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On extension of a solution of the translation equation

The main objective of this paper is to give a necessary and sufficient condition for the extensibility of a solution of the translation equation from a substructure R of an Ehresmann groupoid $(E;\cdot)$ onto the whole groupoid $(E;\cdot)$, without an extension of the fibre Γ. The substructure R satisfies the condition $R \cup R^{-1} = E$.

1. Let E be an arbitrary non-empty set and let "$\cdot$" be an arbitrary partial mapping from the set $E \times E$ into the set E. A pair $(E;\cdot)$ will be called a multiplicative system. The domain of the partial mapping "$\cdot$" will be denoted by $D_{\cdot}$. Let us define the following set

$$E^{0} = \{e \in E: (e,e) \in D_{\cdot} \wedge e \cdot e = e\}.$$

We start with the following definitions.

DEFINITION 1. A multiplicative system $(E;\cdot)$ is an Ehresmann groupoid if the following conditions are fulfilled:

(1) $E^0 = \{e \in E: \bigwedge_{x \in E} [(x,e) \in D. \Rightarrow x \cdot e = x] \wedge$
$\wedge [(e,x) \in D. \Rightarrow e \cdot x = x]\}$,

(2) $\bigwedge_{x,y,z \in E} (x,y) \in D. \wedge (y,z) \in D. \Rightarrow (x,y \cdot z) \in D. \wedge$
$\wedge (x \cdot y, z) \in D.$,

(3) $\bigwedge_{x,y,z \in E} (x,y) \in D. \wedge (x \cdot y, z) \in D. \Rightarrow (y,z) \in D.$,

(4) $\bigwedge_{x,y,z \in E} (y,z) \in D. \wedge (x,y \cdot z) \in D. \Rightarrow (x,y) \in D.$,

(5) $\bigwedge_{x,y,z \in E} (x,y) \in D. \wedge (y,z) \in D. \Rightarrow x \cdot (y \cdot z) = (x \cdot y) \cdot z$,

(6) $\bigwedge_{x \in E} \bigvee^1_{{}_xe, e_x \in E^0} ({}_xe, x) \in D. \wedge (x, e_x) \in D.$,

(7) $\bigwedge_{x \in E} \bigvee^1_{x^{-1} \in E} (x^{-1}, x) \in D. \wedge (x, x^{-1}) \in D. \wedge$
$\wedge x^{-1} \cdot x = e_x \wedge x \cdot x^{-1} = {}_xe$.

In the sequel ${}_xe$ will be called a left unit of the element x and e_x will be called a right unit of the element x. The symbol x^{-1} will denote an inverse element.

DEFINITION 2. A set $A \subset E$ is a substructure of the groupoid $(E;\cdot)$ if and only if $A \cdot A \subset A$, where

$$A \cdot A := \{z \in E: \bigvee_{x,y \in A} (x,y) \in D. \wedge z = x \cdot y\}.$$

If A is a substructure of the Ehresmann groupoid $(E;\cdot)$ then $A^{-1} := \{x \in E: x^{-1} \in A\}$, where x^{-1} is an inverse element.

2. Let Γ be a non-empty set and suppose that $(E;\cdot)$ is a multiplicative system. Let $F: \Gamma \times E \to \Gamma$ be a mapping with the domain $D_F \subset \Gamma \times E$ and the image $Q_F \subset \Gamma$. Let us put

$$Z_F := \{(\alpha, x, y) \in \Gamma \times E^2: (\alpha, x) \in D_F \wedge (F(\alpha, x), y) \in D_F \wedge$$
$$\wedge (x,y) \in D. \wedge (\alpha, x\cdot y) \in D_F\}.$$

DEFINITION 3. A mapping $F: \Gamma \times E \to \Gamma$ is a solution of the translation equation if

$$\bigwedge_{(\alpha,x,y) \in \Gamma \times E^2} (\alpha, x, y) \in Z_F \Longrightarrow F(F(\alpha,x),y) = F(\alpha, x\cdot y).$$

DEFINITION 4. Let A be substructure of the groupoid $(E;\cdot)$. A solution of the translation equation $F: \Gamma \times A \to \Gamma$ is extendable if there exists a solution $\overline{F}$ of the translation equation which is defined on the whole set $\Gamma \times E$ and $\overline{F}\big|_{\Gamma \times A} = F$.

