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Asymptotical interval stability for a linear homogeneous functional equation

1. INTRODUCTION

This paper is a continuation of [1] and it is devoted to investigation of asymptotical interval stability for a linear homogeneous functional equation

$$\varphi[f(x)] = g(x)\varphi(x), \tag{1}$$

where f and g are given functions and φ is an unknown function. We shall be interested in real, continuous solutions of equation (1).

G.A.Shanholt has proved in [4] stability theorems for a difference equation. Similar results for a nonlinear functional equation of first order are presented in [2], [5]. In this paper we will use the definition of asymptotical stability given in [4] and we shall give some necessary and sufficient conditions for asymptotical interval stability for equation (1).

2. PRELIMINARIES

The given functions f and g will be subjected to certain conditions.

Hypothesis (H_1). The function f is defined, strictly increasing and continuous in an interval $I = (0,d)$, $d > 0$ and it fulfils the condition

$$0 < f(x) < x \quad \text{for } x \in I.$$

Hypothesis (H_2). The function g is defined and continuous in the interval I and $g(x) \neq 0$ for $x \in I$.

The following theorem from [3] will be useful in the sequel:

THEOREM 1. If hypothesis (H_1), (H_2) are fulfilled then equation (1) has in I a continuous solution φ depending on an arbitrary function. More precisely, for any $x_0 \in I$ and an arbitrary continuous function $\varphi_0 : I_0 \to R$, where $I_0 := [f(x_0), x_0]$, fulfilling the condition

$$\varphi_0[f(x_0)] = g(x_0)\,\varphi_0(x_0), \tag{2}$$

there exists exactly one continuous solution φ of equation (1) in I such that $\varphi(x) = \varphi_0(x)$ for $x \in I_0$.

We will denote this solution by $\varphi(\cdot, x_0, \varphi_0)$ and the class of continuous functions φ_0 fulfilling the condition (2) by $B(x_0)$. Moreover, we adopt the following notation: $P(a,b)$ is either an open interval (a,b) or a semiclosed interval $[a,b)$ or $(a,b]$ or a closed interval $[a,b]$ where $-\infty < a < b < \infty$. Similarly we will denote the infinite intervals by $P(a,\infty)$, $P(-\infty,b)$. If $a = b$, then the interval

$[a,b]$ we denote by $\{a\}$. Moreover by G_n we will denote the functional sequence

$$G_n(x) := \prod_{i=0}^{n-1} g[f^i(x)] \quad \text{for} \quad x \in I, \quad n \in N.$$

Now we will accept the following definition of interval stability for equation (1) (see [1]).

DEFINITION 1. Let $-\infty < a \leqslant b < \infty$. $P(a,b)$ is stable if for every $\varepsilon > 0$ and $x_0 \in I$ there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that for an arbitrary function $\varphi_0 \in B(x_0)$ fulfilling the inequalities

(3) $$\varphi_0(x) < b + \delta \quad \text{for} \quad x \in I_0,$$

(4) $$\varphi_0(x) > a - \delta \quad \text{for} \quad x \in I_0,$$

the solution $\varphi(\cdot, x_0, \varphi_0)$ of equation (1) fulfils the inequalities

(5) $$\varphi(x, x_0, \varphi_0) < b + \varepsilon \quad \text{for} \quad x \in (0, x_0],$$

(6) $$\varphi(x, x_0, \varphi_0) > a - \varepsilon \quad \text{for} \quad x \in (0, x_0].$$

DEFINITION 2. Let $b \in R$. $P(-\infty, b)$ is stable if for every $\varepsilon > 0$ and $x_0 \in I$ there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that for an arbitrary function $\varphi_0 \in B(x_0)$ fulfilling (3) the solution $\varphi(\cdot, x_0, \varphi_0)$ of equation (1) fulfils (5).

DEFINITION 3. Let $a \in R$. $P(a, \infty)$ is stable if for every $\varepsilon > 0$ and $x_0 \in I$ there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that for an arbitrary function $\varphi_0 \in B(x_0)$ fulfilling (4) the solution $\varphi(\cdot, x_0, \varphi_0)$ of equation (1) fulfils (6).

We will adopt the following definition of attractor and of asymptotic interval stability (see [2], [4]).

