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Higher order jet prolongations type gauge natural bundles over vector bundles

Dedicated to Professor Andrzej Zajtz on the occasion of his 70th birthday with respect and gratitude

Abstract. Let $r \geq 3$ and $m \geq 2$ be natural numbers and E be a vector bundle with m-dimensional basis. We find all gauge natural bundles "similar" to the r-jet prolongation bundle $J^r E$ of E. We also find all gauge natural bundles "similar" to the vector r-tangent bundle $(J^r_{fl}(E, \mathbb{R})_0)^*$ of E.

Introduction

Natural bundles over m-manifolds were introduced by Nijenhuis [31] as a modern approach to geometric objects [1]. The most important results in the theory of natural bundles over m-manifolds is the Palais-Terng finite order theorem [32] and the Epstein-Thurston regularity theorem [7]. The sharp estimation of the order of natural bundles over m-manifolds was obtained by Zajtz [38]. From the order theorem follows that natural bundles over m-manifolds are associated to the principal bundles of repers of higher orders.

Natural bundles on some other categories over manifolds (e.g. all manifolds, fibered manifolds, principal fiber bundles, vector bundles, manifolds with structures) were introduced in [17]. Some sharp order estimations for some such natural bundles can be found in [17] and [22]. A natural bundle over all manifolds of infinite order can be found in [24]. Classifications of some type natural bundles over some categories over manifolds can be found in [35], [37], [3], [12], [23], [25], [26], [30], [16], [4], [6], [27], [18], [28].

Natural bundles make possible to precise define geometric constructions. Investigations, applications and classifications of natural operators (canonical constructions) on sections of natural bundles (geometric objects) were (and actually are) fundamental in the differential geometry. There are over 200 papers on this subject, e.g. [2], [3], [8]-[11], [13]-[15] [19]-[21], [33], [34], [36] and the fundamental monograph of Kolář, Michor and Slovák [17].

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The present paper is a next contribution to the theory of natural bundles. Let us recall the following definition (see for ex. [17], [27]).

Let $F\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ be a covariant functor from the category $\mathcal{VB}_m$ of vector bundles with m-dimensional bases and their vector bundle maps with local diffeomorphisms as base maps. Let $B_{\mathcal{VB}m}\colon \mathcal{VB}_m \longrightarrow \mathcal{M}f_m$ be the base functor.

A gauge natural bundle over $\mathcal{VB}_m$ (called also a gauge bundle functor on $\mathcal{VB}_m$) is a functor F as above satisfying:

(i) (**Base preservation**) $B_{\mathcal{VB}m} \circ F = B_{\mathcal{VB}m}$. Hence the induced projections form a functor transformation $\pi\colon F \longrightarrow B_{\mathcal{VB}m}$.

(ii) (**Localization**) For every inclusion of an open vector subbundle $i_{E|U}\colon E|U \longrightarrow E$, $F(E|U)$ is the restriction $\pi^{-1}(U)$ of $\pi\colon FE \longrightarrow B_{\mathcal{VB}m}(E)$ over U and $Fi_{E|U}$ is the inclusion $\pi^{-1}(U) \longrightarrow FE$.

(iii) (**Regularity**) F transforms smoothly parametrized systems of $\mathcal{VB}_m$-morphisms into smoothly parametrized systems of $\mathcal{VB}_m$-morphisms.

A gauge natural bundle F over $\mathcal{VB}_m$ is *fiber product preserving* if for every fiber product projections $E_1 \stackrel{pr_1}{\longleftarrow} E_1 \times_M E_2 \stackrel{pr_2}{\longrightarrow} E_2$ in the category $\mathcal{VB}_m$ $FE_1 \stackrel{Fpr_1}{\longleftarrow} F(E_1 \times_M E_2) \stackrel{Fpr_2}{\longrightarrow} FE_2$ are fiber product projections in the category $\mathcal{VB}_m$. In other words $F(E_1 \times_M E_2) = F(E_1) \times_M F(E_2)$ modulo the corestriction of (Fpr_1, Fpr_2).

