ANTONI CHRONOWSKI

Some properties of extensions of loops

The paper is a kind of sequel to the paper [3]. In the paper we shall analyse some aspects of the construction of extensions of loops given in [3]. Moreover, we shall consider this construction in the case when loops are groups.

Definitions of quasigroup, loop, subloop, normal subloop, coset, quotient loop are used according to Bruck [2].

DEFINITION 1. (cf. [3]). A loop Σ is said to be an extension of a loop K by a loop L if the following conditions hold:

(i) K is a normal subloop of the loop Σ,

(ii) the quotient loop Σ / K and the loop L are isomorphic.

Let K be a normal subloop of a loop Σ. A mapping $s\colon \Sigma/K \to \Sigma$ is called a selector if it satisfies the following condition:

$$\bigwedge_{M \in \Sigma/K} s(M) \in M.$$

Let L and K be loops. Let $f, g: L \to K$ be arbitrary mappings. By a product fg of the mappings f and g we mean a mapping $fg: L \to K$ defined as follows:

$$(fg)(l) = f(l)g(l)$$

for $l \in L$.

THEOREM 1. Let Σ and L be loops. Let K be a normal subloop of the loop Σ. The loop Σ is an extension of the loop K by the loop L if and only if there exists a mapping $\sigma: L \to \Sigma$ fulfilling the following conditions:

$$(w_1) \quad \bigwedge_{x \in \Sigma} \ \bigvee^{1}_{\langle l,k \rangle \in L \times K} x = \sigma(l)k,$$

$$(w_2) \quad \bigwedge_{l_1, l_2 \in L} \left[\sigma(l_1 l_2)K = (\sigma(l_1)K)(\sigma(l_2)K)\right].$$

P r o o f. If the loop Σ is an extension of the loop K by the loop L, then the mapping $\sigma = s \circ f$, where $s: \Sigma/K \to \Sigma$ is a selector and $f: L \to \Sigma/K$ is an isomorphism, satisfies conditions (w_1) and (w_2) (cf. [3]).

Let a mapping $\sigma: L \to \Sigma$ satisfies conditions (w_1) and (w_2). We define a mapping $f: L \to \Sigma/K$ as follows:

$$f(l) = \sigma(l)K$$

for $l \in L$.

As an easy consequence of conditions (w_1) and (w_2) we obtain that the mapping f is an isomorphism.

Then the loop Σ is an extension of the loop K by the loop L.

THEOREM 2. Let a loop Σ be an extension of a loop K by a loop L. A mapping $\sigma : L \to \Sigma$ satisfies conditions (w_1) and (w_2) if and only if $\sigma = s \circ f$, where $s: \Sigma/K \to \Sigma$ is a selector and $f: L \to \Sigma/K$ is an isomorphism.

P r o o f. Let $\sigma : L \to \Sigma$ satisfies conditions (w_1) and (w_2). We define a mapping $f: L \to \Sigma/K$ as follows:

$$f(l) = \sigma(l)K$$

for $l \in L$. The mapping f is an isomorphism of the loops L and Σ/K. Define a selector $s: \Sigma/K \to \Sigma$ by the rule:

$$s(f(l)) = \sigma(l)$$

for $l \in L$. Hence $\sigma = s \circ f$.

If $\sigma = s \circ f$, where $s: \Sigma/K \to \Sigma$ is a selector and $f: L \to \Sigma/K$ is an isomorphism, then conditions (w_1) and (w_2) are fulfilled (cf. [3]).

LEMMA 1. Let a loop Σ be an extension of a loop K by a loop L. Let $\sigma_1 = s_1 \circ f_1$, where $s_1: \Sigma/K \to \Sigma$ is a selector and $f_1: L \to \Sigma/K$ is an isomorphism. Then for an arbitrary automorphism $\varkappa \in Aut(L)$ and an arbitrary mapping $\delta : L \to K$ the mapping $\sigma = (\sigma_1 \circ \varkappa)\delta$ satisfies conditions (w_1) and (w_2).

