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On the generalized convex functions with respect to the three-parameters family of functions

1. INTRODUCTION

M.C. Peixoto gave in his paper [8] the definition of generalized convex function with respect to a three-parameters family of functions and proved that a function $\psi \in C^3(a,b)$ is a generalized convex function with respect to the family of solutions of the equation $y''' = f(x,y,y',y'')$, if and only if it satisfies the inequality $\psi'''(x) \geqslant f(x,\psi(x),\psi'(x),\psi''(x))$.

In this paper we shall give some equivalent conditions for a two times differentiable function to be a generalized convex function with respect to a three-parameters family of functions of class C^2.

Similar results for the generalized convexity with respect to a two-parameters family of functions may be found in the papers [1], [2], [3], [5], [6] and [9].

2. SOME EQUIVALENT CONDITIONS OF CONVEXITY

We shall assume the following hypothesis:

H. Let F be a three-parameter family of functions defined in an open interval $I = (a,b)$ and satisfying the following conditions:

(i) every function $\varphi \in F$ belongs to $C^2(I)$;

(ii) for every $x_0 \in I$ and for every $y_0, y_1, y_2 \in R$ there is a unique member φ of the family F such that

$$\varphi^{(i)}(x_0) = y_i, \quad i = 0,1,2;$$

(iii) for every three points (x_1,y_1), (x_2,y_2), (x_3,y_3), $a < x_1 < x_2 < x_3 < b$ there is a unique member φ of the family F such that

$$\varphi(x_i) = y_i, \quad i = 1,2,3.$$

DEFINITION. Let hypothesis H be fulfilled. The function ψ is said to be strictly convex with respect to the family F iff for all x_1, x_2, x_3 such that $a < x_1 < x_2 < x_3 < b$ the inequalities

$$\psi(x) > \varphi(x) \quad \text{for} \quad x \in (x_1, x_2)$$

$$\psi(x) < \varphi(x) \quad \text{for} \quad x \in (x_2, x_3)$$

hold, where $\varphi \in F$ is determined by the conditions

$$\varphi(x_i) = \psi(x_i) \qquad i = 1,2,3. \tag{1}$$

We say that ψ is convex with respect to F iff it satisfies weak inequalities $(\leqslant)$ instead of the strong ones.

Similarly we define a strictly concave (concave) function with respect to F by reversing the inequalities.

R e m a r k 1. (see [8]) If a function ψ is strictly convex with respect to F then for all x_1, x_2, x_3 such that $a < x_1 < x_2 < x_3 < b$ the inequalities

$$\psi(x) < \varphi(x) \qquad \text{for } x \in (a, x_1)$$

$$\psi(x) > \varphi(x) \qquad \text{for } x \in (x_3, b)$$

hold, where $\varphi \in F$ is determined by the conditions (1).

THEOREM. Let hypothesis H be fulfilled and let a function $\psi : I \mapsto R$ be twice differentiable in I. Under these assumptions the following statements are equivalent:

(A) the function ψ is strictly convex with respect to the family F;

(B) for every x_1, x_2, such that $a < x_1 < x_2 < b$ the inequalities

$$\psi(x) < \varphi(x) \qquad \text{for } x \in (a, x_1),$$

$$\psi(x) > \varphi(x) \qquad \text{for } x \in (x_1, b) \setminus \{x_2\}$$

hold, where $\varphi \in F$ is determined by the conditions

(2) $\varphi(x_1) = \psi(x_1), \quad \varphi(x_2) = \psi(x_2), \quad \varphi'(x_2) = \psi'(x_2);$

(C) for every $x_0 \in (a,b)$ the inequalities

$$\psi(x) < \varphi(x) \qquad \text{for } x \in (a, x_0)$$

$$\psi(x) > \varphi(x) \qquad \text{for } x \in (x_0, b)$$

hold, where $\varphi \in F$ is determined by the conditions

(3) $\varphi^{(i)}(x_0) = \psi^{(i)}(x_0) \qquad i = 0,1,2;$

(D) for every x_1, x_2, such that $a < x_1 < x_2 < b$ the inequalities

$$\psi(x) < \varphi(x) \qquad \text{for } x \in (a, x_2) \setminus \{x_1\},$$

$$\psi(x) > \varphi(x) \quad \text{for} \quad x \in (x_2, b)$$

hold, where $\varphi \in F$ is determined by the conditions

(4) $\varphi(x_1) = \psi(x_1), \quad \varphi'(x_1) = \psi'(x_1), \quad \varphi(x_2) = \psi(x_2).$

First we shall give some lemmas needed in the proof of the theorem.