The mapping $\overline{F}$ is called an extension of the solution F.

DEFINITION 5. A mapping $F: \Gamma \times E \to \Gamma$ satisfies the identity condition if

$$\bigwedge_{\alpha \in \Gamma} \bigwedge_{e \in E^0} (\alpha, e) \in D_F \Longrightarrow F(\alpha, e) = \alpha.$$

We shall prove the following

THEOREM 1. Let $(E;\cdot)$ be an Ehresmann groupoid and let Γ be a non-empty set. The solution $F: \Gamma \times E \to \Gamma$

of the translation equation satisfies the identity condition if and only if $F(\cdot,x)$ is a bijection from the set Γ onto the set Γ for every $x\in E$.

P r o o f. If the mapping $F(\cdot,x)$ is a bijection from Γ onto Γ for every $x\in E$ then

$$\bigwedge_{\alpha\in\Gamma}\ \bigvee_{\beta\in\Gamma} F(\beta,e)=\alpha$$

for every fixed $e\in E^0$. Therefore $F(\alpha,e)=F(F(\beta,e),e)=$ $=F(\beta,e)=\alpha$ for an arbitrary $\alpha\in\Gamma$ and $e\in E^0$.

Now, assume that $\alpha,\beta\in\Gamma$ and $F(\alpha,x)=F(\beta,x)$ for a fixed $x\in E$. Hence $F(F(\alpha,x),x^{-1})=F(F(\beta,x),x^{-1})$ thus $F(\alpha,x\cdot x^{-1})=F(\beta,x\cdot x^{-1})$ or $F(\alpha,{}_xe)=F(\beta,{}_xe)$, this means that $\alpha=\beta$. Moreover for an arbitrary $x\in E$ and $\alpha\in\Gamma$ there exists $\beta=F(\alpha,x^{-1})$ such that $F(\beta,x)=F(F(\alpha,x^{-1}),x)=$ $=F(\alpha,e_x)=\alpha$. Consequently $F(\cdot,x)$ is a bijection from Γ onto Γ for every $x\in E$, which completes the proof.

The above theorem is not true if the multiplicative system $(E;\cdot)$ has no inverse elements.

R e m a r k 1. If $F:\Gamma\times E\to\Gamma$ is a solution of the translation equation and $(E;\cdot)$ is a multiplicative system without inverse elements then the property

$$\bigwedge_{x\in E} F(\cdot,x):\ \Gamma\xrightarrow[\text{bij}]{}\Gamma$$

is only a sufficient condition for F to fulfil the identity condition.

This fact may be seen from the following

E x a m p l e. Let Γ be the set $\mathbb{R}^+$ of non-negative real numbers. Take $E = \{(x,y) \in \mathbb{R}^2 : x \leqslant y\}$. In the set E we define the operation "$\cdot$" as follows:
$((x,y),(z,t)) \in D_\cdot$ iff $y = z$ and then $(x,y)\cdot(z,t) = (x,t)$.
Let us put $F(\alpha,x,y) = \alpha + y - x$ for $\alpha \in \mathbb{R}^+$ and $(x,y) \in E$. This function $F: \mathbb{R}^+ \times E \to \mathbb{R}^+$ is a solution of the translation equation and F satisfies the identity condition, but for $x < y$ we have $F(\mathbb{R}^+ \times \{x,y\}) = \langle y-x,\infty) \neq \mathbb{R}^+$.

3. A. Grząślewicz has give in [1] the following theorem concerning the extensibility of a solution of the equation

$$H_1(x) \circ H_2(y) = H_3(\overline{x\cdot y}). \tag{1}$$

Let $(E;\cdot)$ be an Ehresmann groupoid and let R be its substructure. If the triplet of functions $(H_1,H_2,H_3) \in [R \to E_1]^3$ */, where $(E_1;\circ)$ is an Ehresmann groupoid and $R \cup R^{-1} = E$, is the solution of the equation (1), then there exists an extension of this solution on the triplet of sets (E,E,E), and it is assigned in an unique manner:

$$\bar{H}_i(x) := \begin{cases} H_i(x) & \text{for } x \in R, \\ H_i({}_xe) \circ [H_i(x^{-1})]_\cdot^{-1} \circ H_i(e_x) & \text{for } x \in R^{-1}, \end{cases}$$

where $i = 1,2,3$.