DEFINITION 4. Let $-\infty < a \leqslant b < \infty$. $P(a,b)$ is an attractor if for every $x_0 \in I$ there exists an $\eta = \eta(x_0) > 0$ such that for an arbitrary function $\varphi_0 \in B(x_0)$ fulfilling the inequalities

(7) $$\varphi_0(x) < b + \eta \qquad \text{for } x \in I_0,$$

(8) $$\varphi_0(x) > a - \eta \qquad \text{for } x \in I_0,$$

the solution $\varphi(\cdot, x_0, \varphi_0)$ of equation (1) fulfils the condition

$$\lim_{n \to \infty} \operatorname{dist}\left(\varphi(f^n(x), x_0, \varphi_0), P(a,b)\right) = 0 \quad \text{for } x \in I_0.$$

DEFINITION 5. Let $b \in R$. $P(-\infty, b)$ is an attractor if for every $x_0 \in I$ there exists an $\eta = \eta(x_0) > 0$ such that for an arbitrary function $\varphi_0 \in B(x_0)$ fulfilling (7) the solution $\varphi(\cdot, x_0, \varphi_0)$ of equation (1) fulfils the condition

$$\lim_{n \to \infty} \operatorname{dist}\left(\varphi(f^n(x), x_0, \varphi_0), P(-\infty, b)\right) = 0 \quad \text{for } x \in I_0.$$

DEFINITION 6. Let $a \in R$. $P(a, \infty)$ is an attractor if for every $x_0 \in I$ there exists an $\eta = \eta(x_0) > 0$ such that for an arbitrary function $\varphi_0 \in B(x_0)$ fulfilling (8) the solution $\varphi(\cdot, x_0, \varphi_0)$ of equation (1) fulfils the condition

$$\lim_{n \to \infty} \operatorname{dist}\left(\varphi(f^n(x), x_0, \varphi_0), P(a, \infty)\right) = 0 \quad \text{for } x \in I_0.$$

DEFINITION 7. Let $-\infty < a \leqslant b < \infty$. $P(a,b)$, $P(-\infty,b)$, $P(a,\infty)$ are asymptotically stable a.s. if $P(a,b)$, $P(-\infty,b)$, $P(a,\infty)$ are stable and they are attractors respectively.

In the end of this section we present theorems from [1] on interval stability which will be useful in the sequel. At first we will assume that f fulfils hypothesis (H_1) and g fulfils the following:

<u>Hypothesis</u> (H_3). The function g is defined and continuous in the interval I and $g(x) > 0$ for $x \in I$.

LEMMA 1. Let hypothesis (H_3) be fulfilled and $0 < a \leqslant b < \infty$ or $-\infty < a \leqslant b < 0$. Then the inequality

$$\min\left\{\frac{a}{b}, \frac{b}{a}\right\} \leqslant g(x) \leqslant \max\left\{\frac{a}{b}, \frac{b}{a}\right\} \quad \text{for } x \in I$$

is a necessary and sufficient condition that for every $x_0 \in I$ and $\delta > 0$ there exists a $y_0 \in (a-\delta, b+\delta)$ such that $g(x_0)y_0 \in (a-\delta, b+\delta)$.

THEOREM 2. Let hypotheses (H_1), (H_3) be fulfilled and either $0 < a < b < \infty$ or $-\infty < a < b < 0$. $P(a,b)$ is stable if and only if one of the following conditions is fulfilled:

(9) $$g(x) = 1 \quad \text{for } x \in I,$$

(10) $$g(x) < \max\left\{\frac{a}{b}, \frac{b}{a}\right\} \quad \text{for } x \in I,$$

(11) $$g(x) > \min\left\{\frac{a}{b}, \frac{b}{a}\right\} \quad \text{for } x \in I.$$

THEOREM 3. Let hypotheses (H_1), (H_3) be fulfilled and $a \neq 0$. $\{a\}$ is stable if and only if one of the following conditions is fulfilled:

(12) There exists an $x_0 \in I$ such that $g(x) = 1$ for $x \in (0, x_0]$ and $g(x) \neq 1$ for $x \in (x_0, d)$,