All fiber product preserving gauge natural bundles over $\mathcal{VB}_m$ are described in [28]. The most important example of the ones is the r-jet prolongation gauge natural bundle $J^r\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$. Another example is the (described in I.1) vector (r)-tangent gauge natural bundle $T^{(r)fl}\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$. Fiber product preserving gauge natural bundles on $\mathcal{VB}_m$ with "similar to $T^{(r)fl}$ and J^r constructions" (Definitions 1 and 2) are also called vector r-tangent gauge natural bundles and r-jet prolongation gauge natural bundles over $\mathcal{VB}_m$, respectively.

Roughly speaking, the main results are the following classification theorems.

Theorem A
Let $m \geq 2$ and $r \geq 3$ be integers. Up to isomorphism there are only two vector r-tangent gauge natural bundles over $\mathcal{VB}_m$.

Theorem B
Let $m \geq 2$ and $r \geq 3$ be integers. Up to isomorphism there are only three r-jet prolongation gauge natural bundles over $\mathcal{VB}_m$.

The above theorems will be detailed formulated later (Theorems 1 and 2 of the present paper). In particular, the two vector r-tangent gauge natural bundles over $\mathcal{VB}_m$ and the three r-jet prolongation gauge natural bundles over $\mathcal{VB}_m$ will be precisely constructed.

All manifolds and maps are assumed to be of class C^∞. Manifolds are assumed to be finite dimensional.

I. The vector r-tangent gauge natural bundles over $\mathcal{VB}_m$

1. The vector (r)-tangent gauge natural bundle over $\mathcal{VB}_m$

Given a $\mathcal{VB}_m$-object $p\colon E \longrightarrow M$, the vector (r)-tangent bundle $T^{(r)fl}E$ of E is the vector bundle

$$T^{(r)fl}E = (J^r_{fl}(E,\mathbb{R})_0)^*$$

over M, where

$$J^r_{fl}(E,\mathbb{R})_0 = \{j^r_x\gamma \mid \ \gamma\colon E \longrightarrow \mathbb{R} \text{ is fiber linear, } \gamma_x = 0,\ x \in M\}.$$

Every $\mathcal{VB}_m$-map $f\colon E_1 \longrightarrow E_2$ covering $\underline{f}\colon M_1 \longrightarrow M_2$ induces a vector bundle map $T^{(r)fl}f\colon T^{(r)fl}E_1 \longrightarrow T^{(r)fl}E_2$ covering $\underline{f}$ such that

$$< T^{(r)fl}f(\omega), j^r_{\underline{f}(x)}\xi > = < \omega, j^r_x(\xi \circ f) >,$$

$\omega \in T^{(r)fl}_x E_1$, $j^r_{\underline{f}(x)}\xi \in J^r_{fl}(E_2,\mathbb{R})_0$, $x \in M_1$.

The correspondence $T^{(r)fl}\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ is a fiber product preserving gauge natural bundle.

2. Another construction of $T^{(r)fl}$

Let $p\colon E \longrightarrow M$ be a $\mathcal{VB}_m$-object. For any m-manifold M and $x \in M$ we have a canonical unital associative algebra homomorphism $t^{(r)}_x\colon J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}((J^r_x(M,\mathbb{R})_0)^*)$ given by

$$t^{(r)}_x(j^r_x\gamma)(\omega)(j^r_x\eta) = \omega(j^r_x(\gamma\eta)),$$

$j^r_x\eta \in J^r_x(M,\mathbb{R})_0$, $j^r_x\gamma \in J^r_x(M,\mathbb{R})$, $\omega \in (J^r_x(M,\mathbb{R})_0)^*$. We have a vector bundle

$$\tilde{T}^{(r)fl}E = \bigcup_{x\in M} \mathrm{Hom}_{t^{(r)}_x}(J^r\mathcal{C}^{\infty,fl}_x(E), (J^r_x(M,\mathbb{R})_0)^*)$$

over M. Here $\mathrm{Hom}_{t^{(r)}_x}(J^r\mathcal{C}^{\infty,fl}_x(E), (J^r_x(M,\mathbb{R})_0)^*)$ is the vector space of all module homomorphisms over $t^{(r)}_x\colon J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}((J^r_x(M,\mathbb{R})_0)^*)$ from the (free) $J^r_x(M,\mathbb{R})$-module $J^r\mathcal{C}^{\infty,fl}_x(E)$ of r-jets at x of germs at x of fiber linear maps $E \longrightarrow \mathbb{R}$ into the $\mathrm{gl}((J^r_x(M,\mathbb{R})_0)^*)$-module $(J^r_x(M,\mathbb{R})_0)^*$. Every

