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On the rectangle proportion

At the 22^{nd} Symposium on functional equations (Oberwolfach, 1984), C. Alsina stated the problem ([1]) of finding all the functions f satisfying the following conditions:

(1) $f\colon (0;+\infty) \times (0;+\infty) = (0;+\infty)^2 \longrightarrow [1;+\infty)$,

(2) $f(x,x) = 1$,

(3) $f(x,y) = f(y,x)$,

(4) $f(x,y) = f(\frac{y^2}{x},y)$.

At the same Symposium the first of the authors of this paper presented (with proof) the general solution of the problem (1) - (4), and showed the form of its general continuous solution (without proof), cf. [5]. In the present paper the missing proof is supplied. Another description has been given by W.Benz in [3].

The so-called rectangle proportion $p(a,b)$ of the rectangle of sides a and b, is defined in the following way:

$$p(a,b) = \frac{\max(a,b)}{\min(a,b)}.$$

The function $f = p$ is a solution to (1) - (4), but it has also the following properties ([2]):

(5) f is continuous on the set $(0;+\infty)^2$,

(6) $f(x,y) = 1 \Rightarrow x = y$,

(7) $f(\lambda x, \lambda y) = f(x,y)$ for $\lambda > 0$,

(8) $x \leqslant \min(y,z) \Rightarrow f(x,y) + f(x,z) = f(x,y+z)$,

(9) $f(x,y) = f(x,\frac{x^2}{y})$,

(10) $f(x,y) = f(y,\frac{y^2}{x})$,

(11) $f(x^2,y^2) = [f(x,y)]^2$.

In the second part of the present paper the above properties are compared to each other and the rectangle proportion is characterized by means of some of them.

I. Let $E = \{(x,y);\ 0 < x < 1 \text{ and } y \geqslant 1\}$ and let $g: E \to [1;+\infty)$ be an arbitrary function.

As it has been proved in [5] the function f defined as follows

(12) $$f(x,y) = \begin{cases} g\left[(x,y)\left(\frac{y}{x}\right)^n\right] & \text{for } 0 < x < y \text{ and an integer } n = n(x,y) \text{, such that } (x,y)\left(\frac{y}{x}\right)^n \in E, \\ 1 & \text{for } x = y \end{cases}$$

and besides $\qquad f(x,y) := f(y,x)$

are well defined and are the only function for which the conditions (1) - (4) hold.

Now we are going to prove the following

THEOREM 1. The function f, defined above in (12), is continuous if and only if the function g is continuous in the set E and

$$\lim_{E \ni (u,v) \to (1;\frac{1}{x})} g(u,v) = g(x,1) \quad \text{for } 0<x<1, \tag{13}$$

and

$$\lim_{E \ni (u,v) \to (1,1)} g(u,v) = 1. \tag{14}$$

P r o o f. A) Let us assume that the function f of the form (12) is continuous. Since $f \backslash E = g$, the function g is continuous on E. We also have, for $0<x<1$, $(u,v) \in E$,

$$\lim_{(u,v) \to (1;\frac{1}{x})} f(u,v) = f(1;\tfrac{1}{x}).$$

Since $(1;\frac{1}{x})(\frac{1}{x})^{-1} \in E$, from the definition of f we obtain:

$$f(1;\tfrac{1}{x}) = g\left[(1;\tfrac{1}{x})(\tfrac{1}{x})^{-1}\right] = g(x;1).$$

Hence (13) is satisfied.

Moreover, again by the continuity of f, we have

$$\lim_{(u,v) \to (1,1)} f(u,v) = 1.$$

But $f(u,v) = g(u,v)$ whenever $(u,v) \in E$, which means that condition (14) is satisfied.