LEMMA 1. (see [4], c.f. [7]) Let $F \subset C^2(I)$, where I is an open interval, be a three-parameter family of functions and the initial value problem (see H(ii)), as well as the boundary value problem (see H(iii)) are uniquely sovable in F. Then

(M1) for every $x_1, x_2, y_0, y_1, y_2 \in R$, such that $a < x_1 < x_2 < b$ there is a unique member φ of F such that

$$\varphi(x_1) = y_1, \quad \varphi'(x_1) = y_0, \quad \varphi(x_2) = y_2;$$

(M2) for every $x_1, x_2, y_0, y_1, y_2 \in R$, such that $a < x_1 < x_2 < b$ there is a unique member φ of F such that

$$\varphi(x_1) = y_1, \quad \varphi(x_2) = y_2, \quad \varphi'(x_2) = y_0$$

(i.e. the mixed problems are also uniquely solvable in F).

The uniqueness of the functions φ determined by the conditions (2) and (4) follows from the above Lemma.

LEMMA 2. (see [8]) Let $a < x_1 < x_2 < b$ and φ_1 and φ_2 be two elements of the family F such that $\varphi_1 \neq \varphi_2$ and $\varphi_1(x_i) = \varphi_2(x_i)$ $i = 1,2.$

Then $\varphi_1(x) > \varphi_2(x)$ for $x \in (x_1, x_2)$ and $\varphi_1(x) < \varphi_2(x)$ for $x \in (a, x_1) \cup (x_2, b)$ (or the reverse inequalities hold).

LEMMA 3. Let functions f, g be defined and two times differentiable in a neighbourhood of a point x_0 and let $f(x_0) = g(x_0)$. Then

a) if $f(x) \geqslant g(x)$ for $x \neq x_0$, then $f'(x_0) = g'(x_0)$;

b) if $f(x) \geqslant g(x)$ for $x > x_0$ (or for $x < x_0$) and $f'(x_0) = g'(x_0)$, then $f''(x_0) \geqslant g''(x_0)$;

c) if $f(x) \geqslant g(x) \geqslant 0$ for $x > x_0$ (or for $x < x_0$) and $f(x_0) = f'(x_0) = f''(x_0) = 0$, then $g'(x_0) = g''(x_0) = 0$.

P r o o f. Part a) is obvious.

We shall prove b) for $x > x_0$. For $x < x_0$ the proof is similar. Let us consider the function $h(x) = f(x) - g(x)$. From the assumptions we have $h(x_0) = 0$, $h'(x_0) = 0$ and $h(x) \geqslant 0$ for $x > x_0$. Hence it follows that there exists a sequence $\{x_n\}$ such that $x_n \mapsto x_0^+$ and $h'(x_n) \geqslant 0$. Therefore $h''(x_0) \geqslant 0$, thus $f''(x_0) \geqslant g''(x_0)$.

Part c). From a) we get $g'(x_0) = 0$, from b) $f''(x_0) \geqslant g''(x_0)$ and $g''(x_0) \geqslant 0$. Hence and from the equality $f''(x_0) = 0$ we have $g''(x_0) = 0$.

LEMMA 4. Let $x_0 \in I$ and let $\varphi_1, \varphi_2 \in F$ be such that $\varphi_1 \neq \varphi_2$, $\varphi_1(x_0) = \varphi_2(x_0)$ and $\varphi_1'(x_0) = \varphi_2'(x_0)$. Then $\varphi_1(x) > \varphi_2(x)$ for $x \in I \setminus \{x_0\}$ (or the reverse inequality holds).