(13) $$g(x) = 1 \quad \text{for } x \in I,$$

(14) $$g(x) < 1 \quad \text{for } x \in I,$$

(15) $$g(x) > 1 \quad \text{for } x \in I.$$

THEOREM 4. Let hypotheses (H_1), (H_3) be fulfilled and $-\infty < b < o < a < \infty$. $P(-\infty,b)$ or $P(a,\infty)$ is stable if and only if

(16) $$g(x) \geqslant 1 \quad \text{for} \quad x \in I.$$

THEOREM 5. Let hypotheses (H_1), (H_3) be fulfilled and either $-\infty \leqslant a \leqslant o < b < \infty$ or $-\infty < a < o \leqslant b < \infty$. $P(a,b)$ is stable if and only if

(17) $$g(x) \leqslant 1 \quad \text{for} \quad x \in I.$$

Now we define the following function:

$$M(x) := \sup_{n \in N} \left\{ \max_{t \in [f(x),x]} G_n(t) \right\} \quad \text{for} \quad x \in I.$$

THEOREM 6. Let hypotheses (H_1), (H_3) be fulfilled. The inequality

(18) $$M(x) < \infty \quad \text{for} \quad x \in I$$

is a necessary and sufficient condition for $\{o\}$, $P(o,\infty)$ and $P(-\infty,o)$ to be stable.

Lastly we will assume that f fulfils hypothesis (H_1) and g fulfils the following:

<u>Hypothesis</u> (H_4). The function g is defined and continuous in the interval I and $g(x) < o$ for $x \in I$.

THEOREM 7. Let hypotheses (H_1), (H_4) be fulfilled. Then the intervals $P(o,b)$ and $P(a,o)$ where $-\infty \leqslant a < o < b < \infty$ and $P(-\infty,b)$ and $P(a,\infty)$ where $-\infty < a < o < b < \infty$ are unstable.

We define two functions:

$$p(x) := \inf_{n \in N} \left\{ \min_{t \in [f(x),x]} G_{2n+1}(t) \right\},$$

$$P(x) := \sup_{n \in N} \left\{ \max_{t \in [f(x),x]} G_{2n}(t) \right\}$$

for $x \in I$.

THEOREM 8. Let hypotheses (H_1), (H_4) be fulfilled. $\{o\}$ is stable if and only if the following conditions are fulfilled:

(19) $$p(x) > -\infty \quad \text{for} \quad x \in I,$$

(20) $$P(x) < \infty \quad \text{for} \quad x \in I.$$

THEOREM 9. Let hypotheses (H_1), (H_4) be fulfilled and $-\infty < a < o < b < \infty$. $P(a,b)$ is stable if and only if

(21) $$\max\left\{\frac{a}{b}, \frac{b}{a}\right\} \leqslant g(x) \quad \text{for} \quad x \in I.$$

3. NECESSARY AND SUFFICIENT CONDITIONS FOR ASYMPTOTICAL INTERVAL STABILITY

3.1. The case $g(x) > o$ for $x \in I$. In this section we will assume that f fulfils hypothesis (H_1) and g fulfils hypothesis (H_3).

We define the following functions:

$$l(x) := \inf_{t \in [f(x),x]} \left\{ \lim_{n \to \infty} G_n(t) \right\},$$

$$L(x) := \sup_{t \in [f(x),x]} \left\{ \lim_{n \to \infty} G_n(t) \right\},$$

$$\bar{L}(x) := \sup_{t \in [f(x),x]} \left\{ \limsup_{n \to \infty} G_n(t) \right\}.$$

Notice that (16) or (17) implies that $\lim_{n \to \infty} G_n(x)$ exists for $x \in I$.

At first we consider $P(a,b)$ where either $-\infty \leqslant a \leqslant o < b < \infty$ or $-\infty < a < o \leqslant b \leqslant \infty$.

