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On continuous solutions of the equation $f(x+3f(x)f(y)+y)=f(x)+f(y)$

In this paper we shall prove that if a function $f\colon R \to R$ is continuous, $f^{-1}(\{0\}) \neq \emptyset$, and fulfils the equation

(1) $$f(x + 3f(x)f(y) + y) = f(x) + f(y),$$

then $f = 0$.

Equation (1) has been considered in papers [2] and [5]. In the paper [5] S.Midura has obtained the general solution of this equation in the class of odd functions.

S.Midura has proved in [5] the following lemmas

LEMMA 1. If function f fulfils equation (1) and $f^{-1}(\{0\}) \setminus \{0\} \neq \emptyset$, then set $f^{-1}(\{0\}) \setminus \{0\}$ is the set of periods of function f.

LEMMA 2. If function f fulfils equation (1) and $f^{-1}(\{0\}) \neq \emptyset$, then set $f^{-1}(\{0\})$ is a subgroup of the group $(R,+)$.

THEOREM 1. The general solution of equation (1) in odd functions class is the set of functions which fulfil Cauchy's equation

$$f(x + y) = f(x) + f(y)$$

and the condition

$$f(f(x)) = 0$$

for any $x, y \in R$.

Equation (1) is a special case of the equation

$$f(\beta_1 \alpha_1, \beta_2 \alpha_1^3 + 3\alpha_1 f(\alpha_1, \alpha_2), f(\beta_1, \beta_2) + \beta_1 \alpha_2) =$$
$$= \alpha_1^2 f(\beta_1, \beta_2) + \beta_1 f(\alpha_1, \alpha_2).$$

The last equation has appeared in [3] where some subsemigroups of the group L_3^1 were appointed.

Let $R^+ = (0, \infty)$, $R^- = (-\infty, 0)$. We shall prove

LEMMA 3. If a function $f: R \to R$ fulfils equation (1), $f(R^+) \subset R^+$, $f(0) = 0$ and f is a continuous function, then $f(R^+) = R^+$.

P r o o f. Because of the continuity of f, the equality $f(0) = 0$ and the inclusion $f(R^+) \subset R^+$, the set $f(R^+)$ may be equal R^+ or $(0, b)$ or $(0, b\rangle$, where $b > 0$. Let us suppose that $f(R^+)$ is a proper interval and let $x_0 \in R^+$ and $f(x_0) = \frac{2}{3} b$. From equation (1) we have

$$f(2x_0 + \tfrac{4}{3} b^2) = \tfrac{4}{3} b,$$

which is impossible. Then we have $f(R^+) = R^+$.

Obviously, we have

LEMMA 4. If a function $f: R \to R$ fulfils equation (1), then so does function $-f$.

LEMMA 5. If a function $f\colon R \to R$ fulfils equation (1), $f(0) = 0$, $f(R^+) = R^+$ and either $f(R^-) = (0,2c)$ or $f(R^-) = (0,2c\rangle$ where $c > 0$, then $f|_{R^+}$ is not an injective function.

P r o o f. Let $f(x_0) = c$, $f(\bar{x}) = c$ for some $x_0 \in R^+$ and $\bar{x} \in R^-$. We have from equation (1) ($x = x_0$, $y = \bar{x}$ and $x = y = x_0$)

$$f(\bar{x} + 3c^2 + x_0) = 2c \quad \text{and} \quad f(2x_0 + 3c^2) = 2c.$$

Now let us put in (1) first $x = x_0$, $y = 2x_0 + 3c^2$ and then $x = x_0$, $y = \bar{x} + 3c^2 + x_0$. Then we have

$$f(x_0 + 3c\ 2c + 2x_0 + 3c^2) = 3c$$

and

$$f(x_0 + 3c\ 2c + \bar{x} + 3c^2 + x_0) = 3c.$$

Since $3c \in f(R^-)$, then $2x_0 + 3c^2 + \bar{x} > 0$. We have also $3x_0 + 9c^2 \neq 2x_0 + 9c^2 + \bar{x}$, which ends the proof.

LEMMA 6. If a function $f\colon R \to R$ fulfils equation (1), $f(0) = 0$, $f(R^+) = R^+$, $f(R^-) = R^-$, then f is an injective function.