P r o o f. At first, we shall prove that the mapping σ we can represent in the form $\sigma = s \circ f$, where $s: \Sigma/K \to \Sigma$ is a selector and $f: L \to \Sigma/K$ is an isomorphism.
The mapping σ can be written as $\sigma = (s_1 \circ f_1 \circ \varkappa)\delta$. Put $f = f_1 \circ \varkappa$. Of course, $f: L \to \Sigma/K$ is an isomorphism and $\sigma = (s_1 \circ f)\delta$.

A mapping $s\colon \Sigma/K \to \Sigma$ defined by the following rule:

$$s(f(l)) = s_1(f(l))\,\delta(l)$$

for $l \in L$ is a selector.

Thus $\sigma(l) = s_1(f(l))\,\delta(l) = s(f(l)) = (s \circ f)(l)$ for an arbitrary $l \in L$, hence $\sigma = s \circ f$. Applying Theorem 2 we get that the mapping σ satisfies conditions (w_1) and (w_2).

THEOREM 3. Let a loop Σ be an extension of a loop K by a loop L. Let $\sigma_1 = s_1 \circ f_1$, where $s_1\colon \Sigma/K \to \Sigma$ is a selector and $f_1\colon L \to \Sigma/K$ is an isomorphism. Let $\sigma\colon L \to \Sigma$ be an arbitrary mapping. The mapping σ satisfies conditions (w_1) and (w_2) if and only if there exist an automorphism $\varkappa \in \mathrm{Aut}(L)$ and a mapping $\delta\colon L \to K$ such that $\sigma = (\sigma_1 \circ \varkappa)\delta$.

P r o o f. If a mapping $\sigma\colon L \to \Sigma$ satisfies conditions (w_1) and (w_2), then according to Theorem 2 $\sigma = s \circ f$, where $s\colon \Sigma/K \to \Sigma$ is a selector and $f\colon L \to \Sigma/K$ is an isomorphism. Notice that the mapping

$$(1) \qquad \varkappa = f_1^{-1} \circ f$$

is an automorphism of the loop L. And so we have $f = f_1 \circ \varkappa$ and $\sigma = s \circ f_1 \circ \varkappa$. Moreover,

$$(2) \qquad s((f_1 \circ \varkappa)(l))K = s_1((f_1 \circ \varkappa)(l))K$$

for $l \in L$. Using equality (2) one can define a mapping $\delta\colon L \to K$ as follows:

$$(3) \qquad s((f_1 \circ \varkappa)(l)) = s_1((f_1 \circ \varkappa)(l))\,\delta(l)$$

for $l \in L$. Since $\sigma = s \circ f_1 \circ \varkappa$ and so $\sigma(l) =$
$= (s \circ f_1 \circ \varkappa)(l) = ((s_1 \circ f_1 \circ \varkappa)(l))\delta(l) = (\sigma_1 \circ \varkappa)(l)\delta(l)$

for $l \in L$. Then $\sigma = (\sigma_1 \circ \varkappa)\delta$, where $\varkappa \in \mathrm{Aut}(L)$ is an automorphism defined by rule (1) and $\delta : L \to K$ is a mapping defined by formula (3).

If there exist an automorphism $\varkappa \in \mathrm{Aut}(L)$ and a mapping $\delta : L \to K$ such that $\sigma = (\sigma_1 \circ \varkappa)\delta$, then from Lemma 1 we get that σ satisfies conditions (w_1) and (w_2).

For ease of reference we now write the following definition (cf. [3]).

Let L and K be loops. Let $\varphi : L \times K \times L \times K \to K$ be any mapping fulfilling the following conditions:

1^0 $\varphi(l, k, 1, 1) = \varphi(1, 1, l, k) = k$,

2^0 $\varphi(1, k_1, 1, k_2) = k_1 k_2$,

3^0 the mapping $\varphi(l_1, k_1, l_2, \cdot) : K \to K$ is a bijection,

4^0 the mapping $\varphi(l_1, \cdot, l_2, k_2) : K \to K$ is a bijection,

for $l, l_1, l_2 \in L$ and $k, k_1, k_2 \in K$.