THEOREM 10. Let hypotheses (H_1), (H_3) be fulfilled and either $-\infty \leqslant a \leqslant o < b < \infty$ or $-\infty < a < o \leqslant b \leqslant \infty$. $P(a,b)$ is a.s. if and only if the following condition is fulfilled

(22) $$g(x) \leqslant 1, \quad L(x) < 1 \quad \text{for} \quad x \in I.$$

P r o o f. Suppose that (22) holds. According to Theorem 5, $P(a,b)$ is stable. Thus it is sufficient to prove that $P(a,b)$ is an attractor. At first we consider the case where $o < L(x) < 1$ for $x \in I$. Let $x_0 \in I$ and we put $\eta(x_0) := \min\left\{\frac{a(L(x_0) - 1)}{L(x_0)}, \frac{b(1 - L(x_0))}{L(x_0)}\right\}$. From (22) we have $\eta(x_0) > o$. Then for $\varphi_0 \in B(x_0)$ such that (7) and (8) hold we have the inequalities:

$$o \leqslant \lim_{n\to\infty} \varphi(f^n(x), x_0, \varphi_0) = \lim_{n\to\infty} G_n(x)\varphi_0(x) <$$

$$< L(x_0)\left[b + \frac{b(1 - L(x_0))}{L(x_0)}\right] < b, \quad \text{when} \quad \varphi_0(x) \geqslant o,$$

$$a < L(x_0)\left[a - \frac{a(L(x_0) - 1)}{L(x_0)}\right] < \lim_{n\to\infty} G_n(x)\, \varphi_0(x) =$$

$$= \lim_{n\to\infty} \varphi(f^n(x), x_0, \varphi_0) < o, \quad \text{whereas for} \quad \varphi_0(x) < o.$$

Thus $P(a,b)$ is an attractor and, consequently, it is a.s.

The case where there exists an $x_0 \in I$ such that $L(x_0) = o$ is very simple. Then for every $\eta > o$ and $\varphi_0 \in B(x_0)$ such that (7) and (8) hold we have $\lim_{n\to\infty} \varphi(f^n(x), x_0, \varphi_0) = o$ for $x \in I_0$ and, consequently, $P(a,b)$ is a.s.

Now let us suppose that $P(a,b)$ is a.s. From Theorem 5 we have (17) which implies inequalities $0 \leqslant L(x) \leqslant 1$ for $x \in I$. Suppose that there exists an $x_0 \in I$ such that $L(x_0) = 1$. Because L is a semi-continuous function there exists a $t \in I_0$ such that $L(x_0) = \lim_{n\to\infty} G_n(t)$. Without any loss of generality we may assume that $t \in (f(x_0), x_0)$. Let us fix an $\eta > 0$. We may take $\varphi_0 \in B(x_0)$ fulfilling (7) and (8) and condition $\varphi_0(t) \notin [a,b]$. Then we have:

$$\lim_{n\to\infty} \varphi(f^n(t), x_0, \varphi_0) = \varphi_0(t) \notin [a,b].$$

This condition contradicts the a.s. of $P(a,b)$.

THEOREM 11. Let hypotheses (H_1), (H_3) be fulfilled and $-\infty < b < 0 < a < \infty$. Each of sets $P(-\infty,b)$, $P(a,\infty)$ is a.s. if and only if

$$g(x) \geqslant 1, \quad l(x) > 1 \quad \text{for } x \in I. \tag{23}$$

P r o o f. Suppose that (23) holds. According to Theorem 4, P - ,b and P a, are stable. We will prove that $P(-\infty,b)$ and $P(a,\infty)$ are attractors. At first we consider the case where $1 < l(x) < \infty$ for $x \in I$. Let $x_0 \in I$ and let us put $\eta(x_0) := \dfrac{a(l(x_0) - 1)}{l(x_0)}$ for $P(a,\infty)$ and $\eta(x_0) := \dfrac{b(1 - l(x_0))}{l(x_0)}$ for $P(-\infty,b)$. We have $\eta(x_0) > 0$ from (23). Then for a $\varphi_0 \in B(x_0)$ such that (8) or (7) holds respectively, we have the inequalities

$$\varphi_0(x) > a - \frac{a(l(x_0) - 1)}{l(x_0)} = \frac{a}{l(x_0)} \qquad \text{for } x \in I_0$$

or

$$\varphi_o(x) < b + \frac{b(1 - l(x_o))}{l(x_o)} = \frac{b}{l(x_o)} \quad \text{for} \quad x \in I_o,$$

(24) $\lim_{n\to\infty} \varphi(f^n(x), x_o, \varphi_o) = \lim_{n\to\infty} G_n(x)\varphi_o(x) \geqslant l(x_o)\varphi_o(x) > a,$

