

Jacek Lech, Tomasz Rybicki

Abstract Lie groups and locally compact topological groups

*Dedicated to Professor Andrzej Zajtz
on the occasion of his 70th birthday*

Abstract. We introduce a notion of abstract Lie group by means of the mapping which plays the role of the evolution operator. We show some basic properties of such groups very similar to the fundamentals of the infinite dimensional Lie theory. Next we give remarkable examples of abstract Lie groups which are not necessarily usual Lie groups. In particular, by making use of Yamabe theorem we prove that any locally compact topological group admits the structure of abstract Lie group and that the Lie algebra and the exponential mapping of it coincide with those determined by the Lie group structure.

1. Introduction

An infinite dimensional Lie group G with its Lie algebra \mathfrak{g} is called regular if there is a bijective evolution mapping

$$\text{Evol}_G^r: \mathcal{C}^\infty(\mathbb{R}, \mathfrak{g}) \longrightarrow \mathcal{C}^\infty((\mathbb{R}, 0), (G, e))$$

such that its evaluation at $1 \in \mathbb{R}$ is smooth. This notion has been introduced by J. Milnor [10] (see also [8]). The right logarithmic derivative δ_G^r is then the inverse of Evol_G^r . Notice that one can use equivalently the left evolution mapping and the left logarithmic derivative to define the regularity. Next the exponential mapping of a regular Lie group $\exp: \mathfrak{g} \longrightarrow G$ is given by

$$\exp(X) = \text{Evol}_G^r(X)(1).$$

In particular, $\exp(tX) = \text{Evol}_G^r(X)(t)$ and clearly $\exp(t+s)X = \exp tX \exp sX$. Let us mention that all known Lie groups are regular (cf. [8]). It seems that

the role of the evolution mapping in the infinite dimensional Lie theory is so important as the role of the exponential mapping in the finite dimensional case.

In this paper we propose a notion generalizing regular Lie groups, namely the notion of abstract Lie groups. In this concept the smooth structure is defined by means of the family of smooth curves $\mathcal{S}^\infty(\mathbb{R}, G)$, and the $\mathcal{S}^\infty(\mathbb{R}, G)$ is defined by the evolution mapping Evol_G^r . The notion of abstract Lie groups generalizes regular Lie groups and is motivated by important examples, cf. Section 3. It is significant that basic properties of infinite dimensional Lie groups can be derived from our definition and new interpretations of the inheritance property, the integrability of Lie subalgebras, and the quotient structures are possible. Here we give only some introductory facts and we omit some proofs as a presentation of the whole setting is beyond the scope of this paper and will be given elsewhere.

We would like to indicate that there are other abstract settings of the infinite dimensional Lie theory. Examples are the following:

- Diffeological groups due to Souriau [16]. A smooth structure is there defined by establishing sets of local smooth mappings from \mathbb{R}^n to G , $n = 1, 2, \dots$, and by imposing some conditions on them. It is possible to define the tangent space $T_e G$, but a Lie algebra structure can be given on some subspace \mathfrak{g} of $T_e G$ only. Then one introduces the exponential mapping on \mathfrak{g} .
- The concept of generalized Lie groups in the sense of Omori [11]. The definition is based on a continuous mapping $\exp: G \rightarrow \mathfrak{g}$ between a topological metric group G and a topological Lie algebra \mathfrak{g} with several technical conditions which mimic essential properties of the exponential map. In particular, it is possible to distinguish the set of differentiable curves.
- Another category of generalized Lie groups was proposed by Chen and Yoh [3]. A clue point is there a description of the Lie algebra $\text{Hom}(\mathbb{R}, G)$ for a topological group. However this framework concerns mainly finite dimensional groups.

Note that all these concepts do not use the regularity and, in view of them, the Lie group structure is inherited by any (closed) subgroup. Consequently it seems that they are not enough refined to give a satisfactory abstract description of the infinite dimensional Lie theory.

As a general framework we will use the concept of smooth structure defined by a family of smooth curves. Let us mention that there exist in the literature similar settings as well as calculi by means of smooth curves, e.g. an interesting calculus of flows on convenient manifolds by A. Zajtš [20].

In this paper we wish to illustrate the introduced concept in the case of LP-groups, i.e. the groups which can be expressed as the projective limits of

an inverse system of finite dimensional Lie groups. We recall basic properties of such groups and endow them with the structure of an abstract Lie group. Consequently, in view of Yamabe theorem, we show that any connected locally compact topological group carries the structure of an abstract Lie group.

For simplicity we will confine our considerations to the \mathcal{C}^∞ smooth category.

2. Smooth spaces

First we introduce the category of smooth spaces.

DEFINITION 2.1

A *smooth space* is a set X endowed with a subset $\mathcal{S}^\infty(\mathbb{R}, X) \subset \text{Map}(\mathbb{R}, X)$ such that constant mappings are in $\mathcal{S}^\infty(\mathbb{R}, X)$ and $c \circ f \in \mathcal{S}^\infty(\mathbb{R}, X)$ whenever $f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ is a smooth reparametrization and $c \in \mathcal{S}^\infty(\mathbb{R}, X)$.

The set $\mathcal{S}^\infty(\mathbb{R}, X)$ is called a *smooth structure* on X . If Y is another smooth space endowed with a smooth structure $\mathcal{S}^\infty(\mathbb{R}, Y)$ then a mapping $f: X \rightarrow Y$ is said to be *smooth* (or a *morphism of smooth structures*) if $f \circ c \in \mathcal{S}^\infty(\mathbb{R}, Y)$ whenever $c \in \mathcal{S}^\infty(\mathbb{R}, X)$. It is clear that the composition of smooth maps is smooth. We write $f_*: \mathcal{S}^\infty(\mathbb{R}, X) \rightarrow \mathcal{S}^\infty(\mathbb{R}, Y)$ for the induced map. Note that in view of Boman theorem [1] this concept extends the usual concept of smoothness.

Any smooth space X is equipped with a natural topology, namely the final topology of $\mathcal{S}^\infty(\mathbb{R}, X)$.

PROPOSITION 2.2

For any smooth spaces X, Y, Z one has

$$\mathcal{S}^\infty(X, \mathcal{S}^\infty(Y, Z)) \cong \mathcal{S}^\infty(X \times Y, Z),$$

i.e., the category of smooth spaces is cartesian closed.

Proof. We endow $\mathcal{S}^\infty(Y, X)$ with a smooth structure as follows:

$$c \in \mathcal{S}^\infty(\mathbb{R}, \mathcal{S}^\infty(Y, Z)) \iff \hat{c} \in \mathcal{S}^\infty(\mathbb{R} \times Y, Z),$$

where $\hat{c}(x, y) = c(x)(y)$. The bijective mapping

$$\alpha: \mathcal{S}^\infty(X, \mathcal{S}^\infty(Y, Z)) \ni c \mapsto \hat{c} \in \mathcal{S}^\infty(X \times Y, Z)$$

is smooth. In fact, for every $\varphi \in \mathcal{S}^\infty(\mathbb{R}, \mathcal{S}^\infty(X, \mathcal{S}^\infty(Y, Z)))$ one has $\hat{\varphi} \in \mathcal{S}^\infty(\mathbb{R} \times X, \mathcal{S}^\infty(Y, Z))$ and $\alpha \circ \hat{\varphi} \in \mathcal{S}^\infty(\mathbb{R} \times X \times Y, Z)$. Consequently $\alpha_*(\varphi) \in \mathcal{S}^\infty(\mathbb{R}, \mathcal{S}^\infty(X \times Y, Z))$. Analogously α^{-1} is smooth as well.

