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Annales Academiae Paedagogicae Cracoviensis

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Elżbieta Bratuszewska

Initial boundary value problem for a mechanical system with local stroke change of stiffness

Abstract. The aim of this paper is to present a new method of solving the initial boundary value problem for a mechanical system with local stroke change of stiffness. The method is based on the theory of distributions.

1. Introduction

We consider the small vibrations of an Euler beam of the length l in its symmetry plane, with stroke change of stiffness of the beam described by the function $\beta\colon \langle 0, l\rangle \longrightarrow \bar{\mathbb{R}}$ defined as follows

$$\beta(x) = \begin{cases} EJ, & x \in \langle 0, x_1) \cup (x_2, l\rangle, \\ 0, & x \in \{x_1, x_2\} \\ +\infty, & x \in (x_1, x_2). \end{cases} \tag{1}$$

The two joints of the beam are located respectively at the points with abscissae x_1 and x_2, and their small vibrations are given by (see [3], [4], [5])

$$\left[\frac{\partial U}{\partial x}(x_i^+, t) - \frac{\partial U}{\partial x}(x_i^-, t)\right] \delta''_{x_i}; \qquad i = 1, 2.$$

The main idea of this formula comes from the paper [3] and is based on a sequential approach. The function $U(x,t)$ is the deflection at the point $x \in \langle 0, l\rangle$ at the moment t, δ''_{x_i} denotes the second derivative of the Dirac distribution δ_{x_i} concentrated at the point x_i.

The small vibrations of the system under consideration are described by the equation

$$\frac{\partial^2}{\partial x^2}\left[\beta(x)\frac{\partial^2 U}{\partial x^2} + \alpha_0 J\frac{\partial^3 U}{\partial x^2 \partial t}\right] + h(x)\frac{\partial^2 U}{\partial t^2} = f(x,t), \tag{2}$$

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where $\alpha_0 J = \text{const}$; $h(x)$ denotes the distribution of the beam masses; $f(x,t)$ is the distribution of the external forces applied to the beam in its symmetry plane.

Since there are the joints respectively at points x_1 and x_2, the function

$$U(.,t) \in C^0(\langle 0,l\rangle) \cap C^4(\langle 0,l\rangle \setminus \{x_1,x_2\})$$

and it is linear in the interval (x_1,x_2) with respect to the absolute stiffness of this element.

We could describe the vibration of our beam by the methods of classical mathematical analysis but this requires taking into consideration the vibrations of three elements of a beam $\langle 0,x_1), (x_1,x_2)$ and $(x_1,l\rangle$. It is both arduous and labour-concerning.

The purpose of this paper is to present a new method of determining the beam vibrations. The point of the matter is that the real situation is modelled as follows. The absolutely stiff part of the beam we describe as the only point of mass located in x_1 (i.e., we let $x_2 = x_1 = \frac{1}{2}(x_1+x_2)$) which bears all dynamical reactions that appear in this stiff part of the beam (this needs a distributional description). The method gives us the possibility to find the discontinuous at x_1 solutions of the substitute beam. It is achieved by introducing δ_{x_1}''' into the initial boundary problem substituting the real problem. Then we return to the real problem by the connection of the points x_1 and x_2 fitting the segment $y(x,t) = p(t)x + q(t)$ in the interval $\langle x_1,x_2\rangle$ to make continuous the solution of the real problem.

The mass of the stiff element of the beam and its dynamical reaction located at the point x_1 are analytically characterized by $\rho F(x_2 - x_1)$ – the mass of the absolutely stiff part of the beam (x_1,x_2) and by $\frac{1}{12}\rho_1 F_1(x_2 - x_1)^2$ – its moment of inertia, computed with respect to the middle point $\frac{1}{2}(x_1 + x_2)$ of this part of the beam. The symbols ρ, ρ_1, F, F_1 stand for the densities and for the cross-section areas of the parts $\langle 0,x_1) \cup (x_2,l\rangle$ and (x_1,x_2) of the beam, respectively. The equation for the function $W(x,t), x \in \langle 0,l\rangle \setminus (x_1,x_2)$ representing vibrations of the substitute beam is of form (10), cf. Section 3.

The knowledge of $W(x,t)$, the solution of the initial boundary problem of the substitute beam, and of the fact that it is impossible to bend the absolutely stiff element (its vibrations are planar) provide the possibility of construction of the function $U(x,t)$, the solution of the real beam in the interval $\langle 0,l\rangle$, via the formula

$$U(x,t) = \begin{cases} W(x,t), & x \in \langle 0,x_1) \cup (x_2,l\rangle \\ a(t)x + b(t), & x \in (x_1,x_2). \end{cases} \tag{3}$$

Since $W(x_1^-,t) = a(t)x_1 + b(t)$, $W(x_2^+,t) = a(t)x_2 + b(t)$ the continuity at x_1 and x_2 yields

$$a(t) = \frac{W(x_2^+,t) - W(x_1^-,t)}{x_2 - x_1}, \qquad b(t) = \frac{x_1 W(x_2^+,t) - x_2 W(x_1^-,t)}{x_2 - x_1}. \tag{4}$$

2. Preliminaries

Let us define the internal damping of the beam

$$\alpha(x) = \begin{cases} \alpha_0\,, & x \in \langle 0, x_1) \cup (x_2, l\rangle \\ 0, & x \in \{x_1, x_2\} \\ \alpha_1\,, & x \in (x_1, x_2) \end{cases} \tag{5}$$

and the stiffness of the beam

$$\beta(x) = \begin{cases} EJ, & x \in \langle 0, x_1) \cup (x_2, l\rangle \\ 0, & x \in \{x_1, x_2\} \\ M, & x \in (x_1, x_2) \end{cases}$$

Here E denotes the Young modulus, J is the axial moment of inertia, M is defined as the stiffness (we assume that $M = \text{const}$).

One edgepoint of the beam is fixed while the other is slidable.

Now let au assume that there are no external forces and therefore, the small transversal vibrations of the beam under consideration are described by the formula

$$\frac{\partial^2}{\partial x^2}\left(\beta(x)\frac{\partial^2 U}{\partial x^2} + \alpha(x)\frac{\partial^3 U}{\partial x^2 \partial t}\right) + h(x)\frac{\partial^2 U}{\partial t^2} = 0 \tag{6}$$

$U(.,t) \in C^0(\langle 0,l\rangle) \cap C^4(\langle 0,x_1) \cup (x_2,l\rangle)$, $U(x,t) = ax + b$, $x \in (x_1,x_2)$, a, b-const.; $h(x) = \rho F + \rho_1 F_1 (x_2 - x_1)\delta_{x_1}$.

The constants a and b are chosen so that $U(.,t) \in C^0(\langle 0,l\rangle)$.