Setting $H_1 = H_2 = H_3$ and treating $(E_1;\circ)$ as a set

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*/ The symbol $[X \to Y]$ denotes a set of all functions F for which $D_F \subset X$ and $\mathrm{C}_F \subset Y$.

of bijections of the fixed set Γ endowed with composition of functions we can consider equation (1) as the translation equation $F(F(\cdot,x),y) = F(\cdot,x\,y)$, where $F: \Gamma \times R \to \Gamma$ and for every $x \in E$ the function $F(\cdot,x) = H_1(x)$ is a bijection from the set Γ onto itself. Considering additionally Remark 1 we get the following

COROLLARY. If $F: \Gamma \times R \to \Gamma$, where R is a substructure of Ehresmann groupoid $(E;\cdot)$ such that $R \cup R^{-1} = E$ and Γ is an arbitrary fixed set, is a solution of the translation equation such that $F(\cdot,x): \Gamma \xrightarrow[\text{bij}]{} \Gamma$ for every $x \in R$ then F can be uniquelly extended to the solution $\overline{F}: \Gamma \times E \to \Gamma$ where

$$\overline{F}(\alpha,x) = \begin{cases} F(\alpha,x) & \text{for } \alpha \in \Gamma \text{ and } x \in R, \\ F^{-1}(\cdot,x^{-1})(\alpha) & \text{for } \alpha \in \Gamma \text{ and } x \in R^{-1}. \end{cases}$$

The assumption that every function of family $\{F(\cdot,x)\}_{x \in R}$ is a bijection of the set Γ is very strong. In this case the problem of extensibility according to Remark 1 is reduced to the solutions fulfilling the identity condition. We shall show that this assumption can be released.

First we shall prove the following

LEMMA. Let R be a substructure of the Ehresmann groupoid $(E;\cdot)$ such that $R \cup R^{-1} = E$, let Γ be an arbitrary fixed non-empty set and let F be an extendable solution of the translation equation defined on the set $\Gamma \times R$.

Then

(2) $\qquad F(\Gamma, {}_xe) = \overline{F}(\Gamma, x^{-1})$ for every $x \in R$,

where $\overline{F}$ is an arbitrary extension of the solution F defined on the set $\Gamma \times E$,

(3) $F(\cdot, x)$ is a one-to-one mapping from the set $F(\Gamma, {}_xe)$ into the set Γ,

(4) the mapping $\overline{F}: \Gamma \times E \to \Gamma$ being the extension of the solution F is uniquely determined.

P r o o f. Let F be an extendable solution of the translation equation defined on the set $\Gamma \times R$ and let $\overline{F}$ be an arbitrary extension of F onto the set $\Gamma \times E$.

If $\alpha \in F(\Gamma, {}_xe)$ then there exists $\beta \in \Gamma$ such that $\alpha = F(\beta, {}_xe)$, thus $\alpha = F(\beta, x \cdot x^{-1}) = \overline{F}(F(\beta, x), x^{-1})$ so $\alpha \in \overline{F}(\Gamma, x^{-1})$. If $\alpha \in \overline{F}(\Gamma, x^{-1})$ then there exists $\beta \in \Gamma$ such that $\alpha = \overline{F}(\beta, x^{-1})$. Thus $\alpha = \overline{F}(\beta, x^{-1} \cdot {}_xe) = F(\overline{F}(\beta, x^{-1}), {}_xe)$ since $e_{x^{-1}} = {}_xe$, whence $\alpha \in F(\Gamma, {}_xe)$. This proves condition (2).

Now, assume that $\alpha, \beta \in F(\Gamma, {}_xe)$ and $F(\alpha, x) = F(\beta, x)$. Then there exists $\gamma, \delta \in \Gamma$ such that $\alpha = F(\gamma, {}_xe)$, $\beta = F(\delta, {}_xe)$ and $F(F(\gamma, {}_xe), x) = F(F(\delta, {}_xe), x)$. Hence $\overline{F}(F(F(\gamma, {}_xe), x), x^{-1}) = \overline{F}(F(F(\delta, {}_xe), x), x^{-1})$ thus $F(F(\gamma, {}_xe), {}_xe) = F(F(\delta, {}_xe), {}_xe)$ or $F(\gamma, {}_xe) = F(\delta, {}_xe)$, this means that $\alpha = \beta$. Consequently $F(\cdot, x)$ is a one-to-one mapping on the set $F(\Gamma, {}_xe)$.