PROPOSITION 2.3

The set of all smooth maps from \mathbb{R} to X coincides with $\mathcal{S}^\infty(\mathbb{R}, X)$, i.e.,

$$\mathcal{S}^\infty(\mathbb{R}, X) = \{c : \mathbb{R} \longrightarrow X \text{ smooth}\}.$$

Proof. For every smooth curve $c \in \mathcal{S}^\infty(\mathbb{R}, X)$ and $f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ the curve $c_*(f) = c \circ f$ is smooth. Conversely, let $c : \mathbb{R} \longrightarrow X$ be smooth. Then $c = c_*(\text{id}_{\mathbb{R}}) \in \mathcal{S}^\infty(\mathbb{R}, X)$ by definition.

PROPOSITION 2.4

Let $\{X_\alpha, p_\alpha^\beta\}$ be an inverse system of smooth spaces. Then its projective limit is naturally endowed with a smooth structure such that

$$\mathcal{S}^\infty(\mathbb{R}, X) = \lim \text{proj } \mathcal{S}^\infty(\mathbb{R}, X_\alpha)$$

is the projective limit of the inverse system $\{\mathcal{S}^\infty(\mathbb{R}, X_\alpha), p_{\alpha*}^\beta\}$.

Proof. By definition $c = \{c_\alpha\} \in \mathcal{S}^\infty(\mathbb{R}, X)$ if and only if for every $\alpha \in A$ $c_\alpha \in \mathcal{S}^\infty(\mathbb{R}, X_\alpha)$ and $(p_\alpha^\beta)_*(c_\beta) = c_\alpha$ for $\alpha \leq \beta$.

Let $c(t) = \{c_\alpha(t)\} = \{c_\alpha\} \in X$ be a constant curve. Then it is smooth as it is componentwise smooth. Likewise, $\mathcal{S}^\infty(\mathbb{R}, X)$ is closed with respect to a smooth reparametrization if it has this property componentwise. Thus $\mathcal{S}^\infty(\mathbb{R}, X)$ is a smooth structure on X .

Next for every $c = \{c_\alpha\} \in \mathcal{S}^\infty(\mathbb{R}, X)$ we see that

$$(p_\alpha^\beta)_*(c_\beta)(t) = p_\alpha^\beta(c_\beta(t)) = c_\alpha(t)$$

so, by definition, $\lim \text{proj } \mathcal{S}^\infty(\mathbb{R}, X_\alpha) = \mathcal{S}^\infty(\mathbb{R}, X)$.

3. Smooth groups and abstract Lie groups

Let G be a group with the multiplication $\mu : G \times G \ni (g, h) \longmapsto gh \in G$ and the inversion $\nu : G \ni g \longmapsto g^{-1} \in G$. If, in addition, G is a smooth space and μ and ν are smooth as morphisms of smooth structures it is called a *smooth group*. Here the product $G \times G$ is given a smooth structure in the obvious way.

Let G be a smooth group. Now we formulate some conditions for G .

(G1) Let \mathfrak{g} be a sequentially complete locally convex topological vector space (s.c.l.c.t.v.s. for short), cf. [10], [11]. Assume that there exists a bijective mapping (called the *right evolution operator*)

$$\text{Evol}_G^r : \mathcal{C}^\infty(\mathbb{R}, \mathfrak{g}) \longrightarrow \mathcal{S}_e^\infty(\mathbb{R}, G),$$

where $\mathcal{S}_e^\infty(\mathbb{R}, G)$ is a space of all mappings from \mathbb{R} to G sending 0 to e . Note that we have the natural right action of G onto the space $\text{Map}(\mathbb{R}, G)$. We assume

that the union of orbits of mappings from $\mathcal{S}_e^\infty(\mathbb{R}, G)$ is equal to $\mathcal{S}^\infty(\mathbb{R}, G)$ and that any translation by $g \in G$ is smooth.

Observe that one can use the convenient vector spaces [8] instead of s.c.l.c.t. v.s.

The inverse of Evol_G^r will be denoted by δ_G^r . The exponential mapping $\exp: \mathfrak{g} \subset \mathcal{C}^\infty(\mathbb{R}, \mathfrak{g}) \longrightarrow G$ is given by $\exp(X) = \text{Evol}_G^r(X)(1)$. Clearly $\exp((t + s)X) = \exp(tX) \exp(sX)$.

(G2) One has

$$\text{Evol}_G^r(\mathfrak{g}) \subset \Lambda(G)$$

where $\Lambda(G) := \{\varphi: \mathbb{R} \longrightarrow G \mid \forall t, s \in \mathbb{R} \varphi(t + s) = \varphi(t)\varphi(s)\}$ is the set of one-parameter subgroups of G . Moreover we assume that

$$\Lambda_0(G) := \text{Evol}_G^r(\mathfrak{g})$$

is the totality of smooth one-parameter subgroups of G .

As a consequence, every smooth homeomorphism $f: G \longrightarrow H$ induces the tangent map $Tf: \Lambda_0(G) \longrightarrow \Lambda_0(H)$.

(G3) The set $\mathcal{S}_e^\infty(\mathbb{R}, G)$ admits a cone structure

$$\mathbb{R} \times \mathcal{S}_e^\infty(\mathbb{R}, G) \ni (\lambda, f) \longmapsto f^\lambda \in \mathcal{S}_e^\infty(\mathbb{R}, G),$$

where $f^\lambda(t) := f(\lambda t)$. We assume that for any $f \in \mathcal{S}_e^\infty(\mathbb{R}, G)$ and $\lambda \in \mathbb{R}$ we have $\delta_G^r(f^\lambda) = \lambda \delta_G^r(f)$.

Putting $\lambda = 0$ we get that the constant e belongs to $\mathcal{S}_e^\infty(\mathbb{R}, G)$ and $\delta_G^r(e) = 0$.

Let $\text{conj}_g: G \ni g \longmapsto hgh^{-1} \in G$ be the conjugation by $g \in G$. Assume that

$$\text{Ad}(g) = T\text{conj}_g.$$

Then we have

$$g(\exp(tX))g^{-1} = \exp(t\text{Ad}(g)X),$$

where \mathfrak{g} is identified with $\Lambda_0(G)$. Clearly for every $g, h \in G$ one has $\text{Ad}(gh) = \text{Ad}(g)\text{Ad}(h)$.

(G4) For any $g \in G$ the mapping $\text{Ad}(g): \mathfrak{g} \longrightarrow \mathfrak{g}$ is smooth.

(G5) For every $f, g \in \mathcal{S}_e^\infty(\mathbb{R}, G)$ we have

$$\delta_G^r(fg) = \delta_G^r f + \text{Ad}(f) \cdot \delta_G^r g.$$

Here the dot denotes the componentwise action.