$\mathcal{VB}_m$-map $f\colon E_1 \longrightarrow E_2$ covering $\underline{f}\colon M_1 \longrightarrow M_2$ induces a vector bundle map $\tilde{T}^{(r)fl}f\colon \tilde{T}^{(r)fl}E_1 \longrightarrow \tilde{T}^{(r)fl}E_2$ covering $\underline{f}$ such that

$$\tilde{T}^{(r)fl}f(\Phi)(j^r_{\underline{f}(x)}\xi)(j^r_{\underline{f}(x)}\gamma) = \Phi(j^r_x(\xi\circ f))(j^r_x(\gamma\circ\underline{f})),$$

$\Phi \in \mathrm{Hom}_{t^{(r)}_x}(J^r\mathcal{C}^{\infty,fl}_x(E_1),(J^r_x(M_1,\mathbb{R})_0)^*)$, $x\in M_1$, $j^r_{\underline{f}(x)}\xi \in J^r\mathcal{C}^{\infty,fl}_{\underline{f}(x)}(E_2)$, $j^r_{\underline{f}(x)}\gamma \in J^r(M_2,\mathbb{R})_0$.

The correspondence $\tilde{T}^{(r)fl}\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ is a fiber product preserving gauge natural bundle.

Lemma 1
We have a natural isomorphism $T^{(r)fl} \tilde{=} \tilde{T}^{(r)fl}$

Proof. Define a base preserving vector bundle map $\Theta\colon T^{(r)fl}E \longrightarrow \tilde{T}^{(r)fl}E$ by

$$\Theta(\omega)(j^r_x\xi)(j^r_x\gamma) = T^{(r)fl}(\xi)(\omega)(j^r_x\gamma) \in \mathbb{R},$$

$\omega \in T^{(r)fl}_x E$, $j^r_x\xi \in J^r\mathcal{C}^{\infty,fl}_x(E)$, $j^r_x\gamma \in J^r_x(M,\mathbb{R})_0$, $x\in M$, where $\xi\colon E\longrightarrow\mathbb{R}$ is considered as $\mathcal{VB}_{m,n}$-map $\xi\colon E\longrightarrow M\times\mathbb{R}$, $\xi(v)=(p(v),\xi(v))$, $v\in E$ and where $\gamma\colon M\longrightarrow\mathbb{R}$ is considered as the fiber linear map $\gamma\colon M\times\mathbb{R}\longrightarrow\mathbb{R}$, $\gamma(m,t)=\gamma(m)t$, $m\in M$, $t\in\mathbb{R}$. Using trivializations of E it is easy to see that Θ is a vector bundle isomorphism.

3. The vector $[r]$-tangent gauge natural bundle over $\mathcal{VB}_m$

Given a $\mathcal{VB}_m$-object $p\colon E\longrightarrow M$ we have the vector bundle

$$T^{[r]fl}E = E\otimes (J^r(M,\mathbb{R})_0)^*$$

over M. Every $\mathcal{VB}_m$-map $f\colon E_1\longrightarrow E_2$ covering $\underline{f}\colon M_1\longrightarrow M_2$ induces (in obvious way) a vector bundle map $T^{[r]fl}f\colon T^{[r]fl}E_1\longrightarrow T^{[r]fl}E_2$ covering $\underline{f}$.

The correspondence $T^{[r]fl}\colon \mathcal{VB}_m\longrightarrow\mathcal{VB}_m$ is a fiber product preserving gauge natural bundle.

4. Another construction of $T^{[r]fl}$

For any m-manifold M and $x\in M$ we have a canonical unital associative algebra homomorphism $t^{[r]}_x\colon J^r_x(M,\mathbb{R})\longrightarrow \mathrm{gl}((J^r_x(M,\mathbb{R})_0)^*)$ given by

$$t^{[r]}_x(j^r_x\gamma)(\omega)=\gamma(x)\omega,$$

$j^r_x\gamma\in J^r_x(M,\mathbb{R})$, $\omega\in(J^r_x(M,\mathbb{R})_0)^*$. Using $t^{[r]}_x$ instead of $t^{(r)}_x$ we can construct similarly as in Section 2 a fiber product preserving gauge natural bundle $\tilde{T}^{[r]fl}\colon \mathcal{VB}_m\longrightarrow\mathcal{VB}_m$.

