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Groups as finite unions of proper subgroups

It is evident that any group is not a union of its two proper subgroups. Moreover, it was proved in [1] and [2] that a group G is the union of three proper subgroups if and only if the Klein 4-group is a homomorphic image of G.

In this note we derive some criterions for groups as unions of at least three proper subgroups. We consider only the irredundant unions, i.e. such that none of the components is contained in the union of all the others. Let l be a natural number, $l \geqslant 2$. A group G is called l-orientable if there exists its subgroup with index l.

THEOREM 1. Let $n \geqslant 3$ and let a group G be an irredundant union of proper subgroups $A_1,\dots,A_n$. Assume the condition:

(H_0) there does not exist l-orientable subgroup, $2 \leqslant l \leqslant n-2$.

Then for the indices of $H = \bigcap_{i=1}^{n} A_i$ and A_i $(i = 1,\dots,n)$

the following equalities are satisfied

(i) $[G : H] = (n - 1)^2$,

(ii) $[A_i : H] = n - 1$, $i = 1 \dots n$,

(iii) $[G : A_i] = n - 1$, $i = 1 \dots n$.

P r o o f. I. In the first part of the proof we show that for any integer s, $2 \leqslant s \leqslant n$, we have $\bigcap_{i=1}^{s} A_i = H$. Thus any intersection of s mutually different factors are equal to each other and to H. The case $s = n$ is trivial. Let $s = n - 1$. We show that

$$(1) \qquad \bigcap_{i=1}^{n-1} A_i \subset A_n.$$

Let x be an arbitrary element such that $x \in A_i$ for $i = 1,\dots,n-1$. We fix y in $A_n - \bigcup_{i=1}^{n-1} A_i$ (such an element exists as G is an irredundant union). Obviously, $y^{-1} \in A_n$, $x \cdot y \in A_n$. Hence $x \in A_n$ and (1) is satisfied. Thus we have

$$(2) \qquad \bigcap_{i=1}^{n-1} A_i = \bigcap_{i=1}^{n} A_i = H.$$

Next we proceed by induction. Let is assume that $1 \leqslant k \leqslant n-3$ and any intersection of s mutually different factors, where $s = n-1, n-2, \dots, n-k$, are equal to each other and to H. We shall prove that any intersection of $n - (k+1)$ factors is also equal to H.

We denote

$$(3) \qquad N := \bigcap_{j=1}^{n-(k+1)} A_j$$

and let x be a fixed element of G such that

$$(4) \qquad x \in A_{n-k} - \bigcup_{j=1}^{n-(k+1)} A_j.$$

The coset $x \cdot N$ is disjoint with $\bigcup_{j=1}^{n-(k+1)} A_j$. In fact, if there are $m \in N$ and i $(i = 1,\dots,n-(k+1))$ such that $x \cdot m \in A_i$ then $m \in A_i$ and therefore $x \in A_i$ which contradicts (4). Hence we have

(5) $$x \cdot N \subset \bigcup_{j=n-k}^{n} A_j.$$

Next, it can be easily verified that

(6) $$N \subset \bigcup_{j=n-k}^{n} (x^{-1} \cdot A_j),$$

(7) $$N \subset \bigcup_{j=n-k}^{n} [(x^{-1} \cdot A_j) \cap N],$$

(8) $$\bigcup_{j=n-k}^{n} [(x^{-1} \cdot A_j) \cap N] =$$

$$= [(x^{-1} \cdot A_{n-k}) \cap N] \cup \bigcup_{j=n-k+1}^{n} [(x^{-1} \cdot A_j) \cap N].$$

But, in view of (4), $x^{-1} \cdot A_{n-k} = A_{n-k}$. From (3) and the inductive assumption we get $A_{n-k} \cap N = H$. This, with (7) and (8) implies