(G6) The map

$$\text{ev}_1 \circ \text{Evol}_G^r: \mathcal{C}^\infty(\mathbb{R}, \mathfrak{g}) \longrightarrow G,$$

where ev_1 is the evaluation at 1 mapping, is smooth as a morphism of smooth structures.

DEFINITION 3.1

A smooth group G is said to be an *abstract Lie group* if the conditions (G1)-(G6) are satisfied.

REMARK 3.2

Equivalently we can use the left evolution and left logarithmic derivative operators. Then the formula (G4) is replaced by

$$\delta_G^l(fg) = \delta_G^l g + \text{Ad}(g^{-1}) \delta_G^l f.$$

Let us mention some examples of abstract Lie groups.

EXAMPLE 3.3

Any regular Lie group [8] satisfies all the above conditions. In particular, they hold for any finite dimensional Lie group.

The following fact is well known.

PROPOSITION 3.4

Let G be a group in the above example, and $\mathfrak{g} = \text{Lie}(G)$. Then $\text{Evol}_G^r(X)(t) = g(t)$ if and only if $g(0) = e$ and

$$\frac{\partial}{\partial t} g(t) = T_e(\mu^{g(t)})X(t),$$

where $\mu^g(x) = \mu(x, g)$, and μ is the multiplication of G .

As immediate consequences we have the following

COROLLARY 3.5

Under the assumptions of Proposition 3.4, for any smooth reparametrization $f \in C^\infty(\mathbb{R}, \mathbb{R})$ and $X \in \mathcal{C}(\mathbb{R}, \mathfrak{g})$

$$\text{Evol}_G^r(X)(f(t)) = \text{Evol}_G^r(f' \cdot (X \circ f))(t) \cdot \text{Evol}_G^r(X)(f(0)).$$

COROLLARY 3.6

Let G, H be as in Proposition 3.4. If $\varphi: G \rightarrow H$ is a smooth homomorphism then the diagram

$$\begin{array}{ccc} C^\infty(\mathbb{R}, \mathfrak{g}) & \xrightarrow{(T\varphi)_*} & \mathcal{C}(\mathbb{R}, \mathfrak{h}) \\ \downarrow \text{Evol}_G^r & & \downarrow \text{Evol}_H^r \\ S_e^\infty(\mathbb{R}, G) & \xrightarrow{\varphi_*} & S_e^\infty(\mathbb{R}, H) \end{array}$$

commutes, where $\mathfrak{h} = \text{Lie}(H)$.

EXAMPLE 3.7

The following situation often arise. Given a regular Lie group G , there is a closed subgroup $H \subset G$ and a Lie algebra \mathfrak{h} such that smooth curves with values in \mathfrak{h} are sent bijectively by Evol_G^r to isotopies with values in H . E.g. this takes place if $G = \text{Diff}(M)$, $\mathfrak{g} = \mathfrak{X}_c(M)$, H is the group of automorphisms of a geometric structure on M , and \mathfrak{h} is the Lie algebra of infinitesimal automorphisms of this structure. Unfortunately, such a bijection does not yield a Lie group structure on H and usually it is very difficult to introduce a Lie group structure on H as the possible construction of such a structure involves a deep insight into the geometry determined by H . However, a common intuition is that such a situation is not bad, and sometimes by abuse one states that H is a Lie group with the Lie algebra \mathfrak{h} . The second named author formalized this intuition in [16] by introducing the concept of a *weak Lie subgroup*.

Let us mention only some examples of weak Lie subgroups:

(i) Let (M, \mathcal{F}) be foliated manifold and $G = \text{Diff}(M)$. Then it was shown in [14] that the subgroup of all leaf preserving diffeomorphisms H is a regular Lie group if \mathcal{F} is a regular foliation. On the other hand, if \mathcal{F} is singular ([18]) then H carries the structure of weak Lie subgroup but it is hopeless to expect that H would admit the usual Lie group structure.

(ii) If $G = \text{Diff}(M)$ and H is the automorphism group of either singular Poisson manifold, or Jacobi manifold, or cosymplectic manifold then H admits the structure of weak Lie subgroup but it would be difficult or impossible to find the usual Lie group structure on H .

(iii) A broad class of groups in differential geometry constitute strict groups. Recall that G is a strict group if it is a subgroup of the group of all smooth bisections of some Lie groupoid.

Recently, it has been shown that for any Lie groupoid its group of all smooth bisections possesses a regular Lie group structure, cf. [15] and references therein. Several subgroups of this group are examples of weak Lie subgroups.

EXAMPLE 3.8

It is well known that $\text{Diff}(M)$ does not admit a (usual) Lie group structure whenever M is a manifold of infinite dimension (cf. [8]). However, under natural assumption the evolution operator exists in this case (cf. [11]) and, consequently, it is possible to endow $\text{Diff}(M)$ with the structure of an abstract Lie group.

EXAMPLE 3.9

It is well known that the quotient of an infinite dimensional Lie group by its normal subgroup need not inherit the Lie group structure. When we consider the structure in the abstract sense the situation is better. It can be shown that the quotient of an abstract Lie group by its normal subgroup satisfies $(G1)$, $(G3)$ - $(G5)$ and the first assertion of $(G2)$.

EXAMPLE 3.10

The basic examples in this paper are LP-groups and locally compact topological groups, cf. Sections 5 and 6.

4. Basic properties of abstract Lie groups

Let $f: G \rightarrow H$ be a smooth homomorphism, where G and H are abstract Lie groups modelled on s.c.l.c.t.v. spaces \mathfrak{g} and \mathfrak{h} , resp. Then, in view of (G2), for any $X \in \mathfrak{g}$ the mapping

$$\mathbb{R} \ni t \mapsto f(\exp_G(tX)) \in H$$

is a smooth one-parameter subgroup of H . There exists a unique $Y \in \mathfrak{h}$ such that $f(\exp_G(tX)) = \exp_H(tY)$. We define $Tf(X) := Y$. It is a linear mapping and we have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{Tf} & \mathfrak{h} \\ \downarrow \exp_G & & \downarrow \exp_H \\ G & \xrightarrow{f} & H \end{array}$$

For all $X \in \mathfrak{g}$ we have the following formulae

$$\text{Ad}(\exp tX)X = X \tag{4.1}$$

and

$$\text{Ad}(g \exp tX)Y = \text{Ad}(\exp t(\text{Ad}(g)X))\text{Ad}(g)Y. \tag{4.2}$$

Indeed,

$$\begin{aligned} g \exp tX \exp tY (\exp tX)^{-1} g^{-1} &= g \exp tX g^{-1} g \exp tY g^{-1} (g \exp tX g^{-1})^{-1} \\ &= \exp t(\text{Ad}(g)X) \exp t(\text{Ad}(g)Y) (\exp t(\text{Ad}(g)X))^{-1} \end{aligned}$$

As usual, we let $\text{ad} := T\text{Ad}$. Then $\text{ad}_X(Y) = [X, Y] = \frac{\partial}{\partial t}|_{t=0} \text{Ad}(\exp tX)Y$.