(25) $\lim_{n\to\infty} \varphi(f^n\ x\ , x_o, \varphi_o) = \lim_{n\to\infty} G_n(x)\varphi_o(x) \leqslant l(x_o)\varphi_o(x) < b.$

Conditions (24) and (25) imply that

$$\lim_{n\to\infty} \mathrm{dist}\big(\varphi(f^n(x), x_o, \varphi_o), P(-\infty, b)\big) = 0 =$$
$$= \lim_{n\to\infty} \mathrm{dist}\big(\varphi(f^n(x), x_o, \varphi_o), P(a, \infty)\big).$$

In case where there exists an $x_o \in I$ such that $l(x_o) = \infty$, it is sufficient to take an η fulfilling inequalities either $o < \eta < a$ or $o < \eta < -b$, respectively.

Now let us suppose that $P(-\infty, b)$ and $P(a, \infty)$ are a.s. From Theorem 4 we have (16) which implies inequalities $1 \leqslant l(x) < \infty$. Suppose that there exists an $x_o \in I$ such that $l(x_o) = 1$. Because l is a semi-continuous function there exists a $t \in I_o$ such that $l(x_o) = \lim_{n\to\infty} G_n(t)$. Without loss of generality we may assume that $t \in (f(x_o), x_o)$. Let us fix $\eta > o$ and we may take $\varphi_o \in B(x_o)$ fulfilling conditions either $\varphi_o(x) > a - \eta$, $\varphi_o(t) < a$ or $\varphi_o(x) < b + \eta$, $\varphi_o(t) > b$ respectively. Then we have

$$\lim_{n\to\infty} \varphi(f^n(t), x_o, \varphi_o) = \lim_{n\to\infty} G_n(t)\varphi_o(t) =$$
$$= l(x_o)\varphi_o(t) = \varphi_o(t) < a,$$

$$\lim_{n\to\infty} \varphi(f^n(t), x_o, \varphi_o) = \lim_{n\to\infty} G_n(t)\varphi_o(t) =$$
$$= l(x_o)\varphi_o(t) = \varphi_o(t) > b,$$

which implies that $P(-\infty, b)$ and $P(a, \infty)$ are not attractors.

An immediate conclusion from Lemma 1 and Theorem 2, 3 is the following:

THEOREM 12. Let hypotheses (H_1), (H_3) be fulfilled and either $0 < a \leqslant b < \infty$ or $-\infty < a \leqslant b < 0$. $P(a,b)$ is a.s. if and only if (10) and (11) hold.

In the end of this section we give a necessary and sufficient conditions for asymptotical stability $\{0\}$, $P(0,\infty)$ and $P(-\infty, 0)$. The following lemma will be useful in the sequel:

LEMMA 2. Let $\{y_{n,k}\}$ be a double sequence of reals. If

$$\limsup_{n\to\infty} y_{n,k} = y_k \quad \text{and} \quad \lim_{k\to\infty} y_k = y_0,$$

then there exist sequences $\{n_i\}$, $\{k_i\}$ of positive integers such that

$$\lim_{i\to\infty} y_{n_i,k_i} = y_0.$$

This lemma is a simple generalization of lemma 4 from [6]. Now we may prove the following:

LEMMA 3. Let hypotheses (H_1), (H_3) be fulfilled and $x_0 \in I$. Then there exist sequences $\{n_i\}$, $\{k_i\}$ and $t_{k_i} \in [f(x_0), x_0]$ such that

$$L(x_0) = \lim_{i\to\infty} G_{n_i}(t_{k_i}). \tag{26}$$

P r o o f. By the definition of function L, there exists a $\{t_k\}$ such that $L(x_0) = \lim_{k\to\infty}\left\{\limsup_{n\to\infty} G_n(t_k)\right\}$.

It is sufficient to put $y_{n,k} := G_n(t_k)$ and from Lemma 2 we have (26).

