JOZEF TABOR

On mappings preserving the stability of the Cauchy functional equation

There are several papers concerning the stability of functional equations and different definitions of the stability are in use (cf. [1], [3], [5], [6], [7], [8], [9]).

By a groupoid we shall understand a pair $(X,\cdot)$, where X is a non-empty set and "$\cdot$" a binary operation on X. If it does not lead to misunderstanding we shall use the same symbol "$\cdot$" for different binary operations or even omit it.

Let E and V be groupoids. By the Cauchy functional equation we mean equation

(1) $$f(x \cdot y) = f(x) \cdot f(y) \quad \text{for } x,y \in E,$$

where $f\colon E \to V$.

DEFINITION 1. (cf. [5], [7]). Let E, V be groupoids and ρ a metric on V. Equation (1) is said to be stable if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $F: E \to V$ which satisfies the inequality

$$\varrho(F(x \cdot y), F(x) \cdot F(y)) < \delta \qquad \text{for } x, y \in E$$

there exists a solution f of equation (1) such that

$$\varrho(F(x), f(x)) < \varepsilon \qquad x \in E.$$

DEFINITION 2. ([6], [3]). Let E, V be groupoids and ϱ a metric on V. Equation (1) is said to be stable if there exists a $K > 0$ such that for every $\varepsilon > 0$ and every $F: E \to V$ which satisfies the inequality

$$\varrho(F(x \cdot y), F(x) \cdot F(y)) < \varepsilon \qquad \text{for } x, y \in E$$

there exists a solution f of equation (1) such that

$$\varrho(F(x), f(x)) < K \qquad \text{for } x \in E.$$

DEFINITION 3. ([7]). Let E be a groupoid, V a group and a topological space simultanously. Let e denote the unit of V. Equation (1) is said to be stable if for every neighbourhood Δ of e there exists a neighbourhood Ω of e such that for every $F: E \to V$ which satisfies the condition

$$F(x) \cdot F(y) \, [F(x \cdot y)]^{-1} \in \Omega \qquad \text{for } x, y \in E$$

there exists a solution f of (1) such that

$$F(x) \cdot [f(x)]^{-1} \in \Delta \qquad \text{for } x \in E.$$

Definition 3, suggested by [8], has been introduced by Z. Moszner (cf. [7]). He established also the relation between definitions 1 and 3. His result reads as follows.

LEMMA 1 ([7]). Let E be a groupoid, let V be a group and a metric space (with metric ϱ) such that the

the right translations are equicontinuous i.e.

$$(2)\quad \underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x,y\in V}{\forall}\ [\varrho(x,y)<\delta \Rightarrow \underset{a\in V}{\forall}\ \varrho(xa,ya)<\varepsilon]$$

Then definitions 1 and 3 are equivalent.

Let $E_1, E_2; V_1, V_2$ be groupoids possessing properties required in definitions 1, 2, 3 respectively. The guestion when the stability of the Cauchy equation for $f\in V_1^{E_1}$ implies the stability of this equation for $f\in V_2^{E_2}$ has been considered in [1] and [7]. In those considerations the following assumption plays an important role (cf. [1], theorem 1, 2, 3):

(i) There exist homomorphisms $I: V_1 \to V_2$, $H: V_2 \to V_1$ such that $I\circ H = \mathrm{id}_{V_2}$.

We give a characterization of this condition in case where V_1, V_2 are groups.

THEOREM 1. Let V_1, V_2 be groups. Then condition (i) is equivalent to the following one:

(ii) There exist a normal subgroup A of V_1 and a subgroup S of V_1 isomorphic to V_2 such that V_1 is the semidirect product of A on S.

P r o o f. Let us assume condition (i) and put

$$A := \ker I, \quad S := \operatorname{im} H.$$

It follows from the equality $I\circ H = \mathrm{id}_{V_2}$ that H is an injection, and hence S is isomorphic to V_2. Further, by the same equality, for every fixed x the set $I^{-1}\{x\}\cap S$ consists of exactly one element, which implies the following conditions

(3) $$A \cap S = \{e\} \quad (e \text{ - unit of } V_1),$$

(4) $$A \cdot S = V_1.$$

Equalities (3) and (4) are equivalent to the fact that V_1 is the semidirect product of A on S (cf. [2], [4]). Thus condition (ii) is valid.