PROPOSITION 4.1

The bracket $[\cdot, \cdot]$ is a Lie algebra bracket on \mathfrak{g} .

Proof. By (G4) $[\cdot, \cdot]$ is linear in the second variable since $\text{ad} = T\text{Ad}$.

Next

$$\begin{aligned} [X + Y, Z] &= \frac{\partial}{\partial t}|_{t=0} \text{Ad}(\exp tX \exp tY)Z \\ &= \frac{\partial}{\partial t}|_{t=0} \text{Ad}(\exp tX) \text{Ad}(\exp tY)Z \\ &= [X, Z] + [Y, Z] \end{aligned}$$

The skew-symmetry follows by (4.1). The Jacobi identity we get by (4.2)

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=0} \text{Ad}(g \exp tX)Y &= \frac{\partial}{\partial t} \Big|_{t=0} \text{Ad}(\exp t(\text{Ad}(g)X))\text{Ad}(g)Y \\ &= [\text{Ad}(g)X, \text{Ad}(g)Y], \end{aligned}$$

where we use the condition (G4).

The following can be shown by a standard argument

PROPOSITION 4.2

If (G, Evol_G^r) and (H, Evol_H^r) are abstract Lie groups, and $\varphi: G \rightarrow H$ be a smooth homomorphism. Then $T\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$ is a Lie algebra homomorphism.

COROLLARY 4.3

The s.c.l.c.t.v.s. \mathfrak{g} in Definition 3.1 is uniquely defined and admits a uniquely defined Lie algebra structure.

From now on \mathfrak{g} will be called the Lie algebra of G and will be denoted by $\text{Lie}(G)$.

It seems that contrary to other abstract settings of Lie theory, our setting enables to introduce an appropriate notion of Lie subgroups in infinite dimension. This notion is strictly connected with the evolution operator.

DEFINITION 4.4

A subgroup $H \subset G$ is said to be a Lie subgroup if there exists a closed subspace $\mathfrak{h} \subset \mathfrak{g}$ such that

$$\text{Evol}_G^r|_{\mathcal{C}^\infty(\mathbb{R}, \mathfrak{h})}: \mathcal{C}^\infty(\mathbb{R}, \mathfrak{h}) \rightarrow \mathcal{S}_e^\infty(\mathbb{R}, G) \cap \text{Map}_e(\mathbb{R}, H)$$

is a bijection. Then we set

$$\text{Evol}_H^r := \text{Evol}_G^r|_{\mathcal{C}^\infty(\mathbb{R}, \mathfrak{h})} \quad \text{and} \quad \mathcal{S}_e^\infty(\mathbb{R}, H) := \mathcal{S}^\infty(\mathbb{R}, G) \cap \text{Map}_e(\mathbb{R}, H).$$

Consequently we have

$$X \in \mathfrak{h} \iff \exp tX \in \mathcal{S}_e^\infty(\mathbb{R}, H) \tag{4.3}$$

i.e. $\Lambda_0(H) = \Lambda_0(G) \cap \mathcal{S}_e^\infty(\mathbb{R}, H)$.

As a consequence one can prove the following

PROPOSITION 4.5

If $H \subset G$ is a Lie subgroup then (H, Evol_H^r) is an abstract Lie group itself. Furthermore, if H_i is a Lie subgroup of G_i , $i = 1, 2$, and $\varphi: G_1 \rightarrow G_2$ is a morphism with $\varphi(H_1) \subset H_2$ then $\varphi|_{H_1}: H_1 \rightarrow H_2$ is also a morphism and $T(\varphi|_{H_1}) = T\varphi|_{\mathfrak{h}}$.

Since a celebrated paper on non-enlargibility by van Est and Korthagen [5] it is well known that the third Lie theorem is, in general, no longer true in the infinite dimensional case. However, there are several important infinite dimensional generalizations, e.g. [4], [13], [19], [7]. In our abstract setting this theorem holds under natural conditions on the Lie algebra.

Let us formulate two conditions related to the integrability in the abstract sense of a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

(A) If $g \in \mathcal{S}_e^\infty(\mathbb{R}, G)$ and for any $t \in \mathbb{R}$ there exists $h \in \mathcal{S}_e^\infty(\mathbb{R}, G)$ with $\delta_G^r(h) \in \mathcal{C}(\mathbb{R}, \mathfrak{h})$ and $h_t = g_t$ then $\delta_G^r(g) \in \mathcal{C}(\mathbb{R}, \mathfrak{h})$. In other words, any G -isotopy with values in H is an H -isotopy, where the group H is defined as in the theorem below.

(B) For any $g \in \mathcal{S}_e^\infty(\mathbb{R}, G)$ with $\delta_G^r(g) \in \mathcal{C}(\mathbb{R}, \mathfrak{h})$ one has $\text{Ad}(g)\mathfrak{h} \subset \mathfrak{h}$.

The following is proved in [16].

THEOREM 4.6

Let \mathfrak{h} be a Lie subalgebra satisfying the conditions (A) and (B). Define

$$H = \{h \in G : h = g_1, \text{ where } g \in \mathcal{S}_e^\infty(\mathbb{R}, G), \delta_G^r(g) \in \mathcal{C}(\mathbb{R}, \mathfrak{h})\}.$$

Then H is a weak Lie subgroup of G with the Lie algebra \mathfrak{h} .

REMARK 4.7

The condition (B) is closely related to the notion of singular foliations [18] whenever $G = \text{Diff}(M)$. Namely it says that the distribution defined by $\mathfrak{h} \subset \mathfrak{X}_c(M)$ integrates to a singular foliation.

EXAMPLE 4.8

Let ω be a symplectic structure on M , and let $\sharp^\omega: \Omega^1(M) \rightarrow \mathfrak{X}(M)$ be the corresponding musical isomorphism.

We set (\mathcal{L} is the Lie derivative)

$$L(M, \omega) = \{X \in \mathfrak{X}_c(M) : \mathcal{L}_X \omega = 0\}$$

$$G(M, \omega) = \{f \in \text{Diff}(M) : f^* \omega = \omega\}.$$

Then $L(M, \omega)$ is a Lie subalgebra of $\mathfrak{X}_c(M)$, and $G(M, \omega)$ is its Lie group in view of a well known result by Weinstein [19].

A smooth path of diffeomorphisms $f(t)$ satisfying $\delta_{\text{Diff}(M)}^r(f)(t) = X(t) \in L(M, \omega)$ for any t is called a symplectic isotopy.

A symplectic isotopy $f(t)$ is *Hamiltonian* (or *exact*) if $\delta_{\text{Diff}(M)}^r(f)(t) = X(t) \in L^*(M, \omega)$ for each t . Here $L^*(M, \omega)$ stands for the Lie algebra Hamiltonian vector fields, i.e. $X \in L^*(M, \omega)$ iff $\iota(X)\omega$ is exact, ι being the insertion.

A diffeomorphism f of M is called *Hamiltonian* if there exists a Hamiltonian isotopy $f(t)$ such that $f(0) = \text{id}$ and $f(1) = f$. The totality of all Hamiltonian diffeomorphisms is denoted by $G^*(M, \omega)$.