B) Let us assume that the function g is continuous in E and it has the properties (13) and (14). Because of the symmetry of the function f we can restrict the proof of its continuity to the points (x,y) such that $y \geqslant x$. It is evident that the set of these points is the union over $\mathbb{Z}$ of the disjoint non-empty subsets

$$Z_n := \left\{(x,y);\ (x,y)\left(\frac{y}{x}\right)^n \in E\right\},\quad n \in \mathbb{Z}.$$

Let us also notice that

$$(x,y)\left(\frac{y}{x}\right)^n \in E \Leftrightarrow x\left(\frac{y}{x}\right)^n < 1 \quad \text{and} \quad y\left(\frac{y}{x}\right)^n \geqslant 1.$$

Hence

$$Z_n := \left\{(x,y);\ x \in (0;1) \quad \text{and} \quad x^{1-\frac{1}{n+1}} \leqslant y < x^{1-\frac{1}{n}}\right\},$$

for $n \in \mathbb{N}$, whereas for $n < -1$

$$Z_n := \left\{(x,y);\ x \in (1;+\infty) \quad \text{and} \quad x^{1-\frac{1}{n}} < y \leqslant x^{1-\frac{1}{n+1}}\right\}.$$

Obviously $Z_0 = E$, and $Z_{-1} = \left\{(x,y);\ x \geqslant 1 \text{ and } y > x^2\right\}$.

Three kinds of points can be distinguished here:

(I) (x_0,y_0) is an interior point of a set Z_n,

(II) $(x_0,y_0) \in Z_n$ and is not an interior point of Z_n,

(III) $x_0 = y_0$.

Case (I). There exists a neighbourhood V of (x_0,y_0) contained in Z_n. Therefore for all the points (x,y) of V, $n = n(x,y)$ in the formula (12) is constant. From the definition of the function f, we obtain for $(x,y) \in V$,

$$f(x,y) = g\left[(x,y)\left(\frac{y}{x}\right)^n\right]$$

and

$$(x,y)\left(\frac{y}{x}\right)^n \in E.$$

Since g is continuous, the function $(x,y) \mapsto (x,y)\left(\frac{y}{x}\right)^n$ is continuous at (x_0,y_0) and $n = n(x,y)$ is constant on V, therefore the composed function f is also continuous at (x_0,y_0).

Case (II). Now (x_0,y_0) lies on the graph of the power function $y = x^{1 - \frac{1}{n+1}}$ which means that

$$(x_0,y_0) = (x_0, x_0^{1 - \frac{1}{n+1}}). \tag{15}$$

If (x,y) tends to (x_0,y_0), the investigations can be restricted to the cases

a) $(x,y) \in Z_n$,

b) $(x,y) \in Z_{n+1}$.

a) If $(x,y) \in Z_n$, then $(x,y)\left(\frac{y}{x}\right)^n \in E$,

and $\lim\limits_{(x,y)\to(x_0,y_0)} (x,y)\left(\frac{y}{x}\right)^n = (x_0,y_0)\left(\frac{y_0}{x_0}\right)^n \in E$

as $(x_0,y_0) \in Z_n$.

Using the definition of f and the continuity of g we obtain:

$$\lim_{Z_n \ni (x,y)\to(x_0,y_0)} f(x,y) = \lim_{Z_n \ni (x,y)\to(x_0,y_0)} g\left[(x,y)\left(\frac{y}{x}\right)^n\right] =$$

$$= g\left[(x_0,y_0)\left(\frac{y_0}{x_0}\right)^n\right] = f(x_0,y_0).$$

b) Let $(x,y) \in Z_{n+1}$. Hence $(x,y)\left(\frac{y}{x}\right)^{n+1} \in E$. However, $(x_0,y_0)\left(\frac{y_0}{x_0}\right)^{n+1} \notin E$ as $(x_0,y_0) \in Z_n$.