P r o o f. It follows from Lemma 1 that $\varphi_1(x) \neq \varphi_2(x)$ for $x \neq x_0$. Suppose that the statement does not hold. Hence either $\varphi_1(x) > \varphi_2(x)$ for $x \in (a, x_0)$ and $\varphi_1(x) < \varphi_2(x)$ for $x \in (x_0, b)$ or $\varphi_1(x) < \varphi_2(x)$ for

$x \in (a,x_0)$ and $\varphi_1(x) > \varphi_2(x)$ for $x \in (x_0,b)$. Let us consider the first case, the other is similar. Applying Lemma 3b) for $f = \varphi_1$, $g = \varphi_2$ and for $f = \varphi_2$, $g = \varphi_1$ we have $\varphi_1''(x_0) \geq \varphi_2''(x_0)$ and $\varphi_2''(x_0) \geq \varphi_1''(x_0)$, respectively. Hence $\varphi_1''(x_0) = \varphi_2''(x_0)$. From H(ii) we get $\varphi_1 = \varphi_2$, a contradiction.

LEMMA 5. Let the assumptions and condition (B) of the Theorem be fulfilled. Then for every x_1, x_2 such that $a < x_1 < x_2 < b$ the inequality

$$\psi(x) < \bar{\varphi}(x) \quad \text{for} \quad x \in (x_1,x_2)$$

holds, where $\bar{\varphi} \in F$ is determined by conditions (4).

P r o o f. Let $a < x_1 < x_2 < b$ and let $\bar{\varphi}$ be determined by conditions (4). Let us assume that the statement is false. Then two cases are possible

a) $\psi(x) \leq \bar{\varphi}(x)$ for all $x \in (x_1,x_2)$ and
$\psi(c) = \bar{\varphi}(c)$ for a $c \in (x_1,x_2)$;

b) there exists a $c \in (x_1,x_2)$ such that $\psi(c) > \bar{\varphi}(c)$.

Case a). From Lemma 3a) we have $\psi'(c) = \bar{\varphi}'(c)$. Let us observe that $\bar{\varphi}$ is just the function determined by (2) with x_1 and $x_2 = c$. By (B) we get $\psi(x) > \bar{\varphi}(x)$ for $x \in (x_1,c)$, contrary to a).

Case b). Let us consider the function φ_1 determined by (2) with $\bar{x}$, c in place of those x_1, x_2, where $\bar{x} = \sup\{x \in [x_1,c) : \psi(x) = \bar{\varphi}(x)\}$. By this definition $\bar{x} \in [x_1,c)$ and

(5) $\psi(\bar{x}) = \bar{\varphi}(\bar{x}) = \varphi_1(\bar{x})$, $\quad \psi(x) > \bar{\varphi}(x)$ for $x \in (\bar{x},c]$,

and from (B) we have

(6) $\psi(x) > \varphi_1(x)$ for $x \in (\bar{x},b)\setminus\{c\}$.

By (6) we have $\bar{\varphi}(x_2) = \psi(x_2) > \varphi_1(x_2)$, but $\bar{\varphi}(c) < \psi(c) = \varphi_1(c)$, therefore there exists a point $d \in (c,x_2)$ such that $\bar{\varphi}(d) = \varphi_1(d)$. The values of functions $\bar{\varphi}$, φ_1 are equal at the points $\bar{x}$ and d and those functions are not identically equal because $\varphi_1(c) = \psi(c) > > \bar{\varphi}(c)$. By Lemma 2 we have $\varphi_1(x) > \bar{\varphi}(x)$ for $x \in (\bar{x},d)$ and from (6) we get

(7) $\psi(x) > \varphi_1(x) > \bar{\varphi}(x)$ for $x \in (\bar{x},c)$.

If $\bar{x} = x_1$, then $\psi(\bar{x}) = \bar{\varphi}(\bar{x})$, $\psi'(\bar{x}) = \bar{\varphi}'(\bar{x})$ and from (7) we get $\varphi_1'(\bar{x}) = \bar{\varphi}'(\bar{x})$. Thus the mixed problem

$$\varphi(\bar{x}) = \psi(\bar{x}), \quad \varphi'(\bar{x}) = \psi'(\bar{x}), \quad \varphi(d) = \bar{\varphi}(d)$$

has two different solutions $\bar{\varphi}$, φ_1, what is impossible.