It can be shown that $G^*(M, \omega)$ is a normal subgroup of $G(M, \omega)$.

Observe that $G^*(M, \omega)$ is a Lie subgroup in the abstract sense. It is a usual Lie group if and only if the group of periods of ω (i.e. the image by the flux homomorphism of the first homotopy group of $G(M, \omega)$) is discrete.

Analogous statements still hold for locally conformal symplectic structures, cf. [7], in view of the existence the flux homomorphism and other invariants, and for regular Poisson manifolds, cf. [14]. For singular Poisson manifolds all groups in question admit only the structure of abstract Lie group.

5. LP-groups and LP-Lie algebras

In this section we recall the concept of LP-groups which can serve as an example of abstract Lie groups.

A topological group G is called an *LP-group* if it is the projective limit of finite dimensional Lie groups, i.e. there exists an inverse system of finite dimensional Lie groups $\{G_\alpha, p_\alpha^\beta\}$ such that $G = \lim \text{proj} \{G_\alpha, p_\alpha^\beta\}$. Here the index set A is directed and p_α^β for $\alpha \leq \beta$ denotes a continuous and open epimorphism, which maps the Lie group G_β onto G_α . In particular, one has $p_\alpha^\beta \circ p_\beta^\gamma = p_\alpha^\gamma$, if $\alpha \leq \beta \leq \gamma$.

For every $\alpha \in A$ there exists a canonical continuous epimorphism $p_\alpha: G \rightarrow G_\alpha$ and it holds $p_\alpha^\beta \circ p_\beta = p_\alpha$, if $\alpha \leq \beta$. The mappings p_α for $\alpha \in A$ can be proved to be open.

Let us denote $\mathfrak{g}_\alpha = \text{Lie}(G_\alpha)$ and let $\exp_\alpha: \mathfrak{g}_\alpha \rightarrow G_\alpha$ be the corresponding exponential map. Then there exists a unique epimorphism $Tp_\alpha^\beta: \mathfrak{g}_\beta \rightarrow \mathfrak{g}_\alpha$ such that $\exp_\beta \circ Tp_\alpha^\beta = p_\alpha^\beta \circ \exp_\alpha$ for $\alpha \leq \beta$. If $\alpha \leq \beta \leq \gamma$, then the equation $p_\alpha^\beta \circ p_\beta^\gamma = p_\alpha^\gamma$ implies $Tp_\alpha^\beta \circ Tp_\beta^\gamma = Tp_\alpha^\gamma$ and $\{\mathfrak{g}_\alpha, Tp_\alpha^\beta\}$ is an inverse system of finite dimensional Lie algebras defining a topological Lie algebra $\mathfrak{g} = \lim \text{proj} \{\mathfrak{g}_\alpha, Tp_\alpha^\beta\}$.

A topological Lie algebra is called an *LP-Lie algebra* if it is the projective limit of an inverse system of finite dimensional Lie algebras. To every LP-group G corresponds an LP-Lie algebra \mathfrak{g} , which we call the *Lie algebra of the LP-group* G and denote by $\mathfrak{g} = \text{Lie}(G)$.

It is well known [9], [6] that the Lie algebra $\text{Lie}(G)$ is independent of an inverse system representing the LP-group G as the projective limit of Lie groups. It follows from the categorical properties that there exists a unique continuous lift $\exp: \mathfrak{g} \rightarrow G$ of the mappings $\exp_\alpha: \mathfrak{g}_\alpha \rightarrow G_\alpha$ for $\alpha \in A$ such that for every $\alpha \in A$ $\exp_\alpha \circ Tp_\alpha = p_\alpha \circ \exp$ holds.

Moreover, for any $\alpha \leq \beta$ the following diagram

$$\begin{array}{ccccc}
 \mathfrak{g}_\alpha & \xleftarrow{Tp_\alpha^\beta} & \mathfrak{g}_\beta & \xleftarrow{Tp_\alpha} & \mathfrak{g} \\
 \downarrow \exp_\alpha & & \downarrow \exp_\beta & & \downarrow \exp \\
 G_\alpha & \xleftarrow{p_\alpha^\beta} & G_\beta & \xleftarrow{p_\alpha} & G
 \end{array}$$

is commutative.

In order to define explicitly \exp , let us assume that G is represented as a closed subgroup of the direct product $\prod G_\alpha$ and that \mathfrak{g} is represented as a closed subalgebra of the direct product $\prod \mathfrak{g}_\alpha$. Then we can define the lifted map \exp in the following way.

Given $X \in \mathfrak{g}$, let $\{X_\alpha = Tp_\alpha(X)\}$ be the corresponding element in $\prod \mathfrak{g}_\alpha$. Then one has $X_\alpha = Tp_\alpha^\beta(X_\beta)$, if $\alpha \leq \beta$, and $\exp X$ is the element of G which corresponds to $\{\exp_\alpha X_\alpha\}$ of $\prod G_\alpha$.

If $\alpha \leq \beta$ then $\exp_\alpha X_\alpha = \exp_\alpha(Tp_\alpha^\beta(X_\beta)) = p_\alpha^\beta(\exp_\beta X_\beta)$ holds and this implies the existence of $\exp X$. Using the commutativity of the above diagram we get the continuity of \exp since the mappings p_α are open and the mappings Tp_α and \exp_α are continuous.

Every $\varphi \in \Lambda(G)$ defines an element $\varphi_\alpha = p_\alpha \circ \varphi \in \Lambda(G_\alpha)$ for every $\alpha \in A$. Since G_α is a Lie group there exists a unique element $X_\alpha \in \mathfrak{g}_\alpha$ such that $\varphi_\alpha(t) = \exp_\alpha tX_\alpha$. Then we have

$$\begin{aligned}
 \exp_\alpha tX_\alpha &= \varphi_\alpha(t) = p_\alpha \circ \varphi(t) = p_\alpha^\beta \circ p_\beta \circ \varphi(t) \\
 &= p_\alpha^\beta(\varphi_\beta(t)) = p_\alpha^\beta(\exp_\beta(tX_\beta)) \\
 &= \exp_\alpha(tTp_\alpha^\beta(X_\beta))
 \end{aligned}$$

for any $\alpha \leq \beta$. This implies that $X_\alpha = Tp_\alpha^\beta(X_\beta)$, if $\alpha \leq \beta$. Consequently the family $\{X_\alpha\}$ defines an element $X \in \mathfrak{g}$ such that $\varphi(t) = \exp(tX)$, $t \in \mathbb{R}$.

On the other hand every $X \in \mathfrak{g}$ defines a one-parameter subgroup $\varphi^X(t) = \exp(tX)$ of the LP-group G . Thus we have proved that every $\varphi \in \Lambda(\mathbb{R}, G)$ is defined by an element $X \in \mathfrak{g}$.

Now we will prove the injectivity of \exp . Let us assume that

$$\exp(tX_1) = \exp(tX_2), \quad X_1, X_2 \in \mathfrak{g}.$$

It follows that

$$\exp_\alpha(tTp_\alpha(X_1)) = p_\alpha(\exp(tX_1)) = p_\alpha(\exp(tX_2)) = \exp_\alpha(tTp_\alpha(X_2))$$

and hence $Tp_\alpha X_1 = Tp_\alpha X_2$ for every $\alpha \in A$. This implies $X_1 = X_2$ and the considered mapping is injective.