P r o o f. For an indirect proof, let $f(x_1) = f(x_2)$ for some $x_1, x_2 \in R$, $x_1 \neq x_2$. Since $f(x) = 0$ iff $x = 0$, we have $x_1 \cdot x_2 \neq 0$. It follows from our assumptions that for every $x \neq 0$ an $\bar{x} \neq 0$ can be chosen such that $x\bar{x} < 0$ and $f(x) = -f(\bar{x})$. Then from equation (1) we have

$$f(x + 3f(x)f(\bar{x}) + \bar{x}) = f(x) + f(\bar{x}) = 0.$$

Hence, as f vanishes only at zero, we obtain

$$x - 3f(x)^2 + \bar{x} = 0,$$

whence

$$\bar{x} = 3f(x)^2 - x$$

and

(2) $$f(x) = -f(3f(x)^2 - x).$$

Let us put in (1) x_1 and $3f(x_2)^2 - x_2$ in the place of x and y, respectively. We obtain

$$f(x_1 + 3f(x_1)f(3f(x_2)^2 - x_2) + 3f(x_2)^2 - x_2) =$$
$$= f(x_1) + f(3f(x_2)^2 - x_2).$$

By virtue of the hypothesis, (2) and the equality above have

$$f(x_1 - 3f(x_1)f(x_1) + 3f(x_2)^2 - x_2) = f(x_1) - f(x_2).$$

The equality $f(x_1) = f(x_2)$ yields

$$f(x_1 - x_2) = 0,$$

which implies $x_1 = x_2$, a contradiction.

Next we shall prove

LEMMA 7. There is no continuous bijection from R onto R which satisfies equation (1).

P r o o f. Let us suppose that there is a bijective solution of (1), say $f_0: R \to R$. Then $F := f^{-1}$ is a continuous function on R and $F(R) = R$. It is easy to see that F fulfils the equation

(3) $$F(u + v) = F(u) + F(v) + 3uv.$$

It has been proved in [4], that the general solution of equation (3) is given by the formula

$$(4) \qquad F(x) = \varphi(x) + \frac{3}{2}x^2,$$

where φ is an arbitrary additive function. The continuity of F implies the continuity of φ. Then φ is linear, as it is additive. It is obvious that any continuous and bijective function belongs to family (4), a contradiction.

Now we are ready to prove the main result of the paper.

THEOREM 2. If a continuous function $f: R \to R$ fulfils equation (1) and $f^{-1}(\{0\}) \neq \emptyset$, then $f = 0$.

P r o o f. Let us assume that there are non-identically equal zero solutions of equation (1) which fulfill the assumptions of the theorem. It follows from the hypothesis, Lemma 2, [1] /p.238/, and the continuity of the function f that two cases are possible:

I) $f^{-1}(\{0\}) = \{0\}$,

II) $f^{-1}(\{0\}) = x_0 Z$ for an $x_0 \in R^+$.

First we shall consider case I. By virtue of Lemma 4 we may consider, without loss of generality, only such functions f for which $f(R^+) \subset R^+$. Now it follows from Lemma 3 that $f(R^+) = R^+$. Hence we have the following possibilities:

Ia) $f|_{R^+}$ is not a bijection from R^+ onto R^+ and $f(R^-) \subset R^+$,

Ib) $f|_{R^+}$ is a bijection from R^+ onto R^+ and $f(R^-) = R^+$, Ic) $f|_{R^+}$ is a bijection from R^+ onto R^+ and

$f(R^-) = (0,\alpha)$, for a $\alpha \in R^+$,

Id) $f(R^-) = R^-$,

Ie) $f(R^-) = (-c,0)$ or $f(R^-) = \langle -c,0)$ for a $c \in R^+$.

The cases Ia) - Ic) are the only possible ones for functions in I such that $f(R^+) = R^+$ and $f(R^-) \subset R^+$. Two remaining cases correspond to the situation where $f(R^+) = R^+$ and $f(R^-) \subset R^-$. No other cases are possible when $f(R^+) = R^+$.

Case Ia). Let us suppose that $0 < x_2 < x_1$ and $f(x_1) = f(x_2)$. Let $b := \max\limits_{\langle 0,x_1\rangle} f$ and $f(y) = b$ for a $y \in \langle 0,x_1\rangle$. We may assume that $y \neq x_1$ (If maximum is attained at x_1, then we can take $y = x_2$). By virtue of continuity of the function f at zero, for $\varepsilon = \frac{x_1-y}{6b}$ there is a $\delta > 0$ such that for a $z_0 \in (0,\delta) \cap \left(0,\frac{x_1-y}{2}\right)$ we have

$$0 < f(z_0) < \frac{x_1-y}{6b}.$$

Hence

$$z_0 < 3bf(z_0) + z_0 < \frac{x_1-y}{2} + z_0 < x_1 - y$$

and then

$$y < y + 3bf(z_0) + z_0 < x_1.$$

By the definition of b, we have

$$f(y + 3bf(z_0) + z_0) \leqslant b.$$

On the other hand, it follows from equation (1), that

$$f(y + 3bf(z_0) + z_0) = f(z_0) + f(y) = f(z_0) + b > b,$$

that amounts to contradiction.