Lemma 2
We have a natural isomorphism $T^{[r]fl} \tilde{=} \tilde{T}^{[r]fl}$.

Proof. The proof is quite similar to the one of Lemma 1.

5. $T^{(r)fl} \neq T^{[r]fl}$

Lemma 3
The gauge natural bundle $T^{(r)fl}\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *and* $T^{[r]fl}\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *are not isomorphic.*

Proof. Let $f\colon \mathbb{R}^{m,n} \longrightarrow \mathbb{R}^{m,n}$ be a $\mathcal{VB}_{m,n}$-map given by

$$f(x,y) = (x, y + x^1 y), \qquad x = (x^1, \dots, x^m) \in \mathbb{R}^m,\ y \in \mathbb{R}^n.$$

Clearly $T_0^{[r]fl} f = \mathrm{id}$ and $T_0^{(r)fl} f \neq \mathrm{id}$. That is why there is no base preserving vector bundle isomorphism $\Theta\colon T^{(r)fl}\mathbb{R}^{m,n} \longrightarrow T^{[r]fl}\mathbb{R}^{m,n}$ commuting with f.

6. The vector r-tangent gauge natural bundles over $\mathcal{VB}_m$

Definition 1
We say that a fiber product preserving gauge natural bundle $F\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ is a vector r-tangent gauge natural bundle if modulo isomorphism F can be constructed similarly as $\tilde{T}^{(r)fl}$ by using eventually another canonical unital associative algebra homomorphism $t_x\colon J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}((J^r_x(M,\mathbb{R})_0)^*)$ instead of $t^{(r)}_x$.

We have the following classification theorem.

Theorem 1
Let $m \geq 2$ *and* $r \geq 3$ *be integers. Up to isomorphism* $T^{(r)fl}\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *and* $T^{[r]fl}\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *are only vector* r*-tangent gauge natural bundles over* $\mathcal{VB}_m$.

Proof. Suppose that for any m-manifold M and $x \in M$

$$t_x\colon J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}((J^r_x(M,\mathbb{R})_0)^*)$$

is a canonical unital associative algebra homomorphism. By the invariance of t_0 with respect to $(\tau^1 x^1, \dots, \tau^m x^m)$ for $\tau^i \neq 0$ we deduce that

$$t_0(j^r_0 x^1)((j^r_0 x^\alpha)^*)(j^r_0 x^2) = 0$$

for $\alpha \neq (1,1,0,\dots,0)$. Then we can write

$$t_0(j^r_0 x^1)(\omega)(j^r_0 x^2) = a\omega(j^r_0(x^1 x^2))$$

for all $\omega \in (J^r_0(\mathbb{R}^m,\mathbb{R})_0)^*$, where $a = t_0(j^r_0 x^1)((j^r_0(x^1x^2))^*)(j^r_0 x^2)$. Then by the invariance of t_0 with respect to 0-preserving diffeomorphisms we have

$$t_0(j^r_0\gamma)(\omega)(j^r_0\eta) = a\omega(j^r_0(\gamma\eta))$$

for all $\omega \in (J^r_0(\mathbb{R}^m,\mathbb{R})_0)^*$ and all $j^r_0\gamma, j^r_0\eta \in J^r_0(\mathbb{R}^m,\mathbb{R})_0$. But t_0 is an algebra homomorphism. Then

$$\begin{aligned} a\omega(j^r_0((x^1)^2x^2)) &= t_0(j^r_0((x^1)^2))(\omega)(j^r_0x^2) \\ &= t_0(j^r_0x^1)(t_0(j^r_0x^1)(\omega))(j^r_0x^2) \\ &= at_0(j^r_0x^1)(\omega)(j^r_0(x^1x^2)) \\ &= a^2\omega(j^r_0((x^1)^2x^2)). \end{aligned}$$

If $r \geq 3$ then $a^2 = a$, i.e. $a = 1$ or $a = 0$. Hence $t_x = t^{(r)}_x$ or $t_x = t^{[r]}_x$.