PROPOSITION 5.1

For any LP-group G , $\Lambda(G)$ is a Lie algebra and the exponential mapping

$$\exp_G: \mathfrak{g} \ni X \longmapsto \varphi^X \in \Lambda(G)$$

is an isomorphism of Lie algebras.

Proof (see also [2]). We have already shown that this mapping is bijective. It remains to prove that it is a homomorphism as well. We use the Trotter formulae. Assume $\varphi_1, \varphi_2 \in \Lambda(G)$. Then for every $\alpha \in A$ holds

$$\begin{aligned} (p_\alpha \circ \varphi_1 + p_\alpha \circ \varphi_2)(t) &= \lim_{k \rightarrow \infty} \left(p_\alpha \circ \varphi_1 \left(\frac{t}{k} \right) p_\alpha \circ \left(\frac{t}{k} \right) \right)^k \\ &= \lim_{k \rightarrow \infty} p_\alpha \left(\varphi_1 \left(\frac{t}{k} \right) \varphi_2 \left(\frac{t}{k} \right) \right)^k, \end{aligned}$$

and

$$\begin{aligned} [p_\alpha \circ \varphi_1, p_\alpha \circ \varphi_2](t^2) &= \lim_{k \rightarrow \infty} \left(p_\alpha \circ \varphi_1 \left(\frac{t}{k} \right) p_\alpha \circ \varphi_2 \left(\frac{t}{k} \right) p_\alpha \circ \varphi_1 \left(-\frac{t}{k} \right) p_\alpha \circ \varphi_2 \left(-\frac{t}{k} \right) \right)^{k^2} \\ &= \lim_{k \rightarrow \infty} p_\alpha \left(\varphi_1 \left(\frac{t}{k} \right) \varphi_2 \left(\frac{t}{k} \right) \varphi_1 \left(-\frac{t}{k} \right) \varphi_2 \left(-\frac{t}{k} \right) \right)^{k^2}. \end{aligned}$$

This implies the existence of the limits

$$\begin{aligned} \lim_{k \rightarrow \infty} \left(\varphi_1 \left(\frac{t}{k} \right) \varphi_2 \left(\frac{t}{k} \right) \right)^k, \\ \lim_{k \rightarrow \infty} p_\alpha \left(\varphi_1 \left(\frac{t}{k} \right) \varphi_2 \left(\frac{t}{k} \right) \varphi_1 \left(-\frac{t}{k} \right) \varphi_2 \left(-\frac{t}{k} \right) \right)^{k^2}. \end{aligned}$$

It is true that $p_\alpha \circ \varphi_i(t) = \exp_\alpha(t(X_i)_\alpha)$ for $i = 1, 2$ and every $\alpha \in A$, so

$$\begin{aligned} p_\alpha \circ (\varphi_1 + \varphi_2)(t) &= p_\alpha \left(\lim_{k \rightarrow \infty} \left(\varphi_1 \left(\frac{t}{k} \right) \varphi_2 \left(\frac{t}{k} \right) \right)^k \right) \\ &= \lim_{k \rightarrow \infty} p_\alpha \left(\varphi_1 \left(\frac{t}{k} \right) \varphi_2 \left(\frac{t}{k} \right) \right)^k \\ &= (p_\alpha \circ \varphi_1 + p_\alpha \circ \varphi_2)(t) \\ &= \exp_\alpha t((X_1)_\alpha + (X_2)_\alpha) \end{aligned}$$

for every $\alpha \in A$.

Moreover

$$\begin{aligned} p_\alpha \circ [\varphi_1, \varphi_2](t^2) &= p_\alpha \left(\lim_{k \rightarrow \infty} p_\alpha \left(\varphi_1 \left(\frac{t}{k} \right) \varphi_2 \left(\frac{t}{k} \right) \varphi_1 \left(-\frac{t}{k} \right) \varphi_2 \left(-\frac{t}{k} \right) \right)^{k^2} \right) \\ &= \lim_{k \rightarrow \infty} p_\alpha \left(\varphi_1 \left(\frac{t}{k} \right) \varphi_2 \left(\frac{t}{k} \right) \varphi_1 \left(-\frac{t}{k} \right) \varphi_2 \left(-\frac{t}{k} \right) \right)^{k^2} \\ &= [p_\alpha \circ \varphi_1, p_\alpha \circ \varphi_2](t^2) \\ &= \exp_\alpha t^2[(X_1)_\alpha, (X_2)_\alpha]. \end{aligned}$$

Let $X_i = \{(X_i)_\alpha\} \in \mathfrak{g}$, $i = 1, 2$. Then $\varphi_i(x) = \exp(tX_i)$, and we have conditions

$$\begin{aligned}
 (\varphi_1 + \varphi_2)(t) &= \exp(t(X_1 + X_2)), \\
 [\varphi_1, \varphi_2](t) &= \exp(t[X_1, X_2]).
 \end{aligned}$$

We get that the mapping \exp_G is an isomorphism.

Finally let us mention the existence of a universal covering group for any LP-group. Let \mathfrak{g} denote an arbitrary LP-Lie algebra and $\{\mathfrak{g}_\alpha, q_\alpha^\beta\}$ an inverse system of finite dimensional Lie algebras such that $\mathfrak{g} = \lim \text{proj} \{\mathfrak{g}_\alpha, q_\alpha^\beta\}$.

Let G_α be a unique connected and simply connected Lie group corresponding to \mathfrak{g}_α . There exists a continuous and open epimorphism $\tilde{p}_\alpha^\beta: \tilde{G}_\beta \rightarrow \tilde{G}_\alpha$, if $\alpha \leq \beta$ such that $T\tilde{p}_\alpha^\beta = q_\alpha^\beta$.

We have that $\{\tilde{G}_\alpha, \tilde{p}_\alpha^\beta\}$ is an inverse system of Lie groups and it defines an LP-group $\tilde{G} = \lim \text{proj} \{\tilde{G}_\alpha, \tilde{p}_\alpha^\beta\}$.

It can be proven that for any LP-Lie algebra \mathfrak{g} corresponds a unique simply connected LP-group \tilde{G} such that $\mathfrak{g} = \text{Lie}(\tilde{G})$ defined in this way [2]. The group $\tilde{G} = \text{Lie}(\mathfrak{g})$ is called the *universal LP-group corresponding to the LP-Lie algebra* \mathfrak{g} .