THEOREM 13. Let hypotheses (H_1), (H_3) be fulfilled. Each of sets $\{o\}$, $P(-\infty,o)$, $P(o,\infty)$ is a.s. if and only if

(27) $$M(x) < \infty , \qquad L(x) = o \qquad \text{for} \quad x \in I.$$

P r o o f. Suppose that (27) holds. From Theorem 6, $\{o\}$, $P(o,\infty)$ and $P(-\infty,o)$ are stable. Let $x_o \in I$ and $\eta > o$. Then we take $\varphi_o \in B(x_o)$ such that $\varphi_o(x) < \eta$ for $P(-\infty,o)$, $\varphi_o(x) > -\eta$ for $P(o,\infty)$ and $-\eta < \varphi_o(x) < \eta$ for $\{o\}$. From (27) we have

$$\limsup_{n\to\infty} \varphi(f^n(x),x_o,\varphi_o) = \limsup_{n\to\infty} G_n(x)\varphi_o(x) = o \quad \text{for} \quad x \in I_o,$$

which implies that the intervals are a.s.

Now let us suppose that $\{o\}$, $P(o,\infty)$ and $P(-\infty,o)$ are asymptotically stable. From Theorem 6 we have (18). Suppose that there exists an $x_o \in I$ such that $L(x_o) > o$. From Lemma 3 there exist sequences $\{n_i\}$, $\{k_i\}$ and $t_{k_i} \in [f(x_o),x_o]$ such that $t_{k_i} \to t_o$ $(i \to \infty)$ and $L(x_o) = \lim_{i\to\infty} G_{n_i}(t_{k_i})$. We take $\varphi_o \in B(x_o)$ fulfilling conditions $\varphi_o(t_o) \neq o$ for $\{o\}$, $\varphi_o(t_o) < o$ for $P(o,\infty)$ and $\varphi_o(t_o) > o$ for $P(-\infty,o)$. Then we have

$$\lim_{i\to\infty} \varphi(f^{n_i}(t_{k_i}),x_o,\varphi_o) = \lim_{i\to\infty} G_{n_i}(t_{k_i})\varphi_o(t_{k_i}) =$$
$$= L(x_o)\,\varphi_o(t_o).$$

This condition contradicts the assumption that $\{o\}$, $P(-\infty,o)$ and $P(o,\infty)$ are attractors.

3.2. The case $g(x) < 0$ for $x \in I$. Now we will assume that f fulfils hypothesis (H_1) and g fulfils hypothesis (H_4). At first we consider intervals $P(a,b)$, $P(-\infty,b)$ and $P(a,\infty)$ where $0 < a \leq b < \infty$ or $-\infty < a \leq b < 0$. If we put $0 < \delta < a$, $0 < \eta < a$ for $P(a,b)$, $P(a,\infty)$, where $0 < a \leq b < \infty$ and $0 < \delta < -b$, $0 < \eta < -b$ for $P(a,b)$ $P(-\infty,b)$ where $-\infty < a \leq b < 0$, then no $\varphi_0 \in B(x_0)$ fulfils inequalities (3), (4) and (7), (8). Thus these intervals are a.s.

From Theorem 7 we also have that intervals $P(0,b)$, $P(a,0)$ where $-\infty < a < 0 < b$, and $P(-\infty,b)$, $P(a,\infty)$ where $-\infty < a < 0 < b < \infty$ are not a.s.

Now we define the following functions

$$R(x) := \sup_{t \in [f(x),x]} \left\{ \lim_{n \to \infty} G_{2n}(t) \right\},$$

$$r(x) := \inf_{t \in [f(x),x]} \left\{ \lim_{n \to \infty} G_{2n+1}(t) \right\},$$

$$\bar{R}(x) := \sup_{t \in [f(x),x]} \left\{ \limsup_{n \to \infty} G_{2n}(t) \right\},$$

$$\bar{r}(x) := \inf_{t \in [f(x),x]} \left\{ \liminf_{n \to \infty} G_{2n+1}(t) \right\}.$$

Notice that (21) implies that $\lim_{n \to \infty} G_{2n}(x)$ and $\lim_{n \to \infty} G_{2n+1}(x)$ exist for $x \in I$. We have a result similar to Lemma 3:

LEMMA 4. Let hypotheses (H_1), (H_4) be fulfilled and $x_0 \in I$. Then there exist sequences $\{n_i\}$, $\{k_i\}$, $\{m_i\}$, $\{l_i\}$ and $t_{k_i} \in I_0$, $t_{l_i} \in I_0$ such that

(28) $$R(x_0) := \lim_{i \to \infty} G_{2n_i}(t_{k_i}),$$

(29) $$r(x_0) := \lim_{i \to \infty} G_{2m_i+1}(t_{l_i}).$$

We have the following

THEOREM 14. Let hypotheses (H_1), (H_4) be fulfilled. The set $\{o\}$ is a.s. if and only if

(30) $-\infty < p(x)$, $P(x) < \infty$, $r(x) = o = R(x)$ for $x \in I$.