The boundary conditions are

$$U(0,t) = 0, \qquad U(l,t) = 0, \qquad \frac{\partial U}{\partial x}(0,t) = 0, \qquad \frac{\partial U}{\partial x}(l,t) = 0. \tag{7}$$

The initial conditions are as follows

$$\begin{aligned} U(x,0) = \varphi_1(x), \qquad & \frac{\partial U}{\partial t}(x,0) = \varphi_2(x) \qquad \text{for } x \in \langle 0,l\rangle \\ \varphi_1(0) = \varphi_1(l), \qquad & \varphi_2(0) = \varphi_2(l). \end{aligned} \tag{8}$$

Let us assume *the physical conditions*:

$$\frac{\partial^2 U}{\partial x^2}(x_i^-,t) = \frac{\partial^2 U}{\partial x^2}(x_i^+,t) = 0, \qquad i = 1,2. \tag{9}$$

3. Method of generalized functions

The initial boundary problem (3), (4), (5), (6) of the vibrations of the substitute beam is described by the formula

$$
\begin{aligned}
&(a) \quad EJ\frac{\partial^4 W}{\partial x^4} + \alpha_0 J\frac{\partial^5 W}{\partial x^4 \partial t} + \left(\rho F + \rho_1 F_1 (x_2 - x_1)\delta_{x_1}\right)\frac{\partial^2 W}{\partial t^2} \\
&(b) \quad + \gamma_1 \left(\frac{\partial^3 W(x_1^+, t)}{\partial x \partial t^2} - \frac{\partial^3 W(x_1^-, t)}{\partial x \partial t^2}\right)\delta''_{x_1} \\
&(c) \quad + \frac{\rho_1 F_1 (x_2 - x_1)^2}{12}\left(\frac{\partial^2 W(x_1^+, t)}{\partial t^2} - \frac{\partial^2 W(x_1^-, t)}{\partial t^2}\right)\delta'_{x_1} \qquad (10) \\
&(d) \quad + \gamma_2 \left(\frac{\partial^5 W(x_1^+, t)}{\partial x^3 \partial t^2} - \frac{\partial^5 W(x_1^-, t)}{\partial x^3 \partial t^2}\right)\delta'''_{x_1} \\
&\quad = 0
\end{aligned}
$$

where

(a) the coefficients at δ_{x_1} and δ'_{x_1} are used to describe the dynamics of the part (x_1, x_2); (see [4])

(b) δ''_{x_1} refers to the joint of the substitute beam; (see [5])

(d) δ'''_{x_1} leads to the discontinuous solution;

(c) δ'_{x_1} characterizes the pair of the forces in x_1; (see [4]),

and γ_1 and γ_2 are the parameters that fit to obtain the continuous solution of the problem and to make the units matching.

We are using the Fourier method to solve the eigenproblem associated to the substitute problem under consideration.

Let us assume (with the constant p and q)

$$W(x,t) = X(x)T(t), \qquad (11)$$

$$a(t) = pT(t), \qquad b(t) = qT(t).$$

Substituting (11) into (6), after some calculations, we obtain

$$\frac{\ddot{T}}{T + \frac{\lambda_0}{E}\dot{T}} = \frac{-EJX^{IV}}{D} = -\omega^2,$$

where

$$\begin{aligned} D = &(\rho F + \rho_1 F_1 \delta_{x_1}(x_2 - x_1))X \\ &+ \gamma_1 (X'(x_1^+) - X'(x_1^-))\delta''_{x_1} \\ &+ \gamma_2 (X'''(x_1^+) - X'''(x_1^-))\delta'''_{x_1} \\ &+ \frac{1}{12}\rho_1 F_1 (x_2 - x_1)^2 (X(x_1^+) - X(x_1^-))\delta'_{x_1} \end{aligned}$$

and the constant $-\omega^2$ is negative to obtain positive eigenvalues.

Hence

$$\ddot{T} + \frac{\alpha_0 \omega^2}{E}\dot{T} + \omega^2 T = 0, \tag{12}$$

$(\dot{} = \frac{d}{dt})$ and

$$\begin{aligned} X^{IV} - \lambda^4 X = &\frac{\rho_1 F_1 (x_2 - x_1)\omega^2}{EJ} X \delta_{x_1} \\ &+ \frac{\rho_1 F_1 (x_2 - x_1)^2 \omega^2}{12EJ}(X(x_1^+) - X(x_1^-))\delta'_{x_1} \\ &+ \frac{\gamma_1 \omega^2}{EJ}(X'(x_1^+) - X'(x_1^-))\delta''_{x_1} \\ &+ \frac{\gamma_2 \omega^2}{EJ}(X'''(x_1^+) - X'''(x_1^-))\delta'''_{x_1}\,, \end{aligned} \tag{13}$$

where $\lambda^4 = \frac{\rho F \omega^2}{EJ}$.

4. The solution of equation (13)

We write, for short,

$$\xi := \frac{x_1 + x_2}{2} = x_1 = x_2\,, \qquad \Theta := \frac{\omega^2}{2EJ}, \qquad \eta := \frac{x_2 - x_1}{2}.$$

The general solution of (13) is given by (we omit the standard calculations)

$$\begin{aligned} X(x) = &P\cos\lambda x + Q\sin\lambda x + R\ \mathrm{ch}\lambda x + S\ \mathrm{sh}\lambda x \\ &+ \frac{4\rho_1 F_1 \eta \Theta}{\lambda^4} X(\xi) H(x - \xi)\left[\mathrm{sh}\lambda(x-\xi) - \sin\lambda(x-\xi)\right] \\ &+ \frac{\rho_1 F_1 \eta^2 \Theta}{3\lambda^2}\left(X'(x_1^+) - X'(x_1^-)\right) H(x-\xi)\left[\mathrm{ch}\lambda(x-\xi) - \cos\lambda(x-\xi)\right] \\ &+ \frac{\gamma_1 \Theta}{\lambda}\left(X(x_1^+) - X(x_1^-)\right) H(x-\xi)\left[\mathrm{sh}\lambda(x-\xi) + \sin\lambda(x-\xi)\right] \\ &+ \gamma_2 \Theta\left(X'''(x_2^+) - X'''(x_2^-)\right) H(x-\xi)\left[\mathrm{ch}\lambda(x-\xi) + \cos\lambda(x-\xi)\right], \end{aligned}$$

where H denotes the Heaviside function of the unit jump, i.e.,

$$H(x-c)=1,\ x>c, \qquad H(x-c)=0,\ x<c, \qquad H(x-c)=\frac{1}{2},\ x=c.$$

According to the idea of the method when adapting the solution of the equation (13) to the intervals $\langle 0,x_1)\cup(x_2,l\rangle$ we get

$$\begin{aligned}
X(x) = {} & P\cos\lambda x + Q\sin\lambda x + R\mathrm{ch}\lambda x + S\mathrm{sh}\lambda x \\
& + \frac{4\rho_1F_1\eta\Theta}{\lambda^4}X(\frac{x_1+x_2}{2})H(x-\frac{x_1+x_2}{2}) \\
& \times\left[\mathrm{sh}\lambda(x-\frac{x_1+x_2}{2})-\sin\lambda(x-\frac{x_1+x_2}{2})\right] \\
& + \frac{3\rho_1F_1\eta^2\Theta}{2\lambda^2}\{H(x-x_2)X(x_2^+)\left[\mathrm{ch}\lambda(x-x_2)-\cos\lambda(x-x_2)\right] \\
& - H(x-x_1)X(x_1^-)\left[\mathrm{ch}\lambda(x-x_1)-\cos\lambda(x-x_1)\right]\}+ \\
& + \gamma_1\Theta\{H(x-x_1)(p-X'(x_1^-))\left[\mathrm{sh}\lambda(x-x_1)+\sin\lambda(x-x_1)\right] \\
& + H(x-x_2)(X'(x_2^+)-p)\left[\mathrm{sh}\lambda(x-x_2)+\sin\lambda(x-x_2)\right]\} \\
& + \gamma_2\Theta\{-H(x-x_1)X'''(x_1^-)\left[\mathrm{ch}\lambda(x-x_1)+\cos\lambda(x-x_1)\right] \\
& + H(x-x_2)X'''(x_2^+)\left[\mathrm{ch}\lambda(x-x_2)+\cos\lambda(x-x_2)\right]\}.
\end{aligned}$$

According to the shape of the function

$$W(x,t)=X(x)T(t)$$

and the initial-boundary conditions given by the formulas (8), (9) we obtain the system of thirteen linear equations with thirteen unknown values

P, Q, R, S, $X(\xi)$, $X(x_1^-)$, $X(x_2^+)$, $X'(x_1^-)$, $X'(x_2^+)$, $X'''(x_1^-)$, $X'''(x_2^+)$, p, q.