(9) $$N \subset H \cup \bigcup_{j=n-k+1}^{n} [(x^{-1} \cdot A_j) \cap N].$$

If $\bigcup_{j=n-k+1}^{n} [(x^{-1} \cdot A_j) \cap N] = \emptyset$, then $N \subset H$ and, in view of definition of H and N, we have $N = H$. On the other hand if $\bigcup_{j=n-k+1}^{n} [(x^{-1} \cdot A_j) \cap N] \neq \emptyset$ then there are $j_1,\dots,j_t$ such that $n-k+1 \leqslant j_i \leqslant n$ for $i = 1,\dots,t$ (so $1 \leqslant t \leqslant k$

and $1 \leqslant t \leqslant n-3$ as $k \leqslant n-3$) and such that

(10) $$(x^{-1} \cdot A_{j_i}) \cap N \neq \emptyset, \quad i = 1,\dots,t$$

and

(11) $$N \subset H \cup [(x^{-1} \cdot A_{j_1}) \cap N] \cup \dots \cup [(x^{-1} \cdot A_{j_t}) \cap N].$$

Then there exist $y_{j_1}, \dots, y_{j_t}$ satisfying

(12) $$y_{j_i} \in x^{-1} \cdot A_{j_i}, \quad y_{j_i} \in N, \quad i = 1,\dots,t.$$

Hence we have

(13) $$x^{-1} \cdot A_{j_i} = y_{j_i} \cdot A_{j_i}, \quad i = 1,\dots,t.$$

Now in view of the fact that N is a subgroup of G,

(14) $$(y_{j_i} \cdot A_{j_i}) \cap N = y_{j_i} \cdot (A_{j_i} \cap N), \quad i = 1,\dots,t.$$

Next, from (10), (13), (14) and the inductive assumption, we get

(15) $$N \subset H \cup (y_{j_1} \cdot H) \cup \dots \cup (y_{j_t} \cdot H).$$

We have also

(16) $$N = H \cup (y_{j_1} \cdot H) \cup \dots \cup (y_{j_t} \cdot H)$$

as H is a subgroup of N and $y_{j_1} \in N, \dots, y_{j_t} \in N$.

All $y_{j_i} \cdot H$ in the right side of (16) are equal to each other and to H. Otherwise the index $[N : H]$ is greater than or equal to 2 (as $1 \leqslant t \leqslant n-3$) and less than or equal to $n-2$, which contradicts the assumption of the theorem.

Therefore we get

(17) $$N = H$$

and the first part of the proof is finished. In particular, we have obtained

(18) $$A_i \cap A_j = H \quad \text{for} \quad i,j = 1,\dots,n, \quad i \neq j$$

and

(19) $$G = H \cup (A_1 - H) \cup \dots \cup (A_n - H),$$

is the union of mutually disjoint components.

II. Let us consider the partition of group A_1 by the left cosets $x \cdot H$. Then the assumption (H_0) implies that this partition consists of at least $n-1$ different cosets. We are going to prove that it has at most $n-1$ different cosets.

Let a_1, a_2 be arbitrary elements of A_1 such that

(20) $$a_1, a_2 \in A_1 - H,$$

(21) $$a_1^{-1} \cdot a_2 \notin H.$$

Hence $a_1 \cdot H \neq a_2 \cdot H$. Let x be an arbitrary element such that

(22) $$x \in A_2 - A_1.$$

Since

$$A_2 - A_1 = A_2 - (A_2 \cap A_1) = A_2 - H,$$

we have

(23) $$x \in A_2 - H.$$

Next, in view of (20) and (22), $x \cdot a_i \notin A_1$, $i = 1,2$, $x \cdot a_i \notin A_2$, $i = 1,2$, which implies $x \cdot a_i \in \bigcup_{j=3}^{n} A_j$, $i = 1,2$. There are $j_1, j_2 \geqslant 3$, $j_1 \neq j_2$, such that $xa_1 \in A_{j_1}$, $xa_2 \in A_{j_2}$. Otherwise, there is $j \geqslant 3$ such that $xa_1, xa_2 \in A_j$ and

$a_1^{-1} \cdot a_2 \in A_1 \cap A_j = H$, which contradicts (21). Hence
$xa_1H \subset A_{j_1}$, $xa_2H \subset A_{j_2}$, $j_1 \neq j_2$.

The above reasoning shows that there are at most $n-2$ different cosets xH, $x \notin H$, in the group A_1, and with the trivial coset H there are exactly $n-1$ such different cosets (bearing in mind (H_0)). Summing up the index $[A_1 : H] = n-1$ (and similarly $[A_i : H] = n-1$, $i = 2, \ldots, n$) and the index $[G : H] = n(n-2) + 1 = (n-1)^2$. This implies (i) and (ii). Finally, the equation (iii) is an immediate consequence of (i) and (ii). Theorem 1 is then proved.