To prove condition (4) we assume that $F_1: \Gamma \times E \to \Gamma$ and $F_2: \Gamma \times E \to \Gamma$ are two extensions of the solution F

defined on the set $\Gamma \times R$. Then for every $x \in R$ and $\alpha \in \Gamma$ we have $F(F_1(\alpha, x^{-1}), x) = F(\alpha, e_x) = F(F_2(\alpha, x^{-1}), x)$. It results from condition (2) that $F_1(\alpha, x^{-1}) \in F(\Gamma, {}_xe)$ and $F_2(\alpha, x^{-1}) \in F(\Gamma, {}_xe)$, so in virtue of (3) we have $F_1(\alpha, x^{-1}) = F_2(\alpha, x^{-1})$. Using $R \cup R^{-1} = E$ we obtain condition (4), which completes the proof.

Now, we shall be concerned with the main theorem of this paper.

THEOREM 2. Let R be a substructure of the Ehresmann groupoid $(E;\cdot)$ such that $R \cup R^{-1} = E$ and let Γ be an arbitrary set. Then the solution F of the translation equation defined on the set $\Gamma \times R$ is extendable to the solution $\bar{F}$ defined on the set $\Gamma \times E$ if and only if for every $x \in R$

(5) $F(\cdot, x)$ is a one-to-one mapping from the set $F(\Gamma, {}_xe)$ onto the set $F(\Gamma, e_x)$,

and then the mapping

$$(6) \quad \bar{F}(\alpha, x) = \begin{cases} F(\alpha, x), & \text{where } \alpha \in \Gamma \text{ and } x \in R, \\ \left[F(\cdot, x^{-1})\big|_{F(\Gamma, e_x)}\right]^{-1}(F(\alpha, {}_xe)), & \text{where } \alpha \in \Gamma \text{ and } x \in R^{-1} \end{cases}$$

is a unique extension of the solution F.

P r o o f. Let us notice that for every $x \in R$

$$(7) \qquad F(\Gamma, x) \subset F(\Gamma, e_x).$$

Really, if $\alpha \in F(\Gamma, x)$ and $\alpha = F(\beta, x)$ then $\alpha = F(\beta, x \cdot e_x) = F(F(\beta, x), e_x)$ so $\alpha \in F(\Gamma, e_x)$.

Now suppose F to be an extendable solution of the translation equation defined on $\Gamma \times R$. It follows from conditions (3) and (7) that $F(\cdot,x)$ is a one-to-one mapping from $F(\Gamma,{}_xe)$ into $F(\Gamma,e_x)$. To prove condition (5) it it is enough to show that for arbitrary $x\in R$ and $\alpha\in F(\Gamma,e_x)$ there exists $\gamma\in F(\Gamma,{}_xe)$ such that $F(\gamma,x)=\alpha$. Since $\alpha\in F(\Gamma,e_x)$ so there exists $\beta\in\Gamma$ such that $\alpha=F(\beta,e_x)$. Put $\gamma=\overline{F}(\beta,x^{-1})$. According to (2) $\gamma\in F(\Gamma,{}_xe)$. Besides $F(\gamma,x)=F(\overline{F}(\beta,x^{-1}),x)=F(\beta,e_x)=\alpha$, which completes the proof of a necessary condition.

Let $F: \Gamma\times R\to\Gamma$ be a solution of the translation equation such that condition (5) is fulfilled. We define a function $\overline{F}: \Gamma\times E\to\Gamma$ by means of condition (6). For $x\in R\cap R^{-1}$ and $\alpha\in\Gamma$ we have

$$\left[F(\cdot,x^{-1})\Big|_{F(\Gamma,e_x)}\right]^{-1}(F(\alpha,{}_xe)) = F(\alpha,x)$$

since $F(F(\alpha,x),x^{-1})=F(\alpha,{}_xe)$ and $F(\alpha,x)\in F(\Gamma,{}_xe)$ according to (7). Considering additionally condition (5) by the fact that ${}_{x^{-1}}e=e_x$, $e_{x^{-1}}={}_xe$ and $R\cup R^{-1}=E$ we can notice that the function $\overline{F}$ is unambiguously defined on the whole set $\Gamma\times E$.