Let $\bar{x} > x_1$. From (7) $\varphi_1'(\bar{x}) \geqslant \bar{\varphi}'(\bar{x})$. If $\varphi_1'(\bar{x}) = \bar{\varphi}'(\bar{x})$ then we proceed as in the case $\bar{x} = x_1$. Let $\varphi_1'(\bar{x}) > \bar{\varphi}'(\bar{x})$. Hence and from the equality $\varphi_1(\bar{x}) = \bar{\varphi}(\bar{x})$ we get that there is an l such that $\varphi_1(x) < \bar{\varphi}(x)$ for $x \in (l,\bar{x})$. From (B) we have $\varphi_1(x) > \psi(x)$ for $x \in (a,\bar{x})$, in particular $\varphi_1(x_1) > \psi(x_1) = \bar{\varphi}(x_1)$. From the continuity of φ_1 and $\bar{\varphi}$ there is a $p \in (x_1,\bar{x})$ such that $\varphi_1(p) = \bar{\varphi}(p)$. Therefore $\varphi_1 = \bar{\varphi}$ as they are members of F passing through the points $(p, \varphi_1(p))$, $(\bar{x}, \varphi_1(\bar{x}))$, $(d, \varphi_1(d))$.

P r o o f o f T h e o r e m. (A) $\Longrightarrow$ (B).

Let $a < x_1 < x_2 < b$ and let $\bar{\varphi}$ be determined by (2). We are going to prove the inequality

(8) $\qquad \psi(x) > \bar{\varphi}(x) \quad$ for $x \in (x_1, x_2)$.

Let us assume that inequality (8) does not hold. Then two cases are possible

a) $\psi(x) \geqslant \bar{\varphi}(x)$ for all $x \in (x_1, x_2)$ and

$\psi(c) = \bar{\varphi}(c)$ for a $c \in (x_1, x_2)$;

b) there exists a $c \in (x_1, x_2)$ such that $\psi(c) < \bar{\varphi}(c)$.

Case a). Let us observe that $\bar{\varphi}$ is just the function determined by conditions (1) with x_1, c, x_2 in place of those x_1, x_2, x_3, therefore by (A) we have $\psi(x) < \bar{\varphi}(x)$ for $x \in (c, x_2)$, contrary to a).

Case b). Let $\bar{x} = \inf\{x \in (c, x_2]: \psi(x) = \bar{\varphi}(x)\}$. Hence $\bar{x} \in (c, x_2]$, $\psi(\bar{x}) = \bar{\varphi}(\bar{x})$ and $\psi(x) < \bar{\varphi}(x)$ for $x \in [c, \bar{x})$.

If $\bar{x} < x_2$, then since ψ is strictly convex with respect to F, we have $\psi(x) > \bar{\varphi}(x)$ for $x \in (x_1, \bar{x})$ and $\psi(x) < \bar{\varphi}(x)$ for $x \in (\bar{x}, x_2)$, because $\bar{\varphi}(x_1) = \psi(x_1)$, $\bar{\varphi}(\bar{x}) = \psi(\bar{x})$ and $\bar{\varphi}(x_2) = \psi(x_2)$. Hence, in particular, $\psi(x) > \bar{\varphi}(x)$ for $x \in [c, \bar{x})$, a contradiction.

Let $\bar{x} = x_2$ and let us consider the function φ_1 determined by (1) with the points x_1, c, $\bar{x}$. From (A) we get the inequality $\psi(x) < \varphi_1(x)$ for $x \in (c, \bar{x})$ and just as in the proof of Lemma 4 (inequality (7)) we have $\psi(x) < \varphi_1(x) < \bar{\varphi}(x)$ for $x \in (c, \bar{x})$. But $\psi(\bar{x}) = \bar{\varphi}(\bar{x})$ and $\psi'(\bar{x}) = \bar{\varphi}'(\bar{x})$, therefore $\bar{\varphi}'(\bar{x}) = \varphi_1'(\bar{x})$. In this way we obtain two different functions $\bar{\varphi}, \varphi_1 \in F$ such that $\bar{\varphi}(x_1) = \varphi_1(x_1)$, $\bar{\varphi}(\bar{x}) = \varphi_1(\bar{x})$ and $\bar{\varphi}'(\bar{x}) = \varphi_1'(\bar{x})$, what by Lemma 1 is impossible. This concludes the proof of inequality (8).