The equations are:

1. $X(0)=0 \Longleftrightarrow P+Q=0,$

2. $X'(0)=0 \Longleftrightarrow R+S=0,$

3. $X(l)=0 \Longleftrightarrow P\cos\lambda l+Q\sin\lambda l+R\mathrm{ch}\lambda l+S\mathrm{ch}\lambda l=0,$

4. $X'(l)=0 \Longleftrightarrow -P\sin\lambda l+Q\cos\lambda l+R\mathrm{sh}\lambda l+S\mathrm{ch}\lambda l=0,$

5. $X''(x_1)=0 \Longleftrightarrow -\lambda^2P\cos\lambda x_1-\lambda^2Q\sin\lambda x_1+\lambda^2R\mathrm{ch}\lambda x_1 + \lambda^2S\mathrm{sh}\lambda x_1-\frac{1}{3}\rho_1F_1\eta^2\Theta X(x_1^-)=0,$

6. $$X''(x_2) = 0 \Longleftrightarrow -\lambda^2 P\cos\lambda x_2 - \lambda^2 Q\sin\lambda x_2 + \lambda^2 R\mathrm{ch}\lambda x_2 + \lambda^2 S\mathrm{sh}\lambda x_2 + \frac{4\rho_1 F_1\eta^2\Theta}{\lambda^2}X(\xi)[\mathrm{sh}\lambda\eta + \sin\lambda\eta] + \gamma_1\Theta[p - X'(x_1^-)]\lambda[\mathrm{sh}2\lambda\eta - \sin 2\lambda\eta] - \gamma_2\Theta X'''(x_1^-)\lambda^2[\mathrm{ch}2\lambda\eta - \cos 2\lambda\eta] = 0,$$

7. $$X(\xi) = P\cos\lambda\xi + Q\sin\lambda\xi + R\mathrm{ch}\lambda\xi + S\mathrm{sh}\lambda\xi + \frac{\rho_1 F_1\eta^2\Theta}{3\lambda^2}X(x_2^+)[\mathrm{ch}\lambda\eta - \cos\lambda\eta] + \frac{\gamma_1\Theta}{\lambda}[X'(x_2^+) - p][\mathrm{sh}\lambda\eta + \sin\lambda\eta] - \gamma_2\Theta X'''(x_2^+)[\mathrm{ch}\lambda\eta + \cos\lambda\eta],$$

8. $$X(x_1^-) = P\cos\lambda x_1 + Q\sin\lambda x_1 + R\mathrm{ch}\lambda x_1 + S\mathrm{sh}x_1,$$

9. $$X(x_2^+) = P\cos\lambda x_2 + Q\sin\lambda x_2 + R\mathrm{ch}\lambda x_2 + S\mathrm{sh}x_2 + \frac{4\rho_1 F_1\eta^2\Theta}{\lambda^4}X(\xi)[\mathrm{sh}\lambda\eta - \sin\lambda\eta] - \frac{\rho_1 F_1\eta^2\Theta}{3\lambda^4}X(x_1^-)[\mathrm{ch}2\lambda\eta + \cos 2\lambda\eta] + \frac{\gamma_1\Theta}{\lambda}[p - X'(x_1^-)][\mathrm{sh}2\lambda\eta + \sin 2\lambda\eta] - \frac{\gamma_2\Theta}{\lambda}X'''(x_1^-)[\mathrm{ch}2\lambda\eta + \cos 2\lambda\eta] + 2\gamma_2\Theta X'''(x_2^+),$$

10. $$X'(x_1^-) = -P\lambda\sin\lambda x_1 + Q\lambda\cos\lambda x_1 + R\lambda\mathrm{sh}\lambda x_1 + S\lambda\mathrm{ch}\lambda x_1,$$

11. $$X'(x_2^+) = -P\lambda\sin\lambda x_2 + Q\lambda\cos\lambda x_2 + R\lambda\mathrm{sh}\lambda x_2 + S\lambda\mathrm{ch}\lambda x_2 + \frac{2\rho_1 F_1\eta\Theta}{\lambda^3}X(\xi)[\mathrm{ch}2\lambda\eta - \cos 2\lambda\eta] + \frac{\gamma_1\Theta}{\lambda}[p - X'(x_1^-)][\mathrm{ch}2\lambda\eta + \cos 2\lambda\eta] + \gamma_1\Theta[X'(x_2^+) - q][\mathrm{ch}2\lambda\eta + \cos 2\lambda\eta] - \frac{\gamma_2\Theta}{\lambda}X'''(x_1^-)\lambda[\mathrm{sh}2\lambda\eta - \sin 2\lambda\eta],$$

12. $$X'''(x_1^-) = \lambda^3 P\sin\lambda x_1 - \lambda^3 Q\cos\lambda x_1 + \lambda^3 P\mathrm{ch}\lambda x_1 + \lambda^3 S\mathrm{sh}\lambda x_1,$$

13. $$X'''(x_2^+) = \lambda^3 P\sin\lambda x_2 - \lambda^3 Q\cos\lambda x_2 + \lambda^3 P\mathrm{ch}\lambda x_2 + \lambda^3 S\mathrm{sh}\lambda x_2 + \frac{2\rho_1 F_1\eta\Theta}{\lambda}X(\xi)[\mathrm{ch}\lambda\eta + \cos\lambda\eta]$$

$$
\begin{aligned}
&-\frac{1}{6}\lambda\rho_1 F_1\eta\Theta X(x_1^-)[\mathrm{sh}2\lambda\eta - \sin 2\lambda\eta] \\
&+\lambda^2\gamma_1\Theta[p - X'(x_1^-)][\mathrm{ch}2\lambda\eta - \cos 2\lambda\eta] \\
&-\lambda^3\gamma_2\Theta X'''(x_1^-)[\mathrm{sh}2\lambda\eta + \sin 2\lambda\eta].
\end{aligned}
$$

The system of the linear equations given above has infinite number of solutions ([2]) and it represents the eigenproblem under consideration. The details of the calculations as well as the explicit solution of the initial boundary problem will be dealt with in a subsequent paper. The problem considered in this paper is also discussed in [5] but in the approach of L. Schwartz [4].

Assuming that the determinant of the matrix of the system 1.-13. of the linear equations is equal to zero we obtain the eigenvalues equation. There is a countable number of such eigenvalues λ_n so we can create an increasing sequence of λ_n . In consequence we put λ_n , T_n , $X_n(x)$, p_n , q_n into the formulas (12) (13) instead of, respectively, λ, T, X, p, q, and form the solution of (10)

$$
W(x,t) = \sum_{n=1}^{\infty} \tilde{X}_n(x)(a_n T_{1n}(t) + b_n T_{2n}(t)),
$$

where $T_{1n}(t)$ and $T_{2n}(t)$ are the linearly independent particular solutions of (12) while

$$
\tilde{X}_n(x) = \begin{cases} X_n(x), & x \in \langle 0, x_1) \cup (x_2, l\rangle \\ p_n x + q_n\,, & x \in (x_1, x_2) \end{cases}.
$$

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Bogdan Choczewski

International meetings organized by Polish schools on functional equations

Abstract. This is an extended version of the talk given on the *10th International Conference on Functional Equations and Inequalities* held in September, 2005, in the Mathematical Research and Conference Center of the Polish Academy of Sciences at Będlewo, cf. [A10]. There is presented, first of all, an outline of the history of the ten meetings, organized in Poland by the Institute of Mathematics of the Pedagogical University of Cracow from 1984 to 2005. The article begins with a description of meetings held in Hungary, prior to the ICFEIs, and ends with informations on other conferences and seminars organized by Polish schools of functional equations in the years 1989 - 2005 (among them, on the *International Symposia on Functional Equations*: *27* (1989), *34* (1996) and *40* (2002)).