THEOREM 2. If $n \geqslant 3$ and the following condition is satisfied:

(H_1) for any l, $2 \leqslant l \leqslant n-1$, there does not exist an l-orientable subgroup of G,

then G is not an irredundant union of n proper subgroups.

P r o o f. Suppose that G is an irredundant union of its proper subgroups $A_1, \ldots, A_n$. The condition (H_1) implies (H_0) in Theorem 1. Hence, $[A_i : H] = n - 1$ from Theorem 1, and A_i is $(n-1)$-orientable which contradicts (H_1).

Let us compare (H_1) with the following condition $(n \geqslant 3)$:

(H_2) for any l, $1 \leqslant l \leqslant n-1$, there exists a l-root of an arbitrary element of G.

LEMMA 1. (H_1) implies (H_2). The conditions are not equivalent.

P r o o f. Let $a \in G$, $a \neq 1$. It is sufficient to show (H_2) for any prime integer l, $2 \leqslant l \leqslant n-1$. By G_a we denote the cyclic subgroup generated by an element a. Group G_a is finite. Otherwise G_a is isomorphic with the additive group of integers and is l-orientable for $l \geqslant 2$. Let m be the rank of G_a. Trivially, G_a is m-orientable and, in view of (H_1), $m \geqslant n > l$. If the prime l divides m, then by Sylow's theorem there exists a subgroup with rank l (and l-orientable) in G_a. Hence l and m are coprime integers, a^l is a generator of G_a, there is s such that $a = (a^l)^s$ and a^s is a l-root of a. Thus (H_1) implies (H_2). The converse is no longer true. As an example we have additive group of rational numbers: clearly l-roots exist and the subgroup of integers is l-orientable for any $l \geqslant 2$.

Suppose that G is finite of rank N and $n \geqslant 3$. Let us consider

(H_3) $(n-1)!$ and N are coprime integers.

It was proved in paper [2] that (H_2) and (H_3) are equivalent.

LEMMA 2. If G is a finite group of rank N then (H_1) and (H_3) are equivalent.

P r o o f. Suppose that (H_1) is satisfied. $(n-1)!$ and N are coprime. In fact, if there is a prime l, $2 \leqslant l \leqslant n-1$, such that l divides N then, due to Sylow's theorem, there is a subgroup of rank l (and l-orientable) in G which contradicts (H_1).

Next, suppose (H_3) and $2 \leqslant l \leqslant n-1$. If there are subgroups X, Y such that $X < Y < G$, $[Y : X] = l$ then $m = \text{rank } Y$ divides N and l divides m. Hence l divides N. But $2 \leqslant l \leqslant n-1$ and l divides $(n-1)!$, which is a contradiction. This ends the proof.

The authors of [1] proved the following theorems.

THEOREM 3. Suppose that kth roots can be taken in the group G for every positive integer k less than a certain n. Then G is not the irredundant union of n proper subgroups.

THEOREM 4. Let G be a finite group of order N, p the smallest prime dividing N, suppose that G is the union of exactly $p+1$ proper subgroups S_i; then at least one of the S's, say S_j has index p. If moreover, this S_j is normal, then all the S_i have index p and p^2 divides N.

Observe that, due to Lemma 1, Theorem 2 is a consequence of Theorem 3. Next, if we put $n = p+1$ in Theorem 4 then the assumption "p is the smallest prime dividing N" gives immediately that $(p-1)!$ and N are coprime. It follows, in view if Lemma 2, that l-orientable subgroups in G do not exist, $2 \leqslant l \leqslant p-1$, and for $n = p+1$ (H_1) is satisfied. Then Theorem 1 implies Theorem 4 in the stronger form as each S_j has index $p = n-1$ even without the assumption that some S_j is normal.

R e f e r e n c e s

[1] Haber S., A. Rosenfeld, Groups as unions of proper subgroups, Amer.Math. Monthly 66 (1959), 491-494.

[2] Scorza G., I gruppi che possono pensarsi come somma di tre loro sottogruppi, Boll. Unione mat. ital. 5 (1926), 216-218.