II. The r-jet prolongation gauge natural bundles over $\mathcal{VB}_m$

7. The (usual) r-jet prolongation gauge natural bundle J^r over $\mathcal{VB}_m$

Given a $\mathcal{VB}_m$-object $p\colon E \longrightarrow M$ the (usual) r-jet prolongation J^rE of E is a vector bundle

$$J^rE = \{j^r_x\sigma \mid \ \sigma \text{ is a local section of } E\ ,\ x \in M\}$$

over M. Every $\mathcal{VB}_m$-map $f\colon E_1 \longrightarrow E_2$ covering $\underline{f}\colon M_1 \longrightarrow M_2$ induces a vector bundle map $J^rf\colon J^rE_1 \longrightarrow J^rE_2$ covering $\underline{f}$ such that

$$J^rf(j^r_x\sigma) = j^r_{\underline{f}(x)}(f \circ \sigma \circ \underline{f}^{-1}), \qquad j^r_x\sigma \in J^rE_1\,.$$

The functor $J^r\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ is a fiber product preserving gauge natural bundle.

8. Another construction of J^r

Let $p\colon E \longrightarrow M$ be a $\mathcal{VB}_m$-object. For any m-manifold M and $x \in M$ we have a canonical unital associative algebra homomorphism $t^{r,m}_x\colon J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}(J^r_x(M,\mathbb{R}))$ given by

$$t^{r,m}_x(j^r_x\eta)(j^r_x\gamma) = j^r_x(\gamma\eta),$$

$j^r_x\eta \in J^r_x(M,\mathbb{R})$, $j^r_x\gamma \in J^r_x(M,\mathbb{R})$. We have a vector bundle

$$\tilde{\tilde{J}}^rE = \bigcup_{x\in M} \mathrm{Hom}_{t^{r,m}_x}(J^r\mathcal{C}^{\infty,fl}_x(E), J^r_x(M,\mathbb{R}))$$

over M. Here $\mathrm{Hom}_{t_x^{r,m}}(J^r\mathcal{C}_x^{\infty,fl}(E), J_x^r(M,\mathbb{R}))$ is the vector space of all module homomorphisms over $t_x^{r,m}\colon J_x^r(M,\mathbb{R}) \longrightarrow \mathrm{gl}(J_x^r(M,\mathbb{R}))$ from the (free) $J_x^r(M,\mathbb{R})$-module $J^r\mathcal{C}_x^{\infty,fl}(E)$ of r-jets at x of germs at x of fiber linear maps $E \longrightarrow \mathbb{R}$ into the $\mathrm{gl}(J_x^r(M,\mathbb{R}))$-module $J_x^r(M,\mathbb{R})$. Every $\mathcal{VB}_m$-map $f\colon E_1 \longrightarrow E_2$ covering $\underline{f}\colon M_1 \longrightarrow M_2$ induces a vector bundle map $\tilde{\tilde{J}}^r f\colon \tilde{\tilde{J}}^r E_1 \longrightarrow \tilde{\tilde{J}}^r E_2$ covering $\underline{f}$ such that

$$\tilde{\tilde{J}}^r f(\Phi)(j^r_{\underline{f}(x)}\xi) = J^r(\underline{f}, \mathrm{id}_{\mathbb{R}}) \circ \Phi(j^r_x(\xi \circ f)),$$

$\Phi \in \mathrm{Hom}_{t_x^{r,m}}(J^r\mathcal{C}_x^{\infty,fl}(E_1), J_x^r(M_1,\mathbb{R}))$, $x \in M_1$, $j^r_{\underline{f}(x)}\xi \in J^r\mathcal{C}^{\infty,fl}_{\underline{f}(x)}(E_2)$.

The correspondence $\tilde{\tilde{J}}^r\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ is a fiber product preserving gauge natural bundle.

Lemma 4
We have a natural isomorphism $J^r \tilde{=} \tilde{\tilde{J}}^r$.

Proof. The proof is similar to the one of Lemma 1. We define an isomorphism $\Theta\colon J^rE \longrightarrow \tilde{\tilde{J}}^rE$ by

$$\Theta(j^r_x\sigma)(j^r_x\xi) = j^r_x(\xi \circ \sigma),$$

$j^r_x\sigma \in J^r_xE$, $j^r_x\xi \in J^r\mathcal{C}_x^{\infty,fl}(E)$.