1. Introduction

The organizing in Poland, by the Institute of Mathematics of the Pedagogical University of Kraków, of the series of the *International Conferences on Functional Equations and Inequalities* (ICFEIs) had been started in 1984. They might be considered as a continuation of meetings held in Hungary in sixties and seventies of the 20th century which were supposed to be satellite ones to the *International Symposia on Functional Equations* (ISFEs). The series of the Symposia had been invented and since 1962 organized by Professor János Aczél (University of Waterloo, ON, Canada), a prominent Hungarian mathematician, who emigrated to Canada in 1964. Thanks mostly to him functional equations became a rapidly developing domain of research in contemporary mathematics.

The "birth" of functional equations in Poland we owe to Professor Stanisław Gołąb and his scientific cooperation with Professor Aczél. Then, in early sixties, a school of functional equations, especially those in a single variable, has been created at the Silesian University in Katowice by Professor Marek Kuczma, encouraged by Professor Gołąb, his Master and Teacher. Marek Kuczma's scientific carreer begun in 1956 at Kraków, where in that time were investigated, mostly by Professor Gołąb and his students: Zenon Moszner and Andrzej Zajtz, mainly functional equations in several variables, connected with the theory of geometric objects. In 1967, Professor Kuczma organized, in cooperation with Professor Gołąb, the first international conference on functional equations in Poland.

The article is edited as follows. We first mention Hungarian conferences (held from 1962 to 1986) and then describe in general, with some statistical data included, all the 10 ICFEIs (from 1984 to 2005). More on the separate ICFEIs is found in the three sections describing the first two, next six and the last two meetings. For each of the ICFEIs we provide informations under the headings: **A. Guests** (names of those colleagues from behind Hungary and Poland who came this time only), **B. Problems** (names of contributors in proposing and solving them during the meeting, with topics labelled only), **C. Papers** (presented on the meeting and printed in the volume in which the respective report is printed). We also remind, according to author's choice, major events characteristic for the meeting in question. The paper is concluded with registering other international meetings of specialists in the fields represented at the ICFEIs, held in Poland after 1967.

2. Conferences in Hungary

In the times of the "iron courtain" in Europe it was an uneasy task for people from the East to attend any conference held on the West. Our coming to the ISFEs in those times was possible for many of us only because Professor Aczél and colleagues stimulated by him were always going to an effort in supporting us in various ways. This situation encouraged members of the Hungarian school of functional equations – students of Professor Aczél and their students – to initiate in sixties a series of international conferences on functional equations and inequalities, paraller to the ISFEs, the first four of which were held in 1962, 1963, 1965 and 1966 at Oberwofach.

The first international conference in Hungary on which functional equations appeared as a separate topic was that held in 1962 at Sárospatak. Then three *Colloquia on Functional Equations* took place in the University of Heavy Industry at Miskolc, in 1966, 1968 and 1970. The main organizer of these meetings was Professor Endre Vincze. The next meetings were organized by the Institute of Mathematics of the University of Debrecen under the chairmanship of Professor Zoltán Daróczy in 1973 and 1979 at Debrecen and in 1986 at Noszvaj. They all were addressed mainly to younger Hungarian and Polish mathematicians, who rather would be unable to participate in the ISFEs as well as in the series of "General Inequalities" held several times at Oberwolfach.

From the report on the *3rd Colloquium at Miskolc* (1970), compiled by M. Hosszú and published in the Aequationes Mathematicae (cf. [B1]), we learn that its Organizing Committee consisted of Professors: Z. Daróczy (Debrecen), A. Moór (Sopron) and Gy. Gáspár, M. Hosszú and E. Vincze (all from Miskolc) and that among 39 participants from 5 countries there were 21 Hungarian and 11 Polish ones (in particular, Professors S. Gołąb and M. Kuczma).

The report on the second *Debrecen Symposium* (1979) was compiled by the secretary of the meeting, K. Lajkó, and published in the Publicationes Mathematicae Debrecen (cf. [B2]). Among 70 participants from 13 countries there were 32 from Hungary (with the memmbers of the Organizing Committee, Professors J. Balázs, Z. Daróczy, I. Fenyö, M. Hosszú, A. Kósa, L. Losonczi, E. Vincze and also I. Kátai, A. Moór and E. Gesztelyi, A. Járai, Gy. Maksa, Zs. Páles, Gy. Szabó, Á. Száz, L. Szekelyhidi); 20 from Poland (with Prof. M. Kuczma and K. Baron, D. Brydak (who came from Port Harcourt, Nigeria), B. Choczewski, S. Czerwik, R. Ger, W. Jarczyk, Z. Kominek, M.E. Kuczma, S. Midura, K. Nikodem, A. Smajdor, S. Wołodźko, M.C. Zdun) and mathematicians known in their fields: D.D. Bainov (Bulgaria, differential-functional equations), W. Benz (West Germany, geometry), S. Bilinski (Yugoslavia, geometry), L. Collatz (West Germany, numerical mathematics), H.-H. Kairies (West Germany, functional equations), P.G. Peljukh (USSR, differential and functional equations), L. Reich (Austria, iteration theory), A.N. Sarkovskii (USSR, functional and differential equations), D.R. Snow (USA, functional equations), W. Walter (West Germany, mathematical analysis).

3. Ten ICFEIs in general (with statistical data)

Despite of a rather difficult situation in Poland, the Institute of Mathematics of the Pedagogical University of Kraków decided to realize in 1984 the idea of Professor Dobiesław Brydak and organized in Poland a conference like the Hungarian ones, presented in the preceding section. The first ICFEI was the beginning of a series of conferences that became biannual since that held in 1991 at Koninki. They have got the nickname "Brydak's conferences", as Professor Brydak was the chairman of the organizing committee of all the nine ICFEIs, and of the scientific committee of the tenth one. To keep the character of Hungarian meetings, the invitations were sent to leading specialists in the field, but also everybody recommended by her or his master could be invited and give a talk on an ICFEI.

Professor Brydak was supported in his organizational work by Professor Bogdan Choczewski (except of 1995, co-chairman in) and – in the years 1984 and 1987 – by Professor Józef Tabor. The following colleagues acted as scientific secretaries: Lech Anczyk (1st), Erwin Turdza (2nd), Barbara Wilk (3rd), Ewa Dudek (4th), Zbigniew Leśniak (5th and 6th), Jacek Chmieliński (from 7th to 10th).

Since 1991 (3rd ICFEI) in the course of preparation of meetings and at their places in the office as the secretary worked Miss Janina Wiercioch and as the technical assistent – Mr Władysław Wilk. They were helped by: Miss Anna Bańdur (1993, 1999 (as Mrs. Grabiec)), Miss Danuta Bogocz (1993), Miss Ewa Dudek (from 1991 to 2001), Dr Paweł Solarz (2003, 2005), Dr Joanna Szczawińska (2003).

In the table below some statistical data concerning all the 10 meetings are collected (the columns “All” and “Foreign” refer to the number of participants, and “P&R” means “Problems&Remarks”). The data are taken from the complete list of participants which has been compiled by Miss J. Wiercioch.