DEFINITION 2. An algebraic structure $(L \times K, \circ)$ with an operation $\circ$ defined by the formula:

$$\langle l_1, k_1 \rangle \circ \langle l_2, k_2 \rangle = \langle l_1 l_2, \varphi(l_1, k_1, l_2, k_2) \rangle$$

for arbitrary pairs $\langle l_1, k_1 \rangle, \langle l_2, k_2 \rangle \in L \times K$ is called a product $\langle L ; K \rangle_\varphi$.

A product $\langle L ; K \rangle_\varphi$ is a loop (cf. [3]).

Let a loop Σ be an extension of a loop K by a loop L. Let $s, s_1 : \Sigma/K \to \Sigma$ be selectors such that $s(K) = s_1(K) = 1$ and let $f, f_1 : L \to \Sigma/K$ be isomorphisms.
We define mappins $\sigma, \sigma_1 : L \to \Sigma$ in the following way:

$$\sigma = s \circ f \quad \text{and} \quad \sigma_1 = s_1 \circ f_1.$$

By means of the mappings σ and σ_1 we define mappings $\varphi, \varphi_1 : L \times K \times L \times K \to K$ by the formulas:

$$(\sigma(l_1)k_1)(\sigma(l_2)k_2) = \sigma(l_1 l_2)\,\varphi(l_1,k_1,l_2,k_2),$$

$$(\sigma_1(l_1)k_1)(\sigma_1(l_2)k_2) = \sigma_1(l_1 l_2)\,\varphi_1(l_1,k_1,l_2,k_2)$$

for $l_1, l_2 \in L$ and $k_1, k_2 \in K$.

The loops $\langle L\ ;\ K\rangle_\varphi$ and $\langle L\ ;\ K\rangle_{\varphi_1}$ are extensions of the loop $K^* = \{\langle 1\ ;\ k\rangle : k \in K\}$ by the loop L (cf. [3]). The extensions $\langle L\ ;\ K\rangle_\varphi$ and $\langle L\ ;\ K\rangle_\varphi$ are isomorphic. Indeed, mappings $F : \Sigma \to \langle L\ ;\ K\rangle_\varphi$ and $F_1 : \Sigma \to \langle L\ ;\ K\rangle_{\varphi_1}$ defined by the rules:

$$F(x) = F(\sigma(l)k) = \langle l, k\rangle,$$

$$F_1(x) = F_1(\sigma_1(l_1)k_1) = \langle l_1, k_1\rangle,$$

for an arbitrary $x = \sigma(l)k = \sigma_1(l_1)k_1 \in \Sigma$ are isomorphisms (cf. [3]). Then the mapping $\psi = F_1 \circ F^{-1}$ is an isomorphism of the extensions $\langle L\ ;\ K\rangle_\varphi$ and $\langle L\ ;\ K\rangle_{\varphi_1}$.

According to Theorem 2 the mappings σ and σ_1 satisfy conditions (w_1) and (w_2). It follows from Theorem 3 that $\sigma = (\sigma_1 \circ \varkappa)\delta$ for some $\varkappa \in \mathrm{Aut}(L)$ and mapping $\delta : L \to K$. In the quotient loops $\langle L\ ;\ K\rangle_\varphi / K^*$ and $\langle L\ ;\ K\rangle_{\varphi_1} / K^*$ all cosets have the same form $\{l\} \times K$ for every $l \in L$. It is easy to see that the loops $\langle L\ ;\ K\rangle_\varphi / K^*$ and $\langle L\ ;\ K\rangle_{\varphi_1} / K^*$ are identical.

A mapping $g : L \to \langle L, K\rangle_\varphi / K^*$ defined as follows:

$$g(l) = \{l\} \times K$$

for $l \in L$ is an isomorphism (cf. [3]). By means of the mapping g and the automorphism $\varkappa \in \mathrm{Aut}(L)$ we define an