It follows form condition (5) that for an arbitrary $x\in R^{-1}$ we have

$$F(\cdot,x^{-1}): F(\Gamma,e_x)\xrightarrow[\text{bij}]{} F(\Gamma,{}_xe),$$

thus

$$\left[F(\cdot,x^{-1})\Big|_{F(\Gamma,e_x)}\right]^{-1}: F(\Gamma,{}_xe)\xrightarrow[\text{bij}]{} F(\Gamma,e_x).$$

The mapping $\overline{F}$ defined by (6) has the following property

(8) $\overline{F}(\cdot,x) : F(\Gamma,{}_x e) \xrightarrow[\text{bij}]{} F(\Gamma,e_x)$ for every $x \in E$.

It results from the definition of the function $\overline{F}$ that $\overline{F}\big|_{\Gamma\times R} = F$.

Thus it must be shown that $\overline{F}$ is a solution of the translation equation. For this purpose we shall distinguish the following cases.

a) $(x,y) \in D_\bullet$, $x \in R$, $y \in R^{-1}$, $x\cdot y \in R$;

b) $(x,y) \in D_\bullet$, $x \in R^{-1}$, $y \in R$, $x\cdot y \in R$;

c) $(x,y) \in D_\bullet$, $x \in R^{-1}$, $y \in R^{-1}$, $x\cdot y \in R^{-1}$;

d) $(x,y) \in D_\bullet$, $x \in R$, $y \in R^{-1}$, $x\cdot y \in R^{-1}$;

e) $(x,y) \in D_\bullet$, $x \in R^{-1}$, $y \in R$, $x\cdot y \in R^{-1}$;

f) $(x,y) \in D_\bullet$, $x \in R$, $y \in R$, $x\cdot y \in R$.

Moreover, if $(x,y) \in D_\bullet$ then $e_x = {}_y e$.

Case a). Let us determine $F(\overline{F}(\overline{F}(\alpha,x),y),y^{-1})$ and $F(\overline{F}(\alpha,x\cdot y),y^{-1})$. We have

$$F(\overline{F}(\overline{F}(\alpha,x),y),y^{-1}) = F(\overline{F}(F(\alpha,x),y),y^{-1}) =$$

$$= F\left(\left[F(\cdot,y^{-1})\Big|_{F(\Gamma,e_y)}\right]^{-1}(F(F(\alpha,x),{}_y e)),y^{-1}\right) =$$

$$= F(F(\alpha,x),e_x) = F(\alpha,x)$$

and

$$F(\overline{F}(\alpha,x\cdot y),y^{-1}) = F(F(\alpha,x\cdot y),y^{-1}) = F(\alpha,x\cdot(y\cdot y^{-1})) =$$

$$= F(\alpha,x\cdot{}_y e) = F(\alpha,x\cdot e_x) = F(\alpha,x).$$

Since $\overline{F}(\overline{F}(\alpha,x),y) \in F(\Gamma,e_y)$ and $\overline{F}(\alpha,x\cdot y) \in F(\Gamma,e_y)$ therefore applying (5) we get $\overline{F}(\overline{F}(\alpha,x),y) = \overline{F}(\alpha,x\cdot y)$.

Case b). Now we determine $\bar{F}(F(\bar{F}(\alpha,x),y),y^{-1})$ and $\bar{F}(\bar{F}(\alpha,x\cdot y),y^{-1})$ for $y^{-1}\in R^{-1}$. Using the equality from case a) we obtain

$$\bar{F}(F(\bar{F}(\alpha,x),y),y^{-1}) = \bar{F}(\bar{F}(\alpha,x),y\cdot y^{-1}) = \bar{F}(\bar{F}(\alpha,x),e_x) =$$
$$= F(\bar{F}(\alpha,x),e_x) = \bar{F}(\alpha,x),$$

because $\bar{F}(\alpha,x)\in F(\Gamma,e_x)$, and $F(\cdot,e_x)\big|_{F(\Gamma,e_x)} = \mathrm{id}_{F(\Gamma,e_x)}$.