From (15) we obtain

$$(16) \qquad (x_0,y_0)\left(\frac{y_0}{x_0}\right)^{n+1} = \left(1;\frac{1}{x_0^{\frac{1}{n+1}}}\right),$$

whereas

$$(17) \qquad (x_0,y_0)\left(\frac{y_0}{x_0}\right)^{n} = \left(x_0^{\frac{1}{n+1}},1\right).$$

If

$$Z_{n+1} \ni (x,y) \to (x_0,y_0) \in Z_n,$$

then (in view of (16)), we have

$$E \ni (u,v) := (x,y)\left(\frac{y}{x}\right)^{n+1} \to (x_0,y_0)\left(\frac{y_0}{x_0}\right)^{n+1} =$$

$$= \left(1,x_0^{-\frac{1}{n+1}}\right) \notin E.$$

Making use of this and of the formulas (12), (13) and (17), we obtain:

$$\lim_{Z_{n+1}\ni(x,y)\to(x_0,y_0)} f(x,y) = \lim_{Z_{n+1}\ni(x,y)\to(x_0,y_0)} g\left[(x,y)\left(\frac{y}{x}\right)^{n+1}\right] =$$

$$= \lim_{E\ni(u,v)\to\left(1;\frac{1}{x_0^{\frac{1}{n+1}}}\right)} g(u,v) = g\left(x_0^{\frac{1}{n+1}},1\right) =$$

$$= g\left[(x_0,y_0)\left(\frac{y_0}{x_0}\right)^{n}\right] = f(x_0,y_0).$$

Case (III) From the definition (12) of the function f we obtain for $x < y$:

$$f(x,y) = g\left[(x,y)\left(\frac{y}{x}\right)^{n}\right] = g(u,v),$$

where

$$(18)\qquad u = u(x,y) := \frac{y^n}{x^{n-1}}, \qquad v = v(x,y) := \frac{y^{n+1}}{x^n}$$

and n is an integer-valued function of x and y such that $(u,v) \in E$. Let us notice that

$$(u,v) \in E \iff \left(\frac{u}{v} \leqslant u < 1 \text{ and } 1 \leqslant v < \frac{v}{u}\right).$$

Thus for $u(x,y)$ and $v(x,y)$ form (18) we obtain:

$$\frac{x}{y} < u(x,y) < 1 \text{ and } 1 \leqslant v(x,y) < \frac{y}{x}.$$

If now $(x,y) \to (x_0,y_0)$, then $(u(x,y), v(x,y)) \to (1,1)$. From this fact and from (14) we have

$$\lim_{(x,y)\to(x_0,x_0)} f(x,y) = \lim_{(u,v)\to(1,1)} g(u,v) = 1 = f(x_0,x_0),$$

according to (12).

If $x = y \to x_0 = y_0$, then $f(x,y) = 1 \to 1 = f(x_0,y_0)$.

In this way we have proved the continuity of the function f in all the cases (I) - (III).

II. The properties of the rectangle proportion we mentioned in the introduction are not independent. In particular, the properties (3) and (7) follow from (4), (9) and (10); (3) is the consequence of (7) and (9) or (7) and (4). Further, the properties (4), (9) and (10) are equivalent to each other whenever the function $f:(0;+\infty)^2 \to \mathbb{R}$ satisfies (3); as well as the conditions (1), (7) and (11) imply (2). We omit the simple proofs of these relations. In view of these relations a question arises which ones from among the properties (1)-(11) may characterize the rectangle proportion p. We shall prove here the following theorem to this effect.

THEOREM 2. If the function $f: (0;+\infty)^2 \to \mathbb{R}$ satisfies one of the following conditions

a) (2), (3), (8) and either (1) or (5)

b) (1), (3), (7), (8), (11)

then f is the rectangle proportion.

The proof of the theorem will be proceded by some lemmas.

LEMMA 1. If a function $f: (0;+\infty)^2 \to \mathbb{R}$ has the property (8) and $a>0$, then the function $f_a := f(a,\cdot)$ satisfies the Jensen functional equation for $x \geqslant a$.