No	Time;place	All	Foreign	Sessions	Talks	P&R	Countries
1.	May 27-June 2, 1984; Sielpia	59	9	8	41	8	8
2.	June 21-27, 1987; Szczawnica	63	24	16	44	10	10
3.	September 3-9, 1991; Koninki	80	21	14	54	9	12
4.	February 15-22, 1993; Krynica	73	16	19	43	17	8
5.	September 4-9, 1995; Złockie	63	27	19	55	7	14
6.	June 1-7, 1997; Złockie	63	18	19	52	9	11
7.	September 12-18, 1999; Złockie	73	28	19	63	14	14
8.	September 10-15, 2001; Złockie	56	16	18	47	7	8
9.	September 7-13, 2003; Złockie	57	22	17	51	8	8
10.	September 11-17, 2005; Będlewo	64	14	20	59	11	10
	Total	650	194	169	509	100	

Here the numbers of attendances: 650 (All) and 194 (Foreign) were obtained as weighted sums with the number of components equal to those of different persons who participate in an ICFE: 230 (All) and 95 (Foreign). Altogether in 10 ICFEIs there were represented 27 countries. The number of Hungarian participants was 61 all, what means that except of 10th ICFEI, it was the greatest from among that of foreign guests, and was equal (consecutively): 2, 8, 4, 5, 8, 5, 8, 8, 12, 1. It was also registered that 58% of foreign participants and 38% of Polish ones took part in a single meeting only.

We now exhibit the names of the participants who attended the maximal number of our conferences (in brackets: the years of missed meetings).

Polish participants.

All ten: Karol Baron, Dobiesław Brydak, Roman Ger, Witold Jarczyk, Zygfryd Kominek, Maciej Sablik.

Nine: Bogdan Choczewski (1995), Kazimierz Nikodem (1999), Andrzej Smajdor (1993), Józef Tabor (2003), Marek Cezary Zdun (2001).

Eight: Jacek Chmieliński (1984, 1987), Marek Czerni (2003, 2005), Zbigniew Leśniak (1984, 1987), Janusz Matkowski (2001, 2003), Zenon Moszner (2003, 2005), Zbigniew Powązka (1989, 2005), Wilhelmina Smajdor (1991, 1993), Janusz Walorski (1984, 1987).

Foreign participants.

Eight: Wolfgang Förg-Rob (Austria; 1984, 1997), Hans-Heinrich Kairies (Germany; 1984, 1987), Peter Volkmann (Germany; 1995, 1997).

Six: Zoltán Daróczy, Károly Lajkó, Zsolt Páles (all from Hungary).

Five: Nicole Brillouët-Belluot (France), Roland Girgensohn (Germany), Laszló Losonczi (Hungary), Gyula Maksa (Hungary), František Neuman (Czech Republic), Adolf Schleiermacher (Germany).

We conclude this section with the names of our late colleagues, indicating in brackets the years they had attended the ICFEI:

Zbigniew Gajda, Silesian University in Katowice (1984, 1987, 1991), †1992;

Zbigniew Wilczyński, Pedagogical University in Rzeszów (1984, 1987, 1991), †1994;

György Szabó, University of Debrecen (1987, 1993, 1995), †1996;

György Targonski, University of Marburg (1991, 1995, 1997), †1998;

Jerzy Popenda, University of Technology in Poznań (1991), †1999;

Eric Russell Love, University of Melbourne (1984, 1987, 1991, 1995), †2001;

Ioana Cioranescu, University of Puerto Rico (1997), †2003;

Wojciech Ślepak, Technical-Humanistic Academy in Bielsko-Biała (1993), †2005.

4. First two meetings

The first ICFEI was held in 1984 at Sielpia (in Kielce region) in the rest house "Komunalni". The welcoming address was given by Professor Zenon Moszner, Rector Magnificus of the Pedagogical University of Kraków. There were 9 foreign (listed below) and 30 Polish participants, cf. [A1].

Our foreign gusests came from 7 countries: Gian Luigi Forti (Italy), Eric Russell Love (Australia), Bertalan Nagy (Hungary), František Neuman (Czechoslovakia), Jürg Rätz (Switzerland), Ludwig Reich (Austria), Endre Vincze (Hungary), Peter Volkmann (West Germany), Roger James Wallace (Australia).

Professor Dieter Keith Ross from the La Trobe University (Australia) died at the age of 51 of a fatal heart attack at Frankfurt Airport on 27th May, 1984, while on his way to our meeting. His lecture was delivered by R.J. Wallace (cf. the abstract in [A1], 243), his student and co-worker from the same University. An obituary note written by E.R. Love is printed in [A1], 191-192.

There were 15 participants of the 1st ICFEI who came also to the 10th ICFEI. Among them Prof. Volkmann was the only foreign guest, and the Polish ones were (actual affiliation): Karol Baron (Uniwersytet Śląski w Katowicach), Dobiesław Brydak (Akademia Pedagogiczna w Krakowie), Bogdan Choczewski (Akademia Górniczo-Hutnicza w Krakowie), Roman Ger (Uniwersytet Śląski w Katowicach), Witold Jarczyk (Uniwersytet Zielonogórski), Zygfyd Kominek (Uniwersytet Śląski w Katowicach), Andrzej Mach (Akademia Pedagogiczna w Kielcach), Janusz Matkowski (Uniwersytet Zielonogórski), Kazimierz Nikodem (Akademia Techniczo-Humanistyczna w Bielsku-Białej), Maciej Sablik (Uniwersytet Śląski w Katowicach), Andrzej Smajdor (Akademia Pedagogiczna w Krakowie), Wilhelmina Smajdor (Politechnika Śląska w Gliwicach), Józef Tabor, (Uniwersytet Rzeszowski), Marek Cezary Zdun (Akademia Pedagogiczna w Krakowie).

Let us note that they all are now leading specialists in the field of functional equations in their centres of research, 7 of them have the scientific title of professor of mathematical sciences, and 6 others have the position of university professors.

B. Problems ([A1], 255-265). G.L. Forti (1. Hyer's stability theorem for non Abelian groups, 2. From number theory, 3. Cauchy's equation on the graph of a given function), J. Gómez-Bayón (functional equation related to the problem of strength of tendons, presented by B. Choczewski, partially solved by F. Neuman), K. Nikodem (respresentation of two functions: Jensen convex majorized by Jensen concave; partially solved by Z. Kominek), L. Reich (determining families of multiplicative functions satisfying a special relation), Z. Skupień (a variant of generalized Aczél-Fuchs problem), J. Tabor (stemming from author's partial solution of Forti's Problem 1.).

C. Papers.

K. Baron, *A remark on the stability of the Cauchy equation*, 7-12.

J. Baster, Z. Moszner, J. Tabor, *On the stability of some class of functional equations*, 13-34.

J. Krzyszkowski, *On ordinary differential inequalities*, 95-108.

E.R. Love, *Inequalities like Opial's inequality*, 109-118.

F. Neuman, *A note on smoothness of the Stäckel transformation*, 147-152.

J. Tabor, *Rational iteration groups*, 153-176.

P. Volkmann, *Caractérisation de la fonction $f(x) = x$ par un système de deux inéquations fonctionnelles*, 177-184.

The 2nd ICFEI was held in 1987 at Szczawnica (a spa in the region of Pieniny mountains) in a health house “Papiernik”. The welcoming address was given by Professor Józef Tabor, Dean of the Faculty of Science and Technology of the Pedagogical University of Kraków. There were 24 foreign and 40 Polish participants, cf. [A2].

Professor Marek Kuczma who was unable to come because of illness, had sent his special greetings for all the participants.