P r o o f. Suppose that (30) holds. From Theorem 8 we have that $\{o\}$ is stable. Let $x_o \in I$ and $\eta > o$. Take a $\varphi_o \in B(x_o)$ such that $-\eta < \varphi_o(x) < \eta$. From (30) we have $\lim_{n\to} \varphi(f^n(x), x_o, \varphi_o) = o$ which implies that $\{o\}$ is a.s.

Now let us suppose that there exists an $x_o \in I$ such that $r(x_o) < o$ or $R(x_o) > o$. Let $\eta > o$ and $\varphi_o \in B(x_o)$ fulfil the condition $-\eta < \varphi_o(x) < \eta$. From Lemma 4 there exist sequences $\{n_i\}$, $\{k_i\}$, $\{m_i\}$, $\{l_i\}$ and $\{t_{k_i}\}, \{t_{l_i}\} \subset I_o$ such that (28) and (29) hold. Without loss of generality we may assume that $t_{k_i} \longrightarrow t_o$ or $t_{l_i} \longrightarrow t_o$ and $\varphi_o(t_o) > o$ where $t_o \in I_o$. Then we have

$$\lim_{i\to\infty} \varphi(f^{2n_i}(t_{k_i}), x_o, \varphi_o) = \lim_{i\to\infty} G_{2n_i}(t_{k_i})\varphi_o(t_{k_i}) = $$

$$= R(x_o)\varphi_o(t_o) > o$$

$$\lim_{i\to\infty} \varphi(f^{2m_i+1}(t_{l_i}), x_o, \varphi_o) = \lim_{i\to\infty} G_{2m_i+1}(t_{l_i})\varphi_o(t_{l_i}) =$$

$$= r(x_o)\varphi_o(t_o) < o.$$

This conditions imply that $\{o\}$ is not an attractor what ends the proof of the theorem.

Finally we consider the case $P(a,b)$ where $-\infty < a < o < b < \infty$. We have two following theorems.

THEOREM 15. Let hypotheses (H_1), (H_4) be fulfilled and $-\infty < a < 0 < b < \infty$ $a \neq -b$. $P(a,b)$ is a.s. if and only if (21) holds.

P r o o f. From Theorem 9 condition (21) is necessary for asymptotical stalibity $P(a,b)$. We will prove that (21) is also sufficient. From Theorem 9, $P(a,b)$ is stable. Let $k := \max\left\{\frac{a}{b}, \frac{b}{a}\right\}$. We have from (21) that $-1 < k < 0$. We may prove by simple induction that

(31) $(k)^{2n+1} \leqslant G_{2n+1}(x) < 0 < G_{2n}(x) \leqslant (k)^{2n}$ $n \in N$, $x \in I$.

Inequalities (31) imply that $\lim_{n\to\infty} G_{2n+1}(x) = 0 = \lim_{n\to\infty} G_{2n}(x)$.

Then for $x_0 \in I$, $\eta > 0$ and φ_0 $B(x_0)$ such that (7), (8) hold we have

(32) $\lim_{n\to\infty} \varphi(f^{2n} x, x_0, \varphi_0) = \lim_{n\to\infty} G_{2n}(x)\varphi_0(x) = 0$ for $x \in I_0$,

(33) $\lim_{n\to\infty} \varphi(f^{2n+1}(x), x_0, \varphi_0) = \lim_{n\to\infty} G_{2n+1}(x)\varphi_0(x) = 0$ for $x \in I_0$.

Conditions (32), (33) imply that $P(a,b)$ is a.s.

THEOREM 16. Let hypotheses (H_1), (H_4) be fulfilled and $-\infty < a < 0 < b < \infty$ $a = -b$. $P(a,b)$ is a.s. if and only if

(34) $-1 \leqslant g(x) < 0$, $-1 < \overline{r}(x) \leqslant 0$, $0 \leqslant \overline{R}(x) < 1$ for $x \in I$.