isomorphism $g_1 : L \to \langle L, K \rangle_\varphi / K^*$ putting $g_1 = g \circ \varkappa$. Notice that $g(l) = \{\langle l,k \rangle : k \in K\}$ and $g_1(l) = \{\langle l_1,k \rangle : l_1 = \varkappa(l) \wedge k \in K\}$ for every $l \in L$. We shall prove that $\psi(g(l)) = g_1(l)$ for $l \in L$. If $\langle l,k \rangle \in g(l)$ then $\psi(\langle l,k \rangle) = F_1(F^{-1}(\langle l,k \rangle)) = F_1(\sigma(l)k)$. Since $\sigma(l) = (\sigma_1 \circ \varkappa)(l)\delta(l) = \sigma_1(l_1)\delta(l)$, where $l_1 = \varkappa(l)$ and so $\psi(\langle l,k \rangle) = F_1(\sigma(l)k) = F_1((\sigma_1(l_1)\delta(l))k) =$
$= F_1(\sigma_1(l_1)\delta(l)) \circ F_1(k) = \langle l_1, \delta(l) \rangle \circ \langle 1,k \rangle =$
$= \langle l_1, \varphi_1(l_1, \delta(l), 1, k) \rangle \in g_1(l)$.
Since the mappings ψ, g, g_1 are isomorphisms, then the inclusion $\psi(g(l)) \subset g_1(l)$ implies the equality $\psi(g(l)) = g_1(l)$ for every $l \in L$. Notice that $\psi(\langle 1, k \rangle) = F_1(F^{-1}(\langle 1, k \rangle)) = F_1(\sigma(1)k) = F_1(\sigma_1(1)k) = \langle 1,k \rangle$ for every $k \in K$.
In the grup theory is known the following definition of an extension of groups (cf. [4]).

DEFINITION 3. A group Σ is said to be an extension of a group K by a group L if the following conditions hold:
(i) K is a normal subgroup of the group Σ,
(ii) the quotient group Σ/K and the group L are isomorphic.

Let a group Σ be an extension of a group K by a group L. Let $s : \Sigma/K \to \Sigma$ be a selector such that $s(K) = 1$ and let $f : L \to \Sigma/K$ be an isomorphism. We define a mapping $\sigma : L \to \Sigma$ as follows:

$$\sigma = s \circ f.$$

Let $\varphi : L \times K \times L \times K \to K$ be a mapping defined by the following rule:

(4) $\left(\sigma(l_1)k_1\right)\left(\sigma(l_2)k_2\right) = \sigma(l_1 l_2)\varphi(l_1,k_1,l_2,k_2)$

for arbitrary $l_1, l_2 \in L$ and $k_1, k_2 \in K$.

We shall prove that the mapping φ has the form:

$$\varphi(l_1,k_1,l_2,k_2) = \lambda(l_1,l_2)\mu(k_1,l_2)k_2$$

for arbitrary $l_1, l_2 \in L$ and $k_1, k_2 \in K$, where mappings $\lambda : L \times L \to K$ and $\mu : K \times L \to K$ satisfy the following conditions:

(a_1) $\lambda(l, 1) = \lambda(1, l) = 1$,

(a_2) $\mu(k, 1) = k$,

(a_3) $\mu(1, l) = 1$,

(a_4) the mapping $\mu(\cdot, l): K \to K$ is a bijection,

(a_5) $\mu(k_1 k_2, l) = \mu(k_1, l)\mu(k_2, l)$,

(a_6) $\mu(k, l_1 l_2) = \lambda(l_1,l_2)\,\mu(\mu(k,l_1),l_2)\,\lambda(l_1,l_2)^{-1}$,

(a_7) $\lambda(l_1,l_2 l_3)\lambda(l_2,l_3) = \lambda(l_1 l_2,l_3)\,\mu(\lambda(l_1,l_2),l_3)$,

for $l, l_1, l_2, l_3 \in L$ and $k, k_1, k_2 \in K$.

In view of Theorem 2 the mapping σ satisfies condition (w_2) which may be written in the form:

(5) $\left(\sigma(l_1)\sigma(l_2)\right)K = \sigma(l_1 l_2)K$

for arbitrary $l_1, l_2 \in L$.

Using (5) we can define a mapping $\lambda : L \times L \to K$ in the following way:

$$\sigma(l_1)\sigma(l_2) = \sigma(l_1 l_2)\lambda(l_1,l_2)$$

for arbitrary $l_1, l_2 \in L$.