A. Guests. Ravi P. Agarwal (Singapore), Jean Dhombres (France), Serge Dubuc (Canada), Detlef Gronau (Austria), Albert W. Marschall (Canada), Gradimir V. Milovanović (Yugoslavia), Jens Schwaiger (Austria), Abe Sklar (USA), Jaroslav Smítal (Czechoslovakia).

B. Problems ([A2], 223-234). Professor Zoltán Daróczy proposed two open problems: a) whether convexity is the difference property in de Bruijn’s sense, b) find continuous solutions of Hosszu’s functional inequality. They were partially solved during the meeting, a) by R. Ger and Z. Kominek and b) by Gy. Maksa, Zs. Páles, Z. Kominek and K. Nikodem. Other problems were posed by S. Dubuc (continuous solutions of a functional equation of infinite order); Zs. Páles (functional inequality with arithmetic and geometric means); L. Székelyhidi (groups on which biadditive functions are products of additive ones).

5. Six biannual meetings, 1991-2001

The 3rd ICFEI was held in 1991 at Koninki (a village at foot of Gorce mountains) in the hostel “Mechanik”. The welcoming address was given by Professor Zenon Uryga, Rector Magnificus of the Pedagogical University of Kraków. There were 21 foreign and 59 Polish participants, cf. [A3].

The meeting took place 3 months after the death, on June 13, 1991, of Professor Marek Kuczma, the founder of the School of Functional Equations at the Silesian University in Katowice. There was a commemorative session on Sunday, September 8, chaired by Roman Ger who presented the biography of Professor Kuczma. Then Marcin E. Kuczma through very personal reminiscences brought the audience nearer to his brother’s reach personality. In this spirit spoke also D. Brydak, B. Choczewski, J. Matkowski, L. Reich, Gy. Targonski and M. Taylor.

A. Guests. Vitor Manuel Carvalho das Neves (Portugal), Alice Chaljub-Simon (France), Bruce R. Ebanks (USA), Mark Taylor (Canada).

B. Problems ([A3], 30-36). G.L. Forti (three problems on a generalized plurality function; and to partial solutions of two of them were presented by Z. Gajda, Z. Kominek and M. Sablik; Z. Moszner (a characterization of solutions to the translation equation); Gy. Targonski (integral equation related to the notion of “phantom iterates”).

C. Papers. The proceedings were published in 1993 in the form of a booklet [A3] by the Institute of Mathematics of the Pedagogical University of Kraków. An obituary note appeared there as a seperate item:

D. Brydak, B. Choczewski, R. Ger, *Marek Kuczma (1935-1991)*, 3-4.

The 4th ICFEI was held in 1993 at Krynica in the house "Walcownik". The welcoming address was given by Professor Stefan Turnau, the Dean of the Faculty of Science and Technology of the Pedagogical University of Kraków. There were 16 foreign and 57 Polish participants, cf. [A4].

It was the only ICFEI held in winter. A lot of snow had created some communicational problems. The meeting was devoted mainly to stability theory of functional equations.

Only on this and on the preceding ICFEI there were several talks given by mathematicians from the University of Technology in Poznań dealing with difference equations, especially with asymptotic and oscillatory properties of their solutions. In 1991 there were 10, and in 1993 – 9 participants from Poznań, who came to both meetings with their leader, Professor Dobiesław Bobrowski.

A. Guests. Wolfgang Sander (Germany).

B. Problems ([A4], 59-68). Z. Daróczy and I. Kátai (characterization of the identity function on $\mathbb{N}$ via multiplicative and additive Cauchy equation, both on restricted domains); G.L. Forti (stability of an equation and the knowledge of the form of its solutions); L. Losonczi (general solution of a fundamental equation of information theory); Z. Powązka (a system of functional equations characterizing the operator of derivation; followed by a remark of M. Rose); Gy. Szabó (constancy of "set-periodic functions in $\mathbb{R}^n$); L. Székelyhi (stability of Hosszu's equation).

C. Papers. The proceedings were published in 1993 in the form of a booklet [A4] by the Institute of Mathematics of the Pedagogical University of Kraków. As a seperate item there appeared Professors Ger's paper (with the bibliography containing 83 items), based on his invited talk:

R. Ger, *A survey of recent results on stability of functional equations*, [A4], 5-36.

The 5th ICFEI was held in 1995 at Muszyna-Złockie in the house "Geofizyk". The meeting was opened by Professor Dobiesław Brydak, the Chairman of the Organizing Committee. There were 27 foreign and 36 Polish participants, cf. [A5].

The house "Geofizyk" became the place of the next four meetings.

A. Guests. Aurelio Cannizzo (Italy), Gregory Derfel (Israel), Mohamed Hmissi (Tunisia), Gorazd Lešnjak (Slovenia), Prasanna K. Sahoo (USA).

B. Problems ([A5], 195-198). I. Corovei (Jensen's functional equation on special groups); W. Jarczyk (on subadditive functions); Á. Száz (on continuous linear maps in normed spaces; solved by Zsolt Páles); Jacek Tabor (student of the Jagiellonian University, son of Józef Tabor – stability of the Cauchy equation on a restricted domain (he gave a talk on Hahn–Banach Theorem and made a remark on the unstability of the Hosszú functional equation on the unit interval)).

C. Papers.

R. Badora, *On the stability of the cosine functional equation*, 5-14.

A. Cannizzo, *Geometrical convexity and the Artin functional equation*, 37-48.

Z. Daróczy, *Functional inequalities for infinite series*, 57-62.

M. Hmissi, *On the functional equation of exit laws for lattice semigroups*, 63-72.

H.-H. Kairies, *Takagi's function and its functional equations*, 73-84.

Pl. Kannappan, T. Riedel, P. Sahoo, *On a generalization of a functional equation associated with Simpson's rule*, 85-102.

J. Matkowski, K. Nikodem, *Convex set-valued functions on* $(0,\infty)$ *and their conjugate*, 103-108.

Z. Moszner, *Sur une problème de Marley*, 109-118.

Z. Powązka, *Sur une équation fonctionnelle associée à l'équation de Jensen*, 119-128.

A. Smajdor, *Superadditive solutions of a functional equation*, 129-138.

Á. Száz, *Oscillation and integration characterizations of bounded a.e. continuous functions*, 139-170.

The 6th ICFEI was held in 1997 at the same place as the 5th ICFEI. The welcoming address was given by Professor Eugeniusz Wachnicki, the Dean of the Faculty of Science and Technology of the Pedagogical University of Kraków. There were 18 foreign and 45 Polish participants, cf. [A6].

In a talk given by Pofessor Bogdan Choczewski there was reminded the 30th anniversary of the first International Conference on Functional Equations ever held in Poland, which had been organized by Professors S. Gołąb and M. Kuczma at Zakopane in October 9-13, 1967 (there were 47 participants, among them 16 who came from East Gemany (1), Hungary (9), Rumania (3) and Yugoslavia (3)).

A. Guests. Ioana Cioranescu (Puerto Rico), Luigi Paganoni (Italy).

B. Problems ([A6], 146-152). B. Choczewski recapitulated an unsolved problem posed by Professor Gołąb in 1967 on determining the inner product with the aid of thenorm ([A6], 146). The problem has been solved by R. Ger ([A6], 147-149). Z. Daróczy quoted a problem posed in a Hungarian journal for high school students on a characterization of affine functions via a functional equation and asked for its elementary solution.