P r o o f. Let $a>0$. By (8) we obtain

(19) $$a \leqslant \min(x,y) \Rightarrow f_a(x) + f_a(y) = f_a(x+y).$$

This is Cauchy's conditional equation. Inserting here $x = y = \frac{z}{2}$, we have

$$a \leqslant \frac{z}{2} \Rightarrow f_a\left(\frac{z}{2}\right) = \frac{1}{2} f_a(z).$$

If $a \leqslant \min(x,y)$, then $a \leqslant \frac{x+y}{2}$ and we get (putting here $z = x+y$):

$$a \leqslant \min(x,y) \Rightarrow f_a\left(\frac{x+y}{2}\right) = \frac{1}{2} f_a(x+y).$$

Hence and from (19) we obtain

(20) $$a \leqslant \min(x,y) \Rightarrow f_a\left(\frac{x+y}{2}\right) = \frac{f_a(x) + f_a(y)}{2},$$

which completes the proof.

LEMMA 2. If a function $f: (0;+\infty)^2 \to \mathbb{R}$ has the property (8) and let $a>0$, then the function $f_a: f(a,\cdot)$ can be extended to an additive function $g_a(x)$ on $\mathbb{R}$.

P r o o f. The function f_a satisfies Cauchy's conditional equation (19). From lemma 1 we know that f_a satisfies Jensen's equation (20) for $x \geqslant a$. Hence, according to Theorem 1 in [4], p.315, there exist a function $g_a: \mathbb{R} \to \mathbb{R}$, and a constant $c_a \in \mathbb{R}$, such that

$$f_a(x) = g_a(x) + c_a \quad \text{for } x \geqslant a, \tag{21}$$

and

$$g_a(x+y) = g_a(x) + g_a(y). \tag{22}$$

If we substitute (21) in (19), we obtain

$$g_a(x) + g_a(y) + 2c_a = g_a(x+y) + c_a.$$

From (22) we conclude that $c_a = 0$, which means that

$$f_a(x) = g_a(x) \quad \text{for } x \geqslant a,$$

i.e. the additive function g_a on $\mathbb{R}$ is actually an extension of f_a.

LEMMA 3. If a function $f: (0;+\infty)^2 \to \mathbb{R}$ satisfies (8) and has the property (1) or (5), then $f_a(x) := f(a,x)$ for $a > 0$ is linear for $x \geqslant a$.

P r o o f. Let f have the property (1), which means that for any $x,y > 0$: $f(x,y) \geqslant 1$. Hence, on the basis of the theorem of Bernstein-Doetsch ([4], p.145), used to the function $-g_a(x)$ form the Lemma 2, g_a is a continuous function.

If f satisfies the condition (5), then the continuity of f_a is assumed, g_a is a function continuous too.

Hence g_a is additive and continuous, thus it is linear and so is the function $f_a(x)$ for $x \geqslant a$.

Proof of Theorem 2. a) Let f have the property (1) or (5) and the property (8). From Lemma 3 we know that

$$f(x,y) = b(x)y \quad \text{for} \quad x \leqslant y \quad \text{and} \quad b(x) \in \mathbb{R}.$$

If f satisfies also (2), then

$$b(x)x = 1 ,$$

which implies

$$f(x,y) = \frac{1}{x} \cdot y \quad \text{for} \quad x \leqslant y,$$

which is the rectangle proportion for $x \leqslant y$.

Since f is symmetric (condition (3)), it is the rectangle proportion for $0 < y < x$, too.

b) Let f have the properties (1), (3), (7), (8), (11).

Properties (1), (8) and Lemma 3 yield

$$f(x,y) = b(x)y \quad \text{for} \quad x \leqslant y.$$

Hence and form (3) we have

$$f(x,y) = \begin{cases} b(x)y & \text{for} \quad x \leqslant y, \\ b(y)x & \text{for} \quad x > y. \end{cases} \tag{23}$$

Let $\lambda > 0$. Then

$$f(\lambda x, \lambda y) = \begin{cases} b(\lambda x)\lambda y & \text{for} \quad x \leqslant y, \\ b(\lambda y)\lambda x & \text{for} \quad x > y. \end{cases}$$

From the property (7) we have

$$b(x)y = b(\lambda x)\lambda y \quad \text{for} \quad x \leqslant y,$$

and hence, for $x = y = 1$ we obtain

$$b(\lambda) = \frac{b(1)}{\lambda}. \tag{24}$$

From (23) we have $f(1,1) = b(1)$ so that (11) yields $b(1) = [b(1)]^2$, i.e. $b(1) = 1$ because of (1).