The inequalities

$$\psi(x) < \bar{\varphi}(x) \qquad \text{for} \quad x \in (a, x_1)$$

$$\psi(x) > \bar{\varphi}(x) \qquad \text{for} \quad x \in (x_2, b)$$

follows from (8), (A) and from Remark 1.

(B) $\Longrightarrow$ (C).

Let $x_0 \in I$ and let $\bar{\varphi}$ be determined by (3). First we are going to prove that

(9) $$\psi(x) < \bar{\varphi}(x) \qquad \text{for} \quad x \in (a, x_0).$$

Assume the contrary. Then either

a) $\psi(x) \leqslant \bar{\varphi}(x)$ for $x \in (a, x_0)$ and $\psi(c) = \bar{\varphi}(c)$ for a $c \in (a, x_0)$,

or

b) there exists a $c \in (a, x_0)$ such that $\psi(c) > \bar{\varphi}(c)$.

To disprove a) let us observe (Lemma 1) that $\bar{\varphi}$ is just the function determined by conditions (2) with $x_1 = c$ and $x_2 = x_0$. By (B) we have $\psi(x) > \bar{\varphi}(x)$ for $x \in (c, x_0)$, a contradiction.

Case b). Let φ_1 be a function determined by (2) with $x_1 = c$ and $x_2 = x_0$. From (B) we have

(10) $$\psi(x) > \varphi_1(x) \quad \text{for} \quad x \in (c, x_0).$$

It follows from the definition of φ_1 that $\varphi_1(x_0) = \bar{\varphi}(x_0)$, $\varphi_1'(x_0) = \bar{\varphi}'(x_0)$ and $\varphi_1(c) = \psi(c) > \bar{\varphi}(c)$. Hence and from Lemma 4 we obtain the inequality

(11) $$\varphi_1(x) > \bar{\varphi}(x) \qquad \text{for} \quad x \in (a, x_0).$$

By (10) and (11) we get

$$\psi(x) > \varphi_1(x) > \bar{\varphi}(x) \qquad \text{for} \quad x \in (c, x_0).$$

From the definition of $\bar{\varphi}$ and from Lemma 3c) used for the functions $f(x) = \psi(x) - \bar{\varphi}(x)$ and $g(x) = \varphi_1(x) - \bar{\varphi}(x)$ we get $\varphi_1''(x_0) = \bar{\varphi}''(x_0)$, what contradicts H(ii).

Now we are going to prove that

$$\psi(x) > \bar{\varphi}(x) \qquad \text{for} \quad x \in (x_0, b).$$

Assume the contrary. Then two cases are possible

a) $\psi(x) \geqslant \bar{\varphi}(x)$ for $x \in (x_0, b)$ and $\psi(c) = \bar{\varphi}(c)$ for a $c \in (x_0, b)$;

b) there exists a $c \in (x_0, b)$ such that $\psi(c) < \bar{\varphi}(c)$.

Case a). Let us observe (Lemma 1) that $\bar{\varphi}$ is just the function determined by (4) with $x_1 = x_0$ and $x_2 = c$. From Lemma 5 we get $\psi(x) < \bar{\varphi}(x)$ for $x \in (x_0, c)$, a contradiction.

Case b). Let φ_2 be a function determined by (4) with $x_1 = x_0$ and $x_2 = c$. By Lemma 5 we have $\psi(x) < \varphi_2(x)$ for $x \in (x_0, c)$. As in the proof of (9) (Case b) we obtain $\psi(x) < \varphi_2(x) < \bar{\varphi}(x)$ for $x \in (x_0, c)$ and $\varphi_2''(x_0) = \bar{\varphi}''(x_0)$. In this way we get two different functions $\varphi_2, \bar{\varphi} \in F$ such that $\varphi_2^{(i)}(x_0) = \bar{\varphi}^{(i)}(x_0)$ $i = 0,1,2$, what by H(ii) is impossible.

$$(C) \Longrightarrow (D).$$

Let $a < x_1 < x_2 < b$ and let $\bar{\varphi}$ be determined by (4). First we shall show that

(12) $$\psi(x) < \bar{\varphi}(x) \qquad \text{for} \quad x \in (x_1, x_2).$$

Let us assume that inequality (12) does not hold. Then either

a) $\psi(x) \leqslant \bar{\varphi}(x)$ for $x \in (x_1, x_2)$ and $\psi(c) = \bar{\varphi}(c)$ for a $c \in (x_1, x_2)$

or

b) there exists a $c \in (x_1, x_2)$ such that $\psi(c) > \bar{\varphi}(c)$.