9. The vertical r-jet prolongation gauge natural bundle J^r_v over $\mathcal{VB}_m$

Given a $\mathcal{VB}_m$-object $p\colon E \longrightarrow M$, the vertical r-jet prolongation J^r_vE of E is a vector bundle

$$J^r_vE = \{j^r_x\sigma \mid \ \sigma\colon M \longrightarrow E_x\,,\ x \in M\}$$

over M. Every $\mathcal{VB}_m$-map $f\colon E_1 \longrightarrow E_2$ covering $\underline{f}\colon M_1 \longrightarrow M_2$ induces a vector bundle map $J^r_vf\colon J^r_vE_1 \longrightarrow J^r_vE_2$ covering $\underline{f}$ such that

$$J^r_vf(j^r_x\sigma) = j^r_{\underline{f}(x)}(f \circ \sigma \circ \underline{f}^{-1})\,, \qquad j^r_x\sigma \in J^r_vE_1\,.$$

The functor $J^r_v\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ is a fiber product preserving gauge natural bundle.

10. Another construction of J^r_v

For any m-manifold M and $x \in M$ we have a canonical unital associative algebra homomorphism $t_x^{r,m,o}\colon J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}(J^r_x(M,\mathbb{R}))$ given by

$$t_x^{r,m,o}(j^r_x\eta)(j^r_x\gamma) = \eta(x)j^r_x\gamma\,,$$

$j^r_x\gamma \in J^r_x(M,\mathbb{R})$, $j^r_x\eta \in J^r_x(M,\mathbb{R})$. Using $t^{r,m,o}_x$ instead of $t^{r,m}_x$ we can construct similarly as in Section 9 a fiber product preserving gauge natural bundle $\tilde{\tilde{J}}^r_v : \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$.

LEMMA 5
We have a natural isomorphism $J^r_v \tilde{=} \tilde{\tilde{J}}^r_v$.

Proof. The proof is quite similar to the one of Lemma 4.

11. The $[r]$-jet prolongation gauge natural bundle $J^{[r]}$ over $\mathcal{VB}_m$

For any m-manifold M and $x \in M$ we have a canonical unital associative algebra homomorphism $t^{[r],m}_x : J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}(J^r_x(M,\mathbb{R}))$ given by

$$t^{[r],m}_x(j^r_x\eta)(j^r_x\gamma) = j^r_x(\eta\gamma) - \gamma(x)j^r_x\eta + \eta(x)\gamma(x)j^r_x 1\,,$$

$j^r_x\gamma \in J^r_x(M,\mathbb{R})$, $j^x\eta \in J^r_x(M,\mathbb{R})$. Using $t^{[r],m}_x$ instead of $t^{r,m}_x$ we can construct similarly as in Section 9 a fiber product preserving gauge natural bundle $J^{[r]} : \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$.

12. J^r, J^r_v and $J^{[r]}$ are not equivalent

LEMMA 6
The gauge natural bundles $J^r : \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *and* $J^r_v : \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *are not isomorphic.*

Proof. Let $f : \mathbb{R}^{m,n} \longrightarrow \mathbb{R}^{m,n}$ be a $\mathcal{VB}_{m,n}$-map given by

$$f(x,y) = (x, y + x^1 y), \qquad x = (x^1,\dots,x^m) \in \mathbb{R}^m,\ y \in \mathbb{R}^n.$$

Clearly $J^r_0 f \neq \mathrm{id}$ and $(J^r_v)_0 f = \mathrm{id}$. That is why there is no base preserving vector bundle isomorphism $\Theta : J^r\mathbb{R}^{m,n} \longrightarrow J^r_v\mathbb{R}^{m,n}$ commuting with f.

LEMMA 7
The gauge natural bundles $J^{[r]} : \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *and* $J^r_v : \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *are not isomorphic.*

Proof. The proof is quite similar to the one of Lemma 6.

LEMMA 8
Let $m \geq 2$. *The gauge natural bundles* $J^{[r]} : \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *and* $J^r : \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ *are not isomorphic.*

Proof. It follows from some result from [29]. More precisely, the vector space of all $\mathcal{VB}_{m,n}$-natural affinors on J^rE is one dimensional and the vector space of all $\mathcal{VB}_{m,n}$-natural affinors on $J^{[r]}E$ is two dimensional.

13. The r-jet prolongation gauge natural bundles over $\mathcal{VB}_m$

Definition 2
We say that a fiber product preserving gauge natural bundle $F\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ is a r-jet prolongation gauge natural bundle if modulo isomorphism F can be constructed similarly as $\tilde{\tilde{J}}^r$ by using eventually another canonical unital associative algebra homomorphism $t_x\colon J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}(J^r_x(M,\mathbb{R}))$ instead of $t^{r,m}_x$.