However

$$\bar{F}(\bar{F}(\alpha,x\cdot y),y^{-1}) = \bar{F}(F(\alpha,x\cdot y),y^{-1}) =$$
$$= \left[F(\cdot,y)\Big|_{F(\Gamma,e_{y^{-1}})}\right]^{-1}(F(F(\alpha,x\cdot y),{}_{y^{-1}}e)) =$$
$$= \left[F(\cdot,y)\Big|_{F(\Gamma,{}_{y}e)}\right]^{-1}F(F(\alpha,x\cdot y),e_y)) =$$
$$= \left[F(\cdot,y)\Big|_{F(\Gamma,e_x)}\right]^{-1}F(\alpha,x\cdot y) =$$
$$= \left[F(\cdot,y)\Big|_{F(\Gamma,e_x)}\right]^{-1}F(\bar{F}(\alpha,x),y) = \bar{F}(\alpha,x),$$

because $\bar{F}(\alpha,x)\in F(\Gamma,e_x)$.

Since $\bar{F}(\bar{F}(\alpha,x),y)\in F(\Gamma,e_y) = F(\Gamma,{}_{y^{-1}}e)$ and $\bar{F}(\alpha,x\cdot y)\in F(\Gamma,{}_{y^{-1}}e)$ therefore in virtue of (8) we get

$\bar{F}(\bar{F}(\alpha,x),y) = \bar{F}(\alpha,x\cdot y)$.

Case c). For $x\in R^{-1}$, $y\in R^{-1}$ and $x\cdot y\in R^{-1}$ we have

$$\bar{F}(\bar{F}(\alpha,x),y) = \left[F(\cdot,y^{-1})\Big|_{F(\Gamma,e_y)}\right]^{-1}(F(\bar{F}(\alpha,x),{}_{y}e)) =$$
$$= \left[F(\cdot,y^{-1})\Big|_{F(\Gamma,e_y)}\right]^{-1}(\bar{F}(\alpha,x)) =$$
$$= \left\{\left[F(\cdot,y^{-1})\Big|_{F(\Gamma,e_y)}\right]^{-1}\circ\left[F(\cdot,x^{-1})\Big|_{F(\Gamma,e_x)}\right]^{-1}\right\}(F(\alpha,{}_{x}e)) =$$

$$= \left[F(\cdot, x^{-1})\Big|_{F(\Gamma, e_x)} \circ F(\cdot, y^{-1})\Big|_{F(\Gamma, e_y)} \right]^{-1} (F(\alpha, {}_x e)) =$$

$$= \left[F(\cdot, y^{-1} \cdot x^{-1})\Big|_{F(\Gamma, e_y)} \right]^{-1} (F(\alpha, {}_x e)) =$$

$$= \left[F(\cdot, (x \cdot y))^{-1}\Big|_{F(\Gamma, e_y)} \right]^{-1} (F(\alpha, {}_x e)) =$$

$$= \bar{F}(\alpha, x \cdot y),$$

since $F(\Gamma, {}_y e) = F(\Gamma, e_x)$, ${}_{x \cdot y} e = {}_x e$ and $e_{x \cdot y} = e_y$ for $(x,y) \in D$. .

In cases d) and e) we argue in a similar way, using a) and b) respectively. Case f) is obvious.

From the above consideration by (4) we obtain that the solution F of the translation equation defined on the set $\Gamma \times R$ and fulfilling condition (5) is uniquely extendable to the solution $\bar{F}$ defined by (6) on the set $\Gamma \times E$.

The above theorem yields a generalization of theorem 1 from [2]. In paper [2] theorem 1 is formulated for a sub-semigroup P of a group $(G; \cdot)$ such that $P \cup P^{-1} = G$. Notice that if $e_x = {}_x e$ then the function $F(\cdot, x)$ is a bijection on its codomain. However, if the structure R has the only one unit then condition (5) is equivalent to the fact that $\{F(\cdot, x)\}_{x \in R}$ is a family of functions being bijections on the common codomain.

R e f e r e n c e s

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