P r o o f. Suppose that (34) holds. From Theorem 9, $P(a,b)$ is stable. We will prove that $P(a,b)$ is an attractor. At first we consider the case where $0 < \overline{R}(x) < 1$ and $-1 < \overline{r}(x) < 0$ for $x \in I$. Let $x_0 \in I$ and we put

$$\eta(x_0) := \min\left\{\frac{a(\overline{R}(x_0) - 1)}{\overline{R}(x_0)}, \frac{a(\overline{r}(x_0) - 1)}{\overline{r}(x_0)}\right\}.$$

From (34) $\eta > 0$. Then for $\varphi_0 \in B(x_0)$ such that (7), (8) hold we have

$$o \leqslant \lim_{n\to\infty} \varphi(f^{2n}(x), x_o, \varphi_o) \leqslant \bar{R}(x_o)\varphi_o(x) <$$

$$< \bar{R}(x_o)\left[b + \frac{a(\bar{R}(x_o) - 1)}{\bar{R}(x_o)}\right] < -a = b$$

for $x \in I_o$, $\varphi_o(x) \geqslant o$,

$$a < \bar{r}(x_o)\left[b + \frac{a(\bar{r}(x_o) - 1)}{\bar{r}(x_o)}\right] < \bar{r}(x_o)\varphi_o(x) \leqslant$$

$$\leqslant \lim_{n\to\infty} \varphi(f^{2n+1}(x), x_o, \varphi_o) \leqslant o$$

for $x \in I_o$, $\varphi_o(x) \geqslant o$,

$$a < \bar{R}(x)\left[a - \frac{a(\bar{R}(x_o) - 1)}{\bar{R}(x_o)}\right] < \bar{R}(x_o)\varphi_o(x) \leqslant$$

$$\lim_{n\to\infty} \varphi(f^{2n}(x), x_o, \varphi_o) \leqslant o$$

for $x \in I_o$, $\varphi_o(x) < o$,

$$o \leqslant \lim_{n\to\infty} \varphi(f^{2n+1}(x), x_o, \varphi_o) = \bar{r}(x_o)\varphi_o(x) <$$

$$< \bar{r}(x_o)\left[a - \frac{a(\bar{r}(x_o) - 1)}{\bar{r}(x_o)}\right] < -a = b$$

for $x \in I_o$, $\varphi_o(x) < o$.

These inequalities imply that $P(a,b)$ is a.s. If there exists an $x_o \in I$ such that $\bar{R}(x_o) = o$ or $\bar{r}(x_o) = o$, then either $\lim_{n\to\infty} \varphi(f^{2n}(x), x_o, \varphi_o) = o$ or $\lim_{n\to\infty} \varphi(f^{2n+1}(x), x_o, \varphi_o) = o$ what also implies that $P(a,b)$ is a.s.

Now we shall prove that (34) is also a necessary condition. Notice that inequality (21) from Theorem 9 implies that $0 \leqslant \bar{R}(x) \leqslant 1$ and $-1 \leqslant \bar{r}(x) \leqslant 0$ for $x \in I$. Suppose that there exists an $x_0 \in I$ such that either $\bar{r}(x_0) = -1$ or $\bar{R}(x_0) = 1$. Since $\bar{r}$ and $\bar{R}$ are semicontinuous there exists a $t \in I_0$ such that $\bar{R}(x_0) = \lim_{n\to\infty} G_{2n}(t)$ or $\bar{r}(x_0) = \lim_{n\to\infty} G_{2n+1}(t)$. Without loss of generality we may assume that $t \in (f(x_0), x_0)$. Then for $\varphi_0 \in B(x_0)$ fulfilling conditions (7), (8) and $\varphi_0(t) < a$ when $\bar{R}(x_0) = 1$, or $\varphi_0(t) > b$ when $\bar{r}(x_0) = 1$, we have

$$\lim_{n\to\infty} \varphi(f^{2n+1}(t), x_0, \varphi_0) = \bar{r}(x_0)\varphi_0(t) = -\varphi_0(t) < -b = a,$$

$$\lim_{n\to\infty} \varphi(f^{2n}(t), x_0, \varphi_0) = \bar{R}(x_0)\varphi_0(t) = \varphi_0(t) < a.$$

Thus $P(a,b)$ is not an attractor , the theorem is proved.

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