We have the following classification theorem.

Theorem 2
Let $m \geq 2$ and $r \geq 3$ be integers. Up to isomorphism $J^r\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$, $J^r_v\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ and $J^{[r]}\colon \mathcal{VB}_m \longrightarrow \mathcal{VB}_m$ are only r-jet prolongation gauge natural bundles over $\mathcal{VB}_m$.

14. A preparation

Lemma 9
For any $\alpha \in \mathbb{R}$ we have a canonical unital algebra homomorphism

$$t^{<\alpha>}_x\colon J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}(J^r_x(M,\mathbb{R}))$$

given by

$$t^{<\alpha>}_x(j^r_x\gamma)(j^r_x\eta) = j^r_x(\gamma\,\eta) + \alpha\eta(x)j^r_x\gamma - \alpha\eta(x)\gamma(x)j^r_x 1\,,$$

where $j^r_x\eta, j^r_x\gamma \in J^r_x(M,\mathbb{R})$, $\dim(M) = m$, $x \in M$. If $\alpha \notin \{-1,0\}$ we have a canonical isomorphism $a_x\colon J^r_x(M,\mathbb{R}) \longrightarrow J^r_x(M,\mathbb{R})$ given by

$$a_x(j^r_x\gamma) = -\frac{\alpha+1}{\alpha} j^r_x\gamma + \gamma(x)j^r_x 1$$

such that $t^{<\alpha>}_x(j^r_x\eta)\circ a_x = a_x \circ t^{r,m}_x(j^r_x\eta)$ for any $j^r_x\eta \in J^r_x(M,\mathbb{R})$. Roughly speaking, for $\alpha \notin \{-1,0\}$ homomorphism $t^{<\alpha>}_x$ is canonically isomorphic to $t^{r,m}_x$.

Proof. It is easy to verify.

15. Proof of Theorem 2

Let for any m-manifold M and $x \in M$

$$t_x\colon J^r_x(M,\mathbb{R}) \longrightarrow \mathrm{gl}(J^r_x(M,\mathbb{R}))$$

be a canonical unital associative algebra homomorphism. Clearly, t_x is uniquely determined by the values

$$t_0(j^r_0\gamma)(j^r_0\eta) \in J^r_0(\mathbb{R}^m,\mathbb{R})$$

for any $j^r_0\gamma, j^r_0\eta \in J^r_0(\mathbb{R}^m, \mathbb{R})$ with $\gamma(0) = 0$. By the invariance of t_0 with respect to 0-preserving diffeomorphisms and the rank theorem and $m \geq 2$ we can assume that

$$j^r_0\gamma = j^r_0 x^1$$

and

$$j^r_0\eta = j^r_0 1 \quad \text{or} \quad j^r_0\eta = j^r_0 x^2.$$

By the invariance of t_0 with respect to $(\tau_1 x^1, \ldots, \tau_m x^m)$ for $\tau_1, \ldots, \tau_m \in \mathbb{R}_+$ we deduce that

$$t_0(j^r_0 x^1)(j^r_0 x^2) = \sigma j^r_0(x^1 x^2)$$

and

$$t(j^r_0 x^1)(j^r_0 1) = \rho j^r_0 x^1$$

for some $\sigma, \rho \in \mathbb{R}$. Then by the invariance of t_0 with respect to 0-preserving diffeomorphisms we have

$$t_0(j^r_0\xi)(j^r_0\eta) = \sigma j^r_0(\xi\eta)$$

and

$$t_0(j^r_0\xi)(j^r_0 1) = \rho j^r_0\xi$$

for all $j^r_0\xi \in J^r_0(\mathbb{R}^m, \mathbb{R})$ and $j^r_0\eta \in J^r_0(\mathbb{R}^m, \mathbb{R})$ and with $\xi(0) = \eta(0) = 0$. Then, since t_0 is an unital homomorphism, it is easy to verify (using the assumptions on r, m) that $\sigma = \rho = 0$ or $\sigma = 1$ and ρ is arbitrary. Therefore $t_x = t^{r,m,o}_x$ or $t_x = t^{<\alpha>}_x$ for $\alpha = \rho - 1$.

Using Lemma 9 we end the proof.

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