C. Papers.

B. Choczewski, *The First International Conference on Functional Equations in Poland: Zakopane 1967*, 5-14.

Z. Kominek, W. Wilczyński, *On sets for which the difference set is the whole space*, 45-52.

Z. Moszner, *L'équation de translation et l'équation de Sincov généralisée*, 53-72.

A. Smajdor, *Concave iteration semigroups of Jensen set-valued functions*, 73-84.

E. Wachnicki, *Sur une équation intégro-fonctionnelle*, 95-104.

The 7th ICFEI was held in 1999 at Złockie in the hotel "Geovita" (the new name of "Geofizyk" after a change of the owner). The welcoming address was given by Professor Eugeniusz Wachnicki, the Dean of the Faculty of Science and Technology of the Pedagogical University of Kraków. There were 28 foreign and 45 Polish participants, cf. [A7].

We were honoured by coming of the dignifies guest, Professor János Aczél (University of Waterloo, ON, Canada), upon whom the University of Technology at Miskolc conferred on September 11, 1999, the degree of Doctor Honoris Causa. It was the fourth of his honorary doctorates, the previous being awarded by the Universities in Karlsruhe (Germany), Graz (Austria) and Katowice (Poland). Prof. Aczél gave us a talk entitled *The strictly monotonic solutions of a functional equation arising from coordination of two ways to measure utility* based on a paper writen jointly with Gy. Maksa and Zs. Páles.

There were present 8 colleagues from the University of Debrecen: Zoltán Boros, Zoltán Daróczy, Tibor Farkas, Attila Gilányi, Károly Lajkó, Gyula Maksa, Zsolt Páles, Lászlo Székelyhidi.

A. Guests. Thomas M.K. Davison (Canada), Lech Maligranda (Sweden), Alexander N. Šarkovskiǐ (Ukraine), Marina Borisovna Vereikina (Ukraine).

B. Problems ([A7], 192-198). J. Aczél (1. Elementary solution of a geometrical problem; 2. G. Pickert's (Germany) on the translation equation (followed by remarks of Z. Moszner); 3. J. Bukszár's (Hungary) on an "integrated Cauchy equation"; 4. A.W. Marschall's (Canada) – on Chini's equation (followed by contributions to its solution of the speaker and by T. Riedel with M. Sablik); Z. Daróczy and Zs. Páles (equivalence of two inequalities related to Jensen's one without any regularity assumptions); T.M.K. Davison (characterizing a special set of polynomials with integer coefficients); L. Maligranda (form of a function occurring in the theory of Orlicz spaces); L. Székelyhidi (when the shape of terms of a sequence involving $2\mathbb{Z}$-periodic functions is preserved after taking the limit).

C. Papers.

T.M.K. Davison, *D'Alembert's functional equation and Chebyshev polynomials*, 31-38.

H.-H. Kairies, *On a Banach space automorphism and its connections to functional equations and continuous nowhere differentiable functions*, 39-48.

Pl. Kannappan, G.H. Kim, *On the stability of the generalized cosine functional equations*, 49-58.

Z. Moszner, *Sur les généralisations du wronskien*, 85-100.

F. Neuman, *Functional and differential equations*, 109-116.

T. Riedel, K. Wallace, *On a Pexider type equation on* Δ^+, 129-138.

A. Wach-Michalik, *On a problem of H.-H. Kairies concerning Euler's Gamma function*, 151-162.

In his closing address of Professor Brydak two our colleagues were commemorated. Professor György Targonski, the creator and organizer of the series of *European Conferences on Iteration Theory*, passed away at the age of 69, on January 10, 1998, in Munich (cf. Section 7). Mr Martin Grinč (Slovakia), Ph. D. student of W. Jarczyk at the Silesian University of Katowice, who was expected to be among us, died of cancer at the age of 28, on January 18, 1999, in Stará L'ubovnă, just after submitting his Ph. D. thesis.

The 8th ICFEI was held in 2001 at the same place as the preceding one. The welcoming address was given by Professor Michał Śliwa, Rector Magnificus of the Pedagogical University of Kraków. There were 16 foreign and 40 Polish participants, cf. [A8].

Our conference was held only 3 weeks after the annual 39th ICFEI (Sandbjerg, Denmark, August 12-18), because of circumstances independent on the organizers of both meetings.

Another coincidence was the Annual Meeting of the Polish Mathematical Society held from September 11 to September 14 at Nowy Sącz (some 40 km far from Złockie). On Wednesday, 12th of September, Professor Ger gave there an invited talk on the development of research in Poland in the domain of functional equations, which was attended by the most of Polish participants of our conference.

The 8th ICFEI will be kept for long time in the memory of all participants also because of that Tuesday, 11th of September, when we have learned on the unconceivable terroristic attack on the World Trade Center in New York City. Next day, after Professor Riedel's (Louisville, KY, USA) talk, we joined our hands in a Solidarity Chain to express sympathy and communion with American people.

B. Problems ([A8], 81-90). Z. Moszner (problem of T. Mitchell (USA), having relations to economics, communicated to the speaker by J. Aczél); Zs. Páles (1. Approximate sandwich theorem; the problem solved jointly by him and L. Székelyhidi (5. Remark in [A8], 83-85)); 2. (ε, δ)-Wright-convex functions); J. Wesołowski (obtaining under weaker assumptions results on solutions of several new functional equations satisfied by bijections preserving independence of random variables, similar to those presented as remarks (7. Problems, [A8], 87-90, cf. also his talk on the matrix case, [A8], 79-80)).

C. Papers.

B. Choczewski, Z. Kominek, *A proof of S. Rolewicz's conjecture*, 5-12.

H.-H. Kairies, *Spectra of certain operators and iterative functional equations*, 13-22.

Z. Moszner, *La fonction d'indice et la fonction exponentielle*, 23-38.

A. Schleiermacher, *Some consequences of a theorem of Liouville*, 39-54.

6. The last two meetings

They are treated here in a concise manner, following the lines of presentation of the former ICFEIs, as the reports on both meetings are printed in the same volume.

The 9th ICFEI was held for the fifth time in Złockie. The welcoming address was given by Professor Eugeniusz Wachnicki, Deputy Rector of the Pedagogical University of Kraków. There were 22 foreign and 35 Polish participants, cf. [A9], 122-125.

The participants have signed a congratulation address to Professor János Aczél who had been promoted to the degree of Doctor Honoris Causa on June, 2003, by the University of Debrecen.

A special letter has been sent to Mrs. Irena Gołąb, wife of Professor Gołąb, on her 100th birthday, celebrated on July, 2003.

Congratulations were also conveyed to Professor Stanisław Midura from Rzeszów, who attended 7 of our conferences, on the occassion of his 70th birthday.

A. Guests. Shin-ichi Nakagiri (Japan), Themistocles M. Rassias (Greece).

B. Problems ([A9], 119-122). I. Corovei (Cluj-Napoca) (on Steffensen's inequality; presented by B. Choczewski; together with remarks); Z. Daróczy (on translative and quasi-commutative operations on groups; followed by M. Sablik's remark); H.-H. Kairies (on a set connected with Schröder's equation); Zs. Páles (three problems concerning $(\frac{1}{3}, \frac{2}{3})$-convexity); Th.M. Rassias (on the stability of the orthogonality equation; together with remarks).

C. Papers.

Sh. Haruki, Sh. Nakagiri, *Functional equations arising from the Cauchy–Riemann equations*, 59-76.

H.-H. Kairies, *On continuous and residual spectra of operators connected with iterative functional equations*, 51-57.

The 10th ICFEI was held at Będlewo in the Mathematical Research and Conference Center, a branch of the Institute of Mathematics of the Polish Academy of Sciences. The welcoming address was given by Professor Marek Cezary Zdun, the Dean of the Faculty of Science and Technology of the Pedagogical University of Kraków. There were 14 foreign and 50 Polish participants, cf. [A10], 162-164.