Using this in (23) we have from (24)

$$f(x,y)=\begin{cases}\frac{y}{x} & \text{for } x \leqslant y,\\ \frac{x}{y} & \text{for } x > y,\end{cases}$$

which means that

$$f(x,y) = \frac{\max(x,y)}{\min(x,y)},$$

which was to be proved.

As we have seen, condition (8) is essentially used in the proof of Theorem 2. But condition (8) alone does not imply the thesis of the Theorem 2, which can be shown on the following example:

E x a m p l e. Let us consider the function $f: (0;+\infty)^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} ay & \text{for } x \leqslant y,\\ x & \text{for } x > y,\end{cases}$$

where $a \in \mathbb{R}$ and $a \neq 0, 1, -1$.

It is evident that f satisfies condition (8), but it is not the rectangle proportion. Note that f does not satisfy any of the conditions (1) - (7), (9) - (11).

On the other hand, the function $f: (0;+\infty)^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases}\left(\frac{y}{x}\right)^2 & \text{for } x \leqslant y,\\ \left(\frac{x}{y}\right)^2 & \text{for } x > y,\end{cases}$$

satisfies the conditions (1) - (7), (9)-(11), but no longer condition (8). This shows that Alsina's conditions (1) - (4)

are not sufficient for a function to be rectangle proportion p. However, from Theorem 2 it follows that conditions (1) - (4) together with (8) do characterize the rectangle proportion p.

R e m a r k. W.Benz proved ([3]) that if a function f satisfies the conditions (1) - (5) and if for every $t > 0$ there exists $\lim_{x \to 0^+} f(x,tx)$ then

$$f(x,y) = h(p(x,y)), \tag{25}$$

where $h: [1;+\infty) \to [1;+\infty)$ is a function such that $h(1) = 1$ and p is the rectangle proportion. In this theorem the assumption of the continuity of the function f is superfluous. For let us put

$$h(t) := \lim_{x \to 0^+} f(x,tx), \quad (x,y) \in E, \quad t = \frac{y}{x}.$$

We have then $t > 1$. Moreover, the function g in formula (12) satisfies for an arbitrary integer n the condition

$$g(x,y) = g\left[\left(\frac{x}{t^n}, \frac{x}{t^{n-1}}\right)t^n\right] = f\left(\frac{x}{t^n}, t\,\frac{x}{t^n}\right) \xrightarrow[n \to \infty]{} h(t) = h\left(\frac{y}{x}\right),$$

because $\frac{x}{t^n} \to 0$ as $n \to \infty$. Hence

$$g(x,y) = h\left(\frac{y}{x}\right) = h(p(x,y)) \qquad \text{for} \quad x < y.$$

The function f is explicitely determined by the function g by means of the formula (12) and the function $h(p(x,y))$ satisfies the conditions (1) - (4), then the condition (25) holds for any positive x, y.

The above argument also shows that it is enough to assume the existance of $\lim_{x \to 0^+} f(x,tx)$ for $t > 1$ only.

B i b l i o g r a p h y

[1] Alsina C., Problem 272, Aequationes Math. 29, (1985), p.100.

[2] Alsina C., E. Trillas, Lecciones de Algebra y Geometria, Editorial Gustavi Gili, Barcelona, 1984.

[3] Benz W., Remark, Aequationes Math. 29 (1985), pp.101-102.

[4] Kuczma M., An introduction to the theory of functional equations and inequalities, Warszawa-Krakow-Katowice, 1985.

[5] Moszner Z., Remark, Aequationes Math. 29 (1985), pp.102-104.