Let us assume now condition (ii) and let $H: V_2 \twoheadrightarrow S$ be an isomorphism. Then conditions (3) and (4) hold and so every $x \in V_1$ can be written uniquely in the form $x = a \cdot s$ with $a \in A$, $s \in S$. We define the projection $\pi : V_1 \twoheadrightarrow S$ as follows:

$$\pi(x) = s \quad \text{for } x = a \cdot s, \quad a \in A, \quad s \in S.$$

Evidently π is an epimorphism of V_1 onto S. We set $I = H^{-1} \circ \pi$. The mapping I is a homomorphism of V_1 onto V_2. Moreover, since π restricted to S is the identity mapping, we have for $a \in V_2$:

$$I \circ H(a) = H^{-1} \circ \pi \circ H(a) = H^{-1} \circ H(a) = a,$$

i.e. $I \circ H = id_{V_2}$. It ends the proof.

We shall consider the problem of the stability on the product of groupoids. Let E, V have the meaning as required in definitions 1, 2, 3, respectively. If the Cauchy equation is stable in the sense of definition k, k = 1,2,3, then we shall write $(E,V) \in CS_k$.

LEMMA 2. Let E_1, E_2 be groupoids with units. Let V abelian group and let ρ be metric on V such that the translations are equicontinuous.

Then

if $(E_1,V)\in CS_k$, $(E_2,V)\in CS_k$, then $(E_1xE_2,V)\in CS_k$, $k=1,3$.

P r o o f. In virtue of Lemma 1 definitions 1 and 3 are equivalent. Therefore we consider only the case $k = 1$.

Suppose that (E_1,V), $(E_2,V)\in CS_1$, fix an $\varepsilon > 0$ and choose, according to (2), δ_1 for $\frac{\varepsilon}{2}$. To (E_1,V) and $\frac{\delta_1}{2}$ in place of ε there exists a δ_2 satisfying definition 1. Similarly to (E_2,V) and $\frac{\delta_1}{2}$ in place of ε there exists a δ_3 satisfying the same definition. Finally there exists also a δ_4 fulfilling condition (2) with $\frac{\delta_1}{2}$ in place of ε. We set $\delta := \min(\delta_2, \delta_3, \delta_4)$. Now let us consider a $G: E_1 x E_2 \to V$ such that

(5) $$\rho[G(x_1x_2,y_1y_2),G(x_1,y_1)G(x_2,y_2)]<\delta$$

for $x_1,x_2\in E_1$, $y_1,y_2\in E_2$.

Let e_1, e_2 be units of E_1, E_2 respectively. We set

$$G_1(x) := G(x,e_2) \quad \text{for} \quad x\in E_1,$$

$$G_2(y) := G(e_1,y) \quad \text{for} \quad y\in E_2.$$

Putting in (5) $y_1 = y_2 = e_2$ and next $x_1 = x_2 = e_1$ we obtain

$$\rho[G_1(x_1x_2),G_1(x_1)G_1(x_2)]<\delta \leqslant \delta_2,$$

$$\rho[G_2(y_1y_2),G_2(y_1)G_2(y_2)]<\delta \leqslant \delta_3.$$

Thus there are homomorphisms $g_1: E_1 \to V$, $g_2: E_2 \to V$ such that

(6) $$\rho[G_1(x),g_1(x)] < \frac{\delta_1}{2} \quad \text{for} \quad x\in E_1,$$

(7) $$\rho[G_2(y),g_2(y)] < \frac{\delta_1}{2} \quad \text{for} \quad y\in E_2.$$