Case Ib). Since $f|_{R^+}$ is continuous and bijective map of R^+ onto R^+ so it is increasing function. Let $y \in R^+$ be fixed. Put $b := \max_{\langle 0,y \rangle} f$. Choose $x_0 \in R^+$, such that $f(-x_0) = b$. Put $m := \max_{\langle -x_0, 0 \rangle} f$. Let $-\bar{x} \in \langle -x_0, 0 \rangle$ and $f(-\bar{x}) = m$. Then we have $m \geqslant b$. It follows from the continuity of the function f at zero, that for $\varepsilon = \frac{\bar{x}}{3m}$ there is $\delta > 0$, such for

$$z_0 \in (0,\delta) \cap (0,\bar{x}) \cap (0,y)$$

we have

$$0 < f(z_0) < \frac{\bar{x}}{3m}.$$

Thus

$$-\bar{x} < z_0 - \bar{x} < -\bar{x} + 3mf(z_0) + z_0 < z_0.$$

Now we have

$$f(-\bar{x} + 3mf(z_0) + z_0) \leqslant m,$$

because $\max_{\langle -\bar{x}, y \rangle} f = m$.

But, from equation (1) we have at the same time

$$f(-\bar{x} + 3mf(z_0) + z_0) = f(-\bar{x}) + f(z_0) = m + f(z_0) > m,$$

which gives a contradiction.

Case Ic). By Lemma 5 it cannot occur.

Case Id). By Lemma 6 f is a bijection of R onto itself. Lemma 7 then says that f, being continuous, cannot satisfy (1).

Case Ie). Let us take an $x_0 \in R^-$ for which $f(x_0) = -\frac{2}{3}c$. Equation (1) ($x = y = x_0$) yields

$$f(x_0 + 3f^2(x_0) + x_0) = 2f(x_0) = -\frac{4}{3}c.$$

On the other hand

$$f(2x_0 + \frac{4c^2}{3}) \geqslant -c$$

and then we have a contradiction once more.

In this way we have proved that there does not exist a function which fulfils the assumptions of the Theorem and the condition I.

Case II. By Lemma 1 f is now periodic and $x_0 \cdot Z - \{0\}$, $x_0 \in R^+$ is the set all its periods. Hence either $f(R) \subset \{0\} \cup R^+$ or $f(R) \subset \{0\} \cup R^-$. By virtue of Lemma 4 we may restrict our considerations to the case where $f(R) \subset \{0\} \cup R^+$.

Let us denote $c := \max\limits_{\langle 0,x_0\rangle} f$. Then the periodicity of the function f implies that $f(R) = \langle 0,c\rangle$, where $c \in R^+$. Let $y_0 \in \langle 0,x_0\rangle$ and $f(y_0) = c$. From the equation (1) we have

$$f(y_0 + 3c^2 + y_0) = 2f(y_0) = 2c,$$

which value cannot be attained by f and in this way Theorem 2 is proved.

R e m a r k 1. In the paper [2] R.Ger has given the general solution $f\colon R \to R$ of the equation

(5) $\quad f(x + 3f(y)^2 - 3f(x)f(y) - y) = f(x) - f(y).$

He has also proved that

LEMMA 8. Any solution $f: R \to R$ of equation (5) satisfies equation (1).

LEMMA 9. Any solution $f: R \to R$ of equation (1) such that $f(R) = -f(R)$ satisfies equation (5).

When we compare Theorem 2 and Lemma 8 we can conclude the following

COROLLARY 1. The function $f = 0$ is the only continuous and vanishing at least at one point solution of equation (5).

R e m a r k 2. In the paper [5] there are shown by an expamle some odd and non-constant solutions of (1). There are additive functions taking only rational values, which are not identically equal to zero but which equal to zero for rational arguments.

Lemma 9 implies that such functions satisfy equation (5). We do not know any example of a function which fulfils (1) and does not fulfil (5). We do not know also if there are functions (even continuous ones) which would fulfil equation (1) and do not vanish.

R e f e r e n c e s

[1] Bourbaki N., Topologie générale, Hermann, Paris (Russian translation: 1969).

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[4] Midura S., Sur la détermination de certains sous-groupes du groupe L_s^1 à l'aide d'équations fonctionnelles. Dissertationes Mathematicae 105, PWN. Warszawa, 1973.
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