q)$ we denote, for $Q=\mathbb{R}$ or $Q=\mathbb{R}_+$, the set of all continuously differentiable functions on Q.

THEOREM 1
Let $f\in C(\mathbb{R})\cap L^p(\exp(-z^2))$. Then

$$|F(f;x,y)-f(y)| \le 3\,\omega\left(f,\mu_x(y)\right)$$

for $x>0$ and $y\in\mathbb{R}$, where

$$\mu_x(y) = \left(y^2\left(1-2\exp(-\sqrt{2}x)+\exp(-2x)\right)+\frac{1}{2}\Big(1-\exp(-2x)\Big)\right)^{\frac{1}{2}}.$$

Proof. First we suppose that f is continuously differentiable on $\mathbb{R}$. We have

$$f(z) = f(y) + \int_y^z f'(\tau)\,d\tau.$$

Hence by equality (3.7) in [2]

$$\int_{-\infty}^{\infty} P(r,y,z)\exp(-z^2)\,dz = 1,$$

Definition 1, Lemma 1 and the Hölder inequality, we obtain

$$\begin{aligned}
&|F(f;x,y)-f(y)| \\
&\quad= \left|\int_0^1 T(x,r)\left(A(f;r,y)-f(y)\right)dr\right| \\
&\quad\le \int_0^1 T(x,r)\,|A(f;r,y)-f(y)|\,dr \\
&\quad\le \int_0^1 T(x,r)\left(\int_{-\infty}^{\infty} P(r,y,z)\exp(-z^2)\,|f(z)-f(y)|\;dz\right)dr \\
&\quad\le \sup_{z\,\in\,\mathbb{R}}|f'(z)|\int_{-\infty}^{\infty}\left(\int_0^1 T(x,r)P(r,y,z)\,dr\right)\exp(-z^2)|z-y|\,dz
\end{aligned}$$

$$\begin{aligned}&\le \sup_{z\in\mathbb{R}} |f'(z)|\,(F(\varphi;x,y))^{\frac12}\,(F(1;x,y))^{\frac12}\\&= \sup_{z\in\mathbb{R}} |f'(z)|\,\mu_x(y)\end{aligned}$$

for $x>0$, $y\in\mathbb{R}$, where $\varphi(z)=(z-y)^2$.

Let $f\in C(\mathbb{R})\cap L^p(\exp(-z^2))$. We have

$$\begin{aligned}f(y)-f_\delta(y) &= \frac{1}{\delta}\int_0^\delta (f(y)-f(y+\tau))\,d\tau,\\ f_\delta'(y) &= \frac{1}{\delta}\left[f(y+\delta)-f(y)\right],\end{aligned}$$

where

$$f_\delta(y)=\frac{1}{\delta}\int_0^\delta f(y+\tau)\,d\tau, \qquad \delta>0,\ y\in\mathbb{R}.$$

This implies that f_δ is continuously differentiable on $\mathbb{R}$. Moreover

$$\sup_{z\in\mathbb{R}}|f(z)-f_\delta(z)|\le\omega(f,\delta), \qquad \sup_{z\in\mathbb{R}}|f_\delta'(z)|\le\delta^{-1}\omega(f,\delta). \tag{2}$$

Observe that

$$|F(f;x,y)-f(y)|\le|F(f-f_\delta;x,y)|+|F(f_\delta;x,y)-f_\delta(y)|+|f_\delta(y)-f(y)|.$$

From (2) and the first part of this proof we get

$$|F(f_\delta;x,y)-f_\delta(y)|\le\delta^{-1}\omega(f,\delta)\,\mu_x(y).$$

By (2) we have

$$\begin{aligned}|F(f-f_\delta;x,y)| &\le F(|f-f_\delta|;x,y)\le\sup_{z\in\mathbb{R}}|f(z)-f_\delta(z)|\le\omega(f,\delta),\\ |f_\delta(y)-f(y)| &\le \omega(f,\delta).\end{aligned}$$

Hence

$$|F(f;x,y)-f(y)|\le 2\,\omega(f,\delta)+\frac{1}{\delta}\,\omega(f,\delta)\,\mu_x(y)$$

for $x>0$, $y\in\mathbb{R}$ and $\delta>0$.

Setting $\delta=\mu_x(y)$ we obtain the assertion of Theorem 1.