Case a). By Lemma 3a) we get $\psi'(c) = \bar{\varphi}'(c)$ and by Lemma 3b) $\psi''(c) \leqslant \bar{\varphi}''(c)$. From (C) we obtain that the case $\psi''(c) = \bar{\varphi}''(c)$ cannot occur, therefore $\psi''(c) < \bar{\varphi}''(c)$. Now let us consider the function φ_1 determined by (3) with $x_0 = c$, $\varphi_1 \neq \bar{\varphi}$ because $\varphi_1''(c) = \psi''(c) < \bar{\varphi}''(c)$. From (C) we have

(13) $$\varphi_1(x) > \psi(x) \quad \text{for } x \in (a,c)$$

and

(14) $$\varphi_1(x) < \psi(x) \quad \text{for } x \in (c,b).$$

From (14) we get $\varphi_1(x) < \bar{\varphi}(x)$ for $x \in (c, x_2)$ and from the definition of φ_1 and from Lemma 4 it follows that $\varphi_1(x) < \bar{\varphi}(x)$ for $x \in I \setminus \{c\}$. Hence, in particular, $\varphi_1(x_1) < \bar{\varphi}(x_1)$, but from (13) we have $\varphi_1(x_1) > \psi(x_1) = \bar{\varphi}(x_1)$, a contradiction.

Case b). Put $\bar{x} = \sup\{x \in [x_1, c) : \psi(x) = \bar{\varphi}(x)\}$. Thus $\bar{x} \in [x_1, c)$ and $\psi(x) > \bar{\varphi}(x)$ for $x \in (\bar{x}, c]$ and $\psi(\bar{x}) = \bar{\varphi}(\bar{x})$. Hence $\psi'(\bar{x}) \geqslant \bar{\varphi}'(\bar{x})$. Let $\varphi_2 \in F$ be determined by (3) with $x_0 = \bar{x}$.

If $\psi'(\bar{x}) > \bar{\varphi}'(\bar{x})$, then $\bar{x} > x_1$ and we proceed as in the proof of Lemma 5 (case b) $\bar{x} > x_1$) and we get the contradiction with the condition H(iii).

Let $\psi'(\bar{x}) = \bar{\varphi}'(\bar{x})$, then by Lemma 3b) we get $\psi''(\bar{x}) \geq \bar{\varphi}''(\bar{x})$ and it follows from (C) that $\psi''(\bar{x}) \neq \bar{\varphi}''(\bar{x})$, i.e.

(15) $$\psi''(\bar{x}) > \bar{\varphi}''(\bar{x}).$$

By (C) we have $\varphi_2(x) < \psi(x)$ for $x \in (\bar{x}, b)$ and in particular

(16) $$\varphi_2(x_2) < \psi(x_2) = \bar{\varphi}(x_2).$$

Lemma 4 and (16) yield $\varphi_2(x) < \bar{\varphi}(x)$ for $x \in I \setminus \{\bar{x}\}$. Thus

(17) $$\varphi''_2(\bar{x}) \leq \bar{\varphi}''(\bar{x}).$$

From (15) and from the definition of φ_2 we have $\varphi''_2(\bar{x}) > \bar{\varphi}''(\bar{x})$, what contradicts (17).

Now we shall show that

$$\psi(x) < \bar{\varphi}(x) \qquad x \in (a, x_1),$$

where $\bar{\varphi}$ is determined by (4).