A mapping $\mu : K \times L \to K$ we define by the rule:

$$\mu(k, l) = \sigma(l)^{-1}k\sigma(l)$$

for arbitrary $l \in L$ and $k \in K$.

We shall prove that the mappings λ and μ satisfy conditions (a_1) - (a_7).

It is easy to check that conditions (a_1) - (a_4) hold.

If $k_1, k_2 \in K$ and $l \in L$, then $\mu(k_1 k_2, l) =$
$= \sigma(l)^{-1} k_1 k_2 \sigma(l) = (\sigma(l)^{-1} k_1 \sigma(l)) (\sigma(l)^{-1} k_2 \sigma(l)) =$
$= \mu(k_1, l) \mu(k_2, l)$ and so condition (a_5) is fulfilled.

If $k \in K$ and $l_1, l_2 \in L$, then

$\mu(k, l_1 l_2) = \sigma(l_1 l_2)^{-1} k \sigma(l_1 l_2) =$
$= [\sigma(l_1) \sigma(l_2) \lambda(l_1, l_2)^{-1}]^{-1} k [\sigma(l_1) \sigma(l_2) \lambda(l_1, l_2)^{-1}] =$
$= \lambda(l_1, l_2) \sigma(l_2)^{-1} (\sigma(l_1)^{-1} k \sigma(l_1)) \sigma(l_2) \lambda(l_1, l_2)^{-1} =$
$= \lambda(l_1, l_2) (\sigma(l_2)^{-1} \mu(k, l_1) \sigma(l_2)) \lambda(l_1, l_2)^{-1} =$
$= \lambda(l_1, l_2) \mu(\mu(k, l_1), l_2) \lambda(l_1, l_2)^{-1}$

and so condition (a_6) is fulfilled.

If $l_1, l_2 \in L$ and $k_1, k_2 \in K$, then

(6) $(\sigma(l_1) k_1)(\sigma(l_2) k_2) = \sigma(l_1 l_2) \lambda(l_1, l_2) \mu(k_1, l_2) k_2.$

Indeed, $(\sigma(l_1) k_1)(\sigma(l_2) k_2) =$
$= \sigma(l_1) \sigma(l_2) (\sigma(l_2)^{-1} k_1 \sigma(l_2)) k_2 =$
$= \sigma(l_1) \sigma(l_2) \mu(k_1, l_2) k_2 = \sigma(l_1 l_2) \lambda(l_1, l_2) \mu(k_1, l_2) k_2.$

If $l_1, l_2, l_3 \in L$ and $k_1, k_2, k_3 \in K$, then

$(\sigma(l_1) k_1)[(\sigma(l_2) k_2)(\sigma(l_3) k_3)] = [(\sigma(l_1) k_1)(\sigma(l_2) k_2)](\sigma(l_3) k_3).$

Using (6) we have:

$(\sigma(l_1) k_1) [(\sigma(l_2) k_2)(\sigma(l_3) k_3)] =$
$= (\sigma(l_1) k_1) [\sigma(l_2 l_3) (\lambda(l_2, l_3) \mu(k_2, l_3) k_3] =$
$= \sigma(l_1 l_2 l_3) \lambda(l_1, l_2 l_3) \mu(k_1, l_2 l_3) \lambda(l_2, l_3) \mu(k_2, l_3) k_3;$

$[(\sigma(l_1) k_1) (\sigma(l_2) k_2)] (\sigma(l_3) k_3) =$
$= [\sigma(l_1 l_2) (\lambda(l_1, l_2) \mu(k_1, l_2) k_2)] (\sigma(l_3) k_3) =$

$= \sigma(l_1 l_2 l_3)\lambda(l_1 l_2, l_3)\mu(\lambda(l_1, l_2)\mu(k_1, l_2)k_2, l_3)k_3.$

Hence, $\lambda(l_1, l_2 l_3)\mu(k_1, l_2 l_3)\lambda(l_2, l_3)\mu(k_2, l_3) =$

$= \lambda(l_1 l_2, l_3)\mu(\lambda(l_1, l_2)\mu(k_1, l_2)k_2, l_3).$

Applying conditions (a_6) and (a_5) to the left side and the right side of the above equality, respectively, we get:

$\lambda(l_1, l_2 l_3)\lambda(l_2, l_3)\mu(\mu(k_1, l_2), l_3)\lambda(l_2, l_3)^{-1}\lambda(l_2, l_3)\mu(k_2, l_3) =$

$= \lambda(l_1 l_2, l_3)\mu(\lambda(l_1, l_2), l_3)\mu(\mu(k_1, l_2), l_3)\mu(k_2, l_3)$

and this means that condition (a_7) hoids.