Starting with a connected LP-group $G = \lim \text{proj} \{G_\alpha, T p_\alpha^\beta\}$ we get its Lie algebra $\mathfrak{g} = \lim \text{proj} \{\mathfrak{g}_\alpha, T p_\alpha^\beta\}$, which is an LP-Lie algebra and hence defines a universal LP-group $\tilde{G} = \lim \text{proj} \{\tilde{G}_\alpha, \tilde{p}_\alpha^\beta\}$. The Lie group \tilde{G}_α is the universal covering group of the connected Lie group G_α for every $\alpha \in A$. Let $\tilde{q}_\alpha: \tilde{G}_\alpha \rightarrow G_\alpha$ denote the covering epimorphism. There exists a lift $\tilde{q}: \tilde{G} \rightarrow G$ of the covering maps \tilde{q}_α , $\alpha \in A$, satisfying the equations $p_\alpha \circ \tilde{q} = \tilde{q}_\alpha \circ \tilde{p}_\alpha$ and the following diagram

$$\begin{array}{ccccc}
 \tilde{G}_\alpha & \xleftarrow{\tilde{p}_\alpha^\beta} & \tilde{G}_\beta & \xleftarrow{\tilde{p}_\alpha} & \tilde{G} \\
 \downarrow \tilde{q}_\alpha & & \downarrow \tilde{q}_\beta & & \downarrow \tilde{q} \\
 G_\alpha & \xleftarrow{p_\alpha^\beta} & G_\beta & \xleftarrow{p_\alpha} & G
 \end{array}$$

is commutative for any $\alpha \leq \beta$.

One can show that the mapping \tilde{q} is a continuous homomorphism which maps \tilde{G} onto a dense algebraic subgroup G_0 of G , but in general \tilde{q} is not a covering mapping and even it need not be surjective.

However, in view of [2], the mapping \tilde{q} is a continuous epimorphism if G is an arcwise connected LP-group.

6. LP-groups as abstract Lie groups

PROPOSITION 6.1

The projective limit of an inverse system of groups carries a natural group structure. Moreover, if $G = \lim \text{proj} \{G_\alpha, p_\alpha^\beta\}$ is the projective limit of smooth groups with smooth homomorphisms p_α^β , then G is a smooth group and the canonical projections $p_\alpha: G \rightarrow G_\alpha$ are smooth.

Proof. It is obvious that

$$\begin{aligned} \mu_\alpha \circ (p_\alpha^\beta \times p_\alpha^\beta) &= p_\alpha^\beta \circ \mu_\beta, \\ \nu_\alpha \circ p_\alpha^\beta &= p_\alpha^\beta \circ \nu_\beta. \end{aligned}$$

Taking $\mu(g_1, g_2) = \{\mu_\alpha((g_1)_\alpha, (g_2)_\alpha)\}$, $e := \{e_\alpha\}$ and $\{f_\alpha\}^{-1} := \{f_\alpha^{-1}\}$ we see that G is a group.

Next, to prove the second assertion we have to show that the mappings $\mu: G \times G \rightarrow G$ and $\nu: G \rightarrow G$ are smooth. Let $f, g \in \mathcal{S}^\infty(\mathbb{R}, G)$. Then $\mu(f, g) := \{\mu_\alpha(f_\alpha, g_\alpha)\}$, $\nu(f) := \{f_\alpha^{-1}\}$, where μ_α is the multiplication of G_α . Consequently, $\mu(f, g) \in \mathcal{S}^\infty(\mathbb{R}, G)$ by definition, and μ is smooth. Likewise, the inversion ν is smooth.

LEMMA 6.2

Let X_α, Y_α be smooth spaces, $\alpha \in A$, and let $X = \lim \text{proj} \{X_\alpha, p_\alpha^\beta\}$, $Y = \lim \text{proj} \{Y_\alpha, q_\alpha^\beta\}$ be the corresponding smooth spaces (cf. Proposition 2.4). If for any α , $f_\alpha: X_\alpha \rightarrow Y_\alpha$ is smooth such that $q_\alpha^\beta \circ f_\beta = f_\alpha \circ p_\alpha^\beta$ for any $\alpha \leq \beta$, then $f = \{f_\alpha\}$ is smooth.

Proof. Let $c = \{c_\alpha\} \in \mathcal{S}^\infty(\mathbb{R}, X)$. Then clearly $f \circ c = \{f_\alpha \circ c_\alpha\} \in \mathcal{S}^\infty(\mathbb{R}, Y)$ and the lemma follows.

We have the following main result.

THEOREM 6.3

Let G be an LP-group. Then G admits the structure of an abstract Lie group. Moreover, the Lie algebra of G coincides with the Lie algebra defined by the abstract Lie group structure of G , and \exp_G coincides with that defined by Evol_G^r .

Proof. Let $G = \lim \text{proj} G_\alpha$, where $\{G_\alpha, p_\alpha^\beta\}$ is an inverse system of finite dimensional Lie groups, be an LP-group.

Let $\mathfrak{g} = \lim \text{proj} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \text{Lie}(G_\alpha)$ is the Lie algebra of G_α . Let us define Evol_G^r as the projective limit of the corresponding inverse system of the mappings $\text{Evol}_{G_\alpha}^r$,

$$\text{Evol}_G^r := \lim \text{proj} \text{Evol}_{G_\alpha}^r.$$

This definition is correct since for any $\alpha \leq \beta$ we have that the following diagram

$$\begin{array}{ccc}
 \mathcal{C}^\infty(\mathbb{R}, \mathfrak{g}_\alpha) & \xleftarrow{(Tp_\alpha^\beta)^*} & \mathcal{C}^\infty(\mathbb{R}, \mathfrak{g}_\beta) \\
 \downarrow \text{Evol}_{G_\alpha}^r & & \downarrow \text{Evol}_{G_\beta}^r \\
 \mathcal{S}_e^\infty(\mathbb{R}, G_\alpha) & \xleftarrow{(p_\alpha^\beta)^*} & \mathcal{S}_e^\infty(\mathbb{R}, G_\beta)
 \end{array}$$

commutes in view of Corollary 3.6.

Evol_G^r is bijective. In fact, this follows by the bijectivity of $\text{Evol}_{G_\alpha}^r$ for any $\alpha \in A$ and the general property of projective limits. Thus we have showed that the set of smooth curves $\mathcal{S}_e^\infty(\mathbb{R}, G)$ is defined by the bijective map Evol_G^r .

In order to show that (G1) is satisfied it suffices to check that

$$\mathcal{S}^\infty(\mathbb{R}, G) = \bigcup_{g \in G} \mathcal{S}_e^\infty(\mathbb{R}, G) \cdot g$$

is indeed a smooth structure. That all constants belong to $\mathcal{S}^\infty(\mathbb{R}, G)$ follows from (G2) proved below. Next let $g \in \mathcal{S}^\infty(\mathbb{R}, G)$ and $f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. By using the translation by $g(0)^{-1} \in G$ and the condition (G1) we may assume that $g \in \mathcal{S}_e^\infty(\mathbb{R}, G)$, i.e. $g = \text{Evol}_G^r(X)$. Next in view of Corollary 3.5 one has

$$\text{Evol}_G^r(X)(f(t)) = \text{Evol}_G^r(f' \cdot (X \circ f))(t) \cdot \text{Evol}_G^r(X)(f(0)).$$

Consequently, $g \circ f = \text{Evol}_G^r(X) \circ f \in \mathcal{S}^\infty(\mathbb{R}, G)$. Thus (G1) is fulfilled.