Similarly we can prove the following theorem for the operator $G(f)$.

Theorem 2

Let $f\in C(\mathbb{R}_+)\cap L^p(z^\alpha\exp(-z))$. Then

$$|G(f;x,y)-f(y)|\le 3\,\omega\left(f,\mu_{\alpha,x}(y)\right)$$

for $x > 0$ *and* $y \geq 0$, *where*

$$\begin{aligned}\mu_{\alpha,x}(y) = \Big(& y^2 \left(1 - 2\exp(-x) + \exp(-\sqrt{2}x)\right) \\ &+ 2(\alpha+2)y \left(\exp(-x) - \exp(-\sqrt{2}x)\right) \\ &+ 2(\alpha+1)y \left(\exp(-x) - 1\right) \\ &+ (\alpha+2)(\alpha+1) \left(1 - 2\exp(-x) + \exp(-\sqrt{2}x)\right)\Big)^{\frac{1}{2}}.\end{aligned}$$

Now we prove

Theorem 3
If $f \in C^1(\mathbb{R}) \cap L^p(\exp(-z^2))$, *then*

$$|F(f;x,y) - f(y)| \leq |f'(y)| \left| y\left(\exp(-\sqrt{2}x) - 1\right)\right| + 2\,\mu_x(y)\,\omega\left(f', \mu_x(y)\right)$$

for $x > 0$ *and* $y \in \mathbb{R}$.

Proof. Let $f \in C^1(\mathbb{R}) \cap L^p(\exp(-z^2))$ and $\psi(z) = |z - y|$. Observe that

$$f(z) - f(y) = f'(y)\,(z-y) + \int_y^z \left(f'(s) - f'(y)\right) ds \tag{3}$$

and

$$\left|\int_y^z \left(f'(s) - f'(y)\right) ds\right| \leq \left|\int_y^z |f'(s) - f'(y)|\, ds\right| \leq \psi(z)\,\omega\left(f', \psi(z)\right).$$

Let $\delta > 0$. We have

$$\omega\left(f', \psi(z)\right) \leq \left(1 + \delta^{-1}\psi(z)\right)\omega\left(f', \delta\right).$$

Hence we get

$$\left|\int_y^z \left(f'(s) - f'(y)\right) ds\right| \leq \psi(z)\,\omega\left(f', \delta\right) + \delta^{-1}\,\psi^2(z)\,\omega\left(f', \delta\right). \tag{4}$$

Applying the Hölder inequality and Lemma 1 we obtain

$$F(\psi;x,y) \leq \left(F\left(\psi^2;x,y\right)\right)^{\frac{1}{2}} \left(F\left(1;x,y\right)\right)^{\frac{1}{2}} = \mu_x(y). \tag{5}$$

From (3), (4), (5) and Lemma 1 we have

$$\begin{aligned}&|F(f;x,y) - f(y)| \\ &\quad= |F\left(f'(y)(z-y);x,y\right)| + \left|F\left(\int_y^z \left(f'(s) - f'(y)\right) ds;x,y\right)\right| \\ &\quad\leq |f'(y)|\,|F(z-y;x,y)| + \omega\left(f',\delta\right) F(\psi;x,y) + \frac{1}{\delta}\,\omega\left(f',\delta\right) F\left(\psi^2;x,y\right) \\ &\quad\leq |f'(y)| \left|y\left(1 - \exp(-\sqrt{2}x)\right)\right| + \omega\left(f',\delta\right)\,\mu_x(y) + \frac{1}{\delta}\,\omega\left(f',\delta\right)\,\mu_x^2(y)\end{aligned}$$

for $x > 0$, $y \in \mathbb{R}$ and $\delta > 0$.

Setting $\delta = \mu_x(y)$ completes the proof of Theorem 3.

For $G(f)$ we obtain a similar result.

Theorem 4
If $f \in C^1(\mathbb{R}_+) \cap L^p(z^\alpha \exp(-z))$, then

$$\begin{aligned}&|G(f;x,y) - f(y)|\\ &\qquad \leq\ |f'(y)|\,|(1+\alpha-y)\,(1-\exp(-x))| + 2\,\mu_{\alpha,x}(y)\,\omega\,(f', \mu_{\alpha,x}(y))\end{aligned}$$

for $x > 0$ and $y \in \mathbb{R}_+$.

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