We define $g: E_1 \times E_2 \to V$ as follows:

$$g(x,y) := g_1(x)g_2(y) \quad \text{for} \quad x \in E_1,\ y \in E_2.$$

Since V is abelian, g satisfies the Cauchy equation. Setting in (5) $x_1 = x$, $y_1 = e_2$, $x_2 = e_1$, $y_2 = y$ we get for $x \in E_1$, $y \in E_2$

$$\varrho[G(x,y), G_1(x)G_2(y)] < \delta \leqslant \delta_4.$$

Hence, in view of (2) (with ε replaced by $\frac{\delta_1}{2}$)

$$\varrho\{G(x,y)[G_2(y)]^{-1}, G_1(x)\} < \frac{\delta_1}{2}.$$

By the above inequality and (6)

$$\varrho\{G(x,y)[G_2(y)]^{-1}, g_1(x)\} < \delta_1$$

and hence

$$\varrho\{G(x,y), g_1(x)G_2(y)\} < \frac{\varepsilon}{2}.$$

Making use of (7) and (2) we get

$$\varrho\{g_1(x)G_2(y), g_1(x)g_2(y)\} < \frac{\varepsilon}{2}.$$

In virtue of the last two equalities

$$\varrho[G(x,y), g_1(x)g_2(y)] < \varepsilon ,$$

i.e.

$$\varrho[G(x,y), g(x,y)] < \varepsilon .$$

LEMMA 3. Let E_1, E_2 be groupoids with units. Let V be an abelian group and let ϱ be metric on V such that the translations satisfy Lipschitz condition with the same constant, i.e.

$$\exists_{M>0} \ \forall_{x,y,a \in V} \ \varrho(xa,ya) \leqslant M\varrho(x,y).$$

Then

if $(E_1,V) \in CS_2$ and $(E_2,V) \in CS_2$, then $(E_1 \times E_2, V) \in CS_2$.

The proof of this lemma is omitted since it is only a slight modification of the proof of Lemma 2.

LEMMA 4. Let E be a groupoid. Let V_1, V_2 be groups and metric spaces and let the group $V_1 \times V_2$ be endowed with the cartesian metric.
Then

if $(E,V_1) \in CS_k$ and $(E,V_2) \in CS_k$, then $(E,V_1 \times V_2) \in CS_k$, for $k = 1,2,3$.

P r o o f. Obvious.

The assumptions of Lemma 4 in the case $k = 1,2$ may be considerably weakned. Instead of assuming that V_1, V_2 are groups we may assume that they are groupoids.

Lommas 2 and 4 directly imply the following

THEOREM 2. Let E_1, E_2 be groupoids with units, let V_1, V_2 be abelian groups and metric spaces such that the translations are equicontinuous. Let $V_1 \times V_2$ be equipped with the cartesian metric.
Then

if $(E_i,V_j) \in CS_k$, $i=1,2$, $j=1,2$, then $(E_1 \times E_2, V_1 \times V_2) \in CS_k$ for $k=1,3$.

Similarly from lemmas 3 and 4 we obtain

THEOREM 3. Let E_1, E_2 be groupoids with units, let V_1, V_2 be abelian groups and metric spaces such that the translations satisfy Lipschitz condition with the same constant. Let $V_1 \times V_2$ be equipped with the cartesian metric.

Then

if $(E_i, V_j) \in CS_2$, $i=1,2$, $j=1,2$, then $(E_1 \times E_2, V_1 \times V_2) \in CS_2$. We illustrate our considerations by two examples.

E x a m p l e 1. Let us consider the multiplicative group $GL(n,R)$ of real non-singular nxn-matrices and the subgroup $GL^+(n,R) = \{A \in GL(n,R): \det A > 0\}$. Let V be a groupoid and a metric space such that $(GL(n,R),V) \in CS_i$ ($i = 1$ or $i = 2$ or $i = 3$). $GL(n,R)$ is the semidirect product of $GL^+(n,R)$ on some two elements subgroup of $GL(n,R)$. Hence, in view of [1] and Theorem 1, $(GL^+(n,R),V) \in CS_i$.

E x a m p l e 2. Let V be a real vector space, R the additive group of real numbers, Z the additive group of integers, R^+ the multiplicative group of positive real numbers. In R and Z we consider the usual metric. By the theorem of Hyers $(V,R) \in CS_1$. Futhermore $(R^+,R) \in CS_1$ (cf. [7]). Since Z is a discrete space $(V,Z) \in CS_1$ and $(R^+,Z) \in CS_1$. Clearly the translations in R and in Z are equicontinuous. So by Theorem 2 $(V \times R^+, R \times Z) \in CS_1$.

R e f e r e n c e s

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