Specifically, the following diagram

$$\begin{array}{ccccc}
 \mathcal{C}^\infty(\mathbb{R}, \mathfrak{g}_\alpha) & \xleftarrow{(Tp_\alpha^\beta)^*} & \mathcal{C}^\infty(\mathbb{R}, \mathfrak{g}_\beta) & \xleftarrow{(Tp_\beta)^*} & \mathcal{C}^\infty(\mathbb{R}, \mathfrak{g}) \\
 \downarrow \text{Evol}_{G_\alpha}^r & & \downarrow \text{Evol}_{G_\beta}^r & & \downarrow \text{Evol}_G^r \\
 \mathcal{S}_e^\infty(\mathbb{R}, G_\alpha) & \xleftarrow{(p_\alpha^\beta)^*} & \mathcal{S}_e^\infty(\mathbb{R}, G_\beta) & \xleftarrow{(p_\beta)^*} & \mathcal{S}_e^\infty(\mathbb{R}, G)
 \end{array}$$

is commutative for any $\alpha \leq \beta$.

To check (G3) let $f \in \mathcal{S}_e^\infty(\mathbb{R}, G)$ and let $f^\lambda \in \text{Map}(\mathbb{R}, G)$ is given by $f^\lambda(t) := f(\lambda t) = \{f_\alpha(\lambda t)\}$ for $\lambda \in \mathbb{R}$. Then, due to (G1), $f^\lambda \in \mathcal{S}_e^\infty(\mathbb{R}, G)$ and we have

$$\delta_G^r(f^\lambda) = \{\delta_{G_\alpha}^r(f_\alpha^\lambda)\} = \{\lambda \delta_{G_\alpha}^r(f_\alpha)\} = \lambda \{\delta_{G_\alpha}^r(f_\alpha)\} = \lambda \delta_G^r(f).$$

Next the condition (G4) follows from Lemma 6.2 as for $g = \{g_\alpha\} \in G$ one has $\text{Ad}_G(g) = \{\text{Ad}_{G_\alpha}(g_\alpha)\}$.

The condition (G5) holds by the componentwise computation

$$\begin{aligned}
 \delta_G^r(fg) &= \{\delta_{G_\alpha}^r(f_\alpha g_\alpha)\} = \{\delta_{G_\alpha}^r(f_\alpha)\} + \{\text{Ad}_{G_\alpha}(f_\alpha) \delta_{G_\alpha}^r(g_\alpha)\} \\
 &= \{\delta_{G_\alpha}^r(f_\alpha)\} + \{\text{Ad}_{G_\alpha}(f_\alpha)\} \{\delta_{G_\alpha}^r(g_\alpha)\} \\
 &= \delta_G^r(f) + \text{Ad}_G(f) \delta_G^r(g).
 \end{aligned}$$

In order to show (G6) let us observe that the following diagram

$$\begin{array}{ccccc}
 \mathcal{S}^\infty(\mathbb{R}, G_\alpha) & \xleftarrow{(p_\alpha^\beta)_*} & \mathcal{S}^\infty(\mathbb{R}, G_\beta) & \xleftarrow{(p_\beta)_*} & \mathcal{S}^\infty(\mathbb{R}, G) \\
 \downarrow \text{ev}_1^\alpha & & \downarrow \text{ev}_1^\beta & & \downarrow \text{ev}_1 \\
 G_\alpha & \xleftarrow{p_\alpha^\beta} & G_\beta & \xleftarrow{p_\beta} & G
 \end{array}$$

is commutative, where $\text{ev}_1 := \lim \text{proj } \text{ev}_1^\alpha$, since

$$\begin{aligned}
 p_\beta \circ \text{ev}_1(\varphi) &= p_\beta(\{\text{ev}_1^\alpha(\varphi_\alpha)\}) = \text{ev}_1^\beta(\varphi_\beta) = \varphi_\beta(1) = p_{\beta_*}(\varphi(1)) \\
 &= \text{ev}_1^\beta(p_{\beta_*}(\varphi)).
 \end{aligned}$$

Hence

$$\text{ev}_1 \circ \text{Evol}_G^r(\{f_\alpha\}) = \{\text{ev}_1^\alpha \circ \text{Evol}_{G_\alpha}^r(f_\alpha)\} \in G,$$

and in view of Lemma 6.2 the condition follows.

Finally we have to prove (G2). The first assertion is fulfilled by the definition of Evol_G^r . Now let $\varphi \in \Lambda_0(G)$. Then $\varphi = \{\varphi_\alpha\}$ and $\varphi_\alpha \in \Lambda_0(G_\alpha)$. For any $\alpha \leq \beta$ one has $\varphi_\alpha = p_\alpha^\beta \circ \varphi_\beta$. If $\varphi_\alpha = \exp_{G_\alpha}(X_\alpha)$ then for $X = \{X_\alpha\}$ we have

$$\frac{d}{dt} \Big|_0 p_\alpha^\beta(\exp_{G_\beta}(tX_\beta)) = Tp_\alpha^\beta(X_\beta).$$

On the other hand

$$\frac{d}{dt} \Big|_0 \exp_{G_\alpha}(tTp_\alpha^\beta(X_\beta)) = T \exp_{G_\alpha} . Tp_\alpha^\beta(X_\beta) = Tp_\alpha^\beta(X_\beta).$$

and

$$p_\alpha^\beta(\exp_{G_\beta}(0)) = e_\alpha = \exp_{G_\alpha}(0).$$

So we have

$$p_\alpha^\beta(\exp_{G_\beta}(tX_\beta)) = \exp_{G_\alpha}(tTp_\alpha^\beta(X_\beta))$$

if $\alpha \leq \beta$. Consequently, we have $X = \{X_\alpha\} \in \mathfrak{g} = \lim \text{proj } \mathfrak{g}_\alpha$ and $\exp_G(X) = \varphi$ as required. (G2) is then proved.

From the above considerations it is clear that $\mathfrak{g} = \lim \text{proj } \mathfrak{g}_\alpha = \text{Lie}(G)$. Moreover for $X \in \mathfrak{g}$

$$\exp_G(X) = \text{Evol}_G^r(X)(1) = \{\text{Evol}_{G_\alpha}^r(X_\alpha)(1)\} = \{\exp_{G_\alpha}(X_\alpha)\}$$

so we have $\exp_G = \lim \text{proj } \exp_{G_\alpha}$.

From the above proof it follows the following

COROLLARY 6.4

If G is an LP-group then $\Lambda(G) = \Lambda_0(G)$.

Finally we have

THEOREM 6.5

Let G be a connected locally compact topological group. Then G admits the structure of an abstract Lie group. Furthermore its Lie algebra and the exponential mapping are the same as those determined by the abstract Lie group structure.

Proof. In fact, by Yamabe theorem such a G can be expressed as the projective limit of an inverse system of finite dimensional Lie groups. Moreover, one defines $\mathfrak{g} = \text{Lie}(G)$ and \exp_G independently of the inverse system chosen, so Theorem 6.3 applies to G .

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*Faculty of Applied Mathematics
AGH University of Science and Technology
al. Mickiewicza 30
PL-30-059 Kraków
Poland
E-mail: lechjace@wms.mat.agh.edu.pl
E-mail: tomasz@uci.agh.edu.pl*