Let $\varphi_3 \in F$ be determined by (3) with $x_0 = x_1$. From (C) we get $\psi(x) > \varphi_3(x)$ for $x \in (x_1, b)$ and form (12) $\varphi_3(x) < \bar{\varphi}(x)$ for $x \in (x_1, x_2)$. Hence and by Lemma 4 we obtain $\varphi_3(x) < \bar{\varphi}(x)$ for $x \in (a, x_1)$, from (C) $\psi(x) < \varphi_3(x)$ for $x \in (a, x_1)$. Thus $\psi(x) < \bar{\varphi}(x)$ for $x \in (a, x_1)$.

The inequality

$$\psi(x) > \bar{\varphi}(x) \qquad \text{for } x \in (x_2, b)$$

follows from (12), (A) and from lemmas 3 and 4.

$$(D) \Longrightarrow (A).$$

Let $a < x_1 < x_2 < x_3 < b$ and let $\bar{\varphi}$ be determined by (1). We shall prove that

(18) $$\psi(x) > \bar{\varphi}(x) \quad \text{for} \quad x \in (x_1, x_2).$$

Again, let us assume that inequality (18) does not hold. We shall consider two cases

a) $\psi(x) \geqslant \bar{\varphi}(x)$ for $x \in (x_1, x_2)$ and $\psi(c) = \bar{\varphi}(c)$ for a $c \in (x_1, x_2)$;

b) there exists a $c \in (x_1, x_2)$ such that $\psi(c) < \bar{\varphi}(c)$.

Case a). By Lemma 3a) we have $\psi'(c) = \bar{\varphi}'(c)$. Thus $\bar{\varphi}$ satisfies (4) with $x_1 = c$ and x_2, so that, by (D), we get $\psi(x) < \bar{\varphi}(x)$ for $x \in (c, x_2)$, contrary to the inequality in a).

Case b). Let $\bar{x} = \sup\{x \in [x_1, c): \psi(x) = \bar{\varphi}(x)\}$. By this definition $\bar{x} \in [x_1, c)$, $\psi(\bar{x}) = \bar{\varphi}(\bar{x})$ and $\psi(x) < \bar{\varphi}(x)$ for $x \in (\bar{x}, c)$, whence $\psi'(\bar{x}) \leqslant \bar{\varphi}'(\bar{x})$. If we had $\psi'(\bar{x}) = \bar{\varphi}'(\bar{x})$, then $\bar{\varphi}$ would satisfy (4) with $x_1 = \bar{x}$, $x_2 = x_3$ and the first inequality of (D) would contradict (1). Therefore

(19) $$\psi'(\bar{x}) < \bar{\varphi}'(\bar{x}).$$

Now let us consider the function φ_1 determined by (4) with $x_1 = \bar{x}$ and x_2. Since $\varphi_1'(\bar{x}) = \psi'(\bar{x})$ and $\varphi_1(\bar{x}) = \bar{\varphi}(\bar{x})$, we see from (19) and from the continuity of φ_1 and $\bar{\varphi}$ that $\varphi_1(x) < \bar{\varphi}(x)$ for x from a right neighbourhood of $\bar{x}$. From (D) we have the inequalities

(20) $$\psi(x) < \varphi_1(x) \quad \text{for} \quad x \in (\bar{x}, x_2)$$

and $\psi(x) > \varphi_1(x)$ for $x \in (x_2, b)$. The latter yields $\psi(x_3) > \varphi_1(x_3)$. Hence and from the equality $\psi(x_3) = \bar{\varphi}(x_3)$ we have

(21) $$\bar{\varphi}(x_3) > \varphi_1(x_3).$$

Applying Lemma 2 for φ_1 and $\bar{\varphi}$ with x_1 replaced by $\bar{x}$, we get $\varphi_1(x) < \bar{\varphi}(x)$ for $x \in (\bar{x}, x_2)$ and $\varphi_1(x) > \bar{\varphi}(x)$ for $x \in (a, \bar{x}) \cup (x_2, b)$, in particular $\varphi_1(x_3) > \bar{\varphi}(x_3)$, what contradicts (21).

The inequality

$$\psi(x) < \bar{\varphi}(x) \qquad \text{for} \quad x \in (x_2, x_3)$$

follows from (18) and from (D).

R e m a r k 2. By a suitable change of inequalities that appear in the Theorem we get the conditions equivalent to the fact that ψ is convex, or concave or strictly concave with respect to F.

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