Comparing equalities (4) i (6) we obtain

$$\varphi(l_1, k_1, l_2, k_2) = \lambda(l_1, l_2)\mu(k_1, l_2)k_2$$

for $l_1, l_2 \in L$ and $k_1, k_2 \in K$.

It follows from [3] that the group Σ and the loop $\langle L;K\rangle_\varphi$, where φ is a mapping defined by formula (4) are isomorphic, thus $\langle L ; K\rangle_\varphi$ is a group.

If a mapping $\varphi : L \times K \times L \times K \to K$ has the form

(7) $$\varphi(l_1, k_1, l_2, k_2) = \lambda(l_1, l_2)\mu(k_1, l_2)k_2$$

for arbitrary $l_1, l_2 \in L$ and $k_1, k_2 \in K$, where mappings $\lambda : L \times L \to K$ and $\mu : K \times L \to K$ satisfy conditions (a_1) - (a_7), then $\langle L ; K\rangle_\varphi$ is a group.

It is easy to check that conditions 1° - 4° of Definition 2 are fulfilled, then $\langle L ; K\rangle_\varphi$ is a loop.

We shall show that the operation $\circ$ in the loop $\langle L ; K\rangle_\varphi$ is associative.

If $l_1, l_2, l_3 \in L$ and $k_1, k_2, k_3 \in K$, then

$\langle l_1, k_1\rangle \circ [\langle l_2, k_2\rangle \circ \langle l_3, k_3\rangle] =$

$= \langle l_1 l_2 l_3, \varphi(l_1, k_1, l_2 l_3, \varphi(l_2, k_2, l_3, k_3))\rangle$

and $[\langle l_1,k_1\rangle \circ \langle l_2,k_2\rangle] \circ \langle l_3,k_3\rangle =$

$= \langle l_1 l_2 l_3, \varphi(l_1 l_2, \varphi(l_1,k_1,l_2,k_2),l_3,k_3)\rangle.$

Applying (a_6), (a_7) and (a_5) we obtain:

$$\varphi(l_1,k_1,l_2 l_3, \varphi(l_2,k_2,l_3,k_3)) =$$
$$= \lambda(l_1,l_2 l_3)\mu(k_1,l_2 l_3)\lambda(l_2,l_3)\mu(k_2,l_3)k_3 =$$
$$=\lambda(l_1,l_2 l_3)\lambda(l_2,l_3)\mu(\mu(k_1,l_2),l_3)\lambda(l_2,l_3)^{-1}\lambda(l_2,l_3)\mu(k_2,l_3)k_3=$$
$$= \lambda(l_1,l_2 l_3)\lambda(l_2,l_3)\mu(\mu(k_1,l_2),l_3)\mu(k_2,l_3)k_3 =$$
$$= \lambda(l_1 l_2,l_3)\mu(\lambda(l_1,l_2),l_3)\mu(\mu(k_1,l_2),l_3)\mu(k_2,l_3)k_3 =$$
$$= \lambda(l_1 l_2,l_3)\mu(\lambda(l_1,l_2)\mu(k_1,l_2)k_2,l_3)k_3 =$$
$$= \varphi(l_1 l_2, \varphi(l_1,k_1,l_2,k_3),l_2,k_3).$$

Thus $\langle L ; K\rangle_\varphi$ is a group.

It follows from the considerations in [3] that the group $\langle L ; K\rangle_\varphi$, where the mapping φ has form (7), is an extension up to isomorphism of the group K by the group L.

In this way the problem of determination of all extensions of a group K by a group L can be reduced to the construction of all products $\langle L ; K\rangle_\varphi$, where the mapping φ has form (7).

R e f e r e n c e s

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