The Organizing Committee of the jubilee meeting acted this time under the chairmanship of Professor Janusz Brzdęk and consisted of Dr. Paweł Solarz, Miss Janina Wiercioch and Mr Władysław Wilk. The members of the Scientific Committee were Professors: Dobiesław Brydak (Chairman), Bogdan Choczewski, Roman Ger, Marek Cezary Zdun and Dr. Jacek Chmieliński (Secretary).

The talks focused on the following topics: stability theory of functional equations, functional equations in several variables, iterative functional equations, functional inequalities (in particular: convexity); multifunctions, iteration theory; applications. They were in programme of almost all ICFEIs; with stability, multifunctions, iteration theory being the topics attracting recently the attention of more researchers.

The jubilee of the 10th ICFEI, joined with that of the 70th birthdays of Professors Dobiesław Brydak and Bogdan Choczewski, the organizers of the previous Conferences, was celebrated on a festive banquet on Thursday, September 15, 2005. Professor Roman Ger made a hearthy speech on this occassion, and the jubilees obtained special addresses signed by all the participants of the 10th ICFEI in which one reads, among others: *The Conference has always been very important for Polish Mathematicians working in the field of functional equations. The international significance of this is shown by the enclosed list of countries from which the participants of all the 10 Conferences have originated.*

In his closing address Professor Choczewski announced that Professor Brydak has been named Honorary Chairman of the Scientific Committe of the next ICFEI. He also congratulated Professor Marian Kwapisz (from the Casimir the Great University in Bydgoszcz, present also on the 6th ICFEI) on the occassion of his 70th birthday.

A. Guests. Lyudmila Efremova (Russia), Boris Paneah (Israel), Vasile D. Popa (Romania), Vsevolod Sakbaev (Russia), Ekaterina Shulman (Israel), Bing Xu (China) (Professor P.V. Subrahmanyan (Indian Institute of Technology Madras, Chennai, India) who had an insufficient visa was stopped on the Polish border on his way to Będlewo.)

B. Problems ([A10], 155-162). V. Mityushev (1. Linear iterative functional equation with singularity; followed by a remark by J. Walorski on its solvability, 2. Functional equation arisen in the diffraction theory); K. Nikodem (concerning λ-convexity); G. Toader (of Matkowski-Sutô type).

C. Papers.

M. Piszczek, *Second Hukuhara derivative and cosine family of linear set-valued functions*, 87-98.

The current year was also the jubilee year for two distinguished colleagues from the Pedagogical University of Kraków.

Professor Zenon Moszner, who had attended all the ICFEIs except of the last two, observed the anniversary of his 75th birthday. The 50th anniversary of his work at the same University and of his 70th birthday were celebrated by his University in 2000. A special volume of the *Annales Academiae Paedagogicae Cracoviensis Studia Mathematica* [B3] dedicated to him contains 28 papers of 34 authors (other 8 papers with dedication, of 9 authors, are found in [A7]).

Professor Andrzej Zajtz, who had participated in the first and in the second ICFEI celebrated his 70th birthday. During a special meeting held in October, 2005, at the Institute of Mathematics of the Pedagogical University in Kraków a volume of the *Annales Academiae Paedagogicae Cracoviensis Studia Mathematica* [B4], containing 19 papers (of 30 authors) dedicated to the Jubilee had been presented to him by Professors Jacek Gancarzewicz and Eugeniusz Wachnicki, the editors of the volume.

7. Other relevant international meetings held in Poland

For the sake of completeness, we register below and comment other meetings on functional equations, inequalities, and related topics which had been organized in Poland after the Conference at Zakopane in 1967 (cf. Sections 1, 5, and [B12]).

The 27th International Symposium on Functional Equations, 1989. Held from August 14 to 24, first at Bielsko-Biała, then at Katowice and Kraków (cf. [B5]), it was organized by the Bielsko-Biała Branch of the Łodź Technical University, by the Silesian University of Katowice, and by the Pedagogical University of Kraków. The Organizing Committee consisted of Professors J. Aczél (Waterloo, ON, Canada), B. Choczewski (Kraków), R. Ger (Katowice), J. Matkowski (Bielsko-Biała), J. Rätz (Bern) and A. Sklar (Chicago). Dr. L. Anczyk acted as secretary of the symposium. There were 8 sponsoring institutions. The 64 participants came from 13 countries (25 from Poland).

It was the first ISFE ever held in East Europe. There were 59 talks and 36 contributions in problems-and-remarks sessions. Three special sessions were organized: *Functional Equations in the Teaching of Mathematics* – by Professor C. Alsina (6 contributors); *Functional Equations and Functional Inequalities in Functional Analysis* – by Professors J. Matkowski and J. Rätz (6 contributors) and *Some Applications of Functional Equations Outside Mathematics* – by Professors J. Aczél and F. Stehling (5 contributors).

For the last time in his life Professor Marek Kuczma was able to participate actively in a conference. On the Katowice session of the symposium he gave us a talk *A functional equation reflecting a mean value property "without mean values and derivatives" in fields of characteristics 2* ([B5], 285-286), was the co-author of Professor Aczél's talk ([B5], 265-266) and (with T.M.K. Davison) – of a remark to this talk (35. Remark, [B5], 322).

The Polish-Austrian Seminar on Functional Equations and Iteration Theory, 1994. Held from October 26 to 31 at Cieszyn (cf. [B6]), it was the last of three meetings gathering representatives of Austrian and Polish schools of researchers in the domain, mainly from Graz and Katowice. The Seminar was attended by 34 persons: 9 from Austria, 1 from Canada and 24 from Poland. There were 29 papers presented, and 19 of them (of 27 authors), were published in the special volume [B6] of the *Annales Mathematicae Silesianae*, in 5 thematic groups: functional equations in several variables (5 papers), iterative functional equations (4), Hyers–Ulam stability (4), iteration theory (3), set-valued functions (3).

The first two, Austrian-Polish Seminars, were held at the Karl-Franzens Universität Graz, in May, 1986 and in October, 1991. The leaders of the native groups participating in the three Seminars, and their main organizers, were Professors Ludwig Reich (Graz) and Roman Ger (Katowice). A part of 1991 Seminar, held 4 months after Professor Kuczma's death, was dedicated to his memory, and his scientific legacy was presented in the special lectures delivered by: Karol Baron *M. Kuczma's papers on iterative functional equations* ([B7], 1-6), Bogdan Choczewski *Papers of Marek Kuczma written in the last decade of his life* ([B7], 7-16) and Roman Ger *M. Kuczma's papers on functional equations in several variables* ([B7], 17-28).

The 34th International Symposium on Functional Equations, 1996. Held form June 10 to 19 at Wisła-Jawornik (cf. [B8]), it was organized by the Silesian University of Katowice under the honorary patronage of the Polish Parliamentary Commission on Education, Science and Technical Progress. The Organizing Committee consisted of Professors J. Aczél, W. Benz (Hamburg), R. Ger and J. Rätz. Dr. R. Badora acted as secretary of the symposium. There were 9 sponsoring institutions. The 67 participants came from 15 countries (27 from Poland).

The formal opening of the Symposium was preceded by the ceremonial award of the degree of Doctor Honoris Causa to Professor János Aczél by the Silesian University of Katowice. There were 59 talks and 28 contributions in problems-and-remarks sessions. Professor Kazimierz Nikodem presented a survey talk *Functional equations for set-valued functions.* Two special sessions were organized: *Selection Models* – by Professor J. Aczél (4 contributors) and *Hyers-Ulam Stability* – by Professor R. Ger and J. Tabor (9 contributors).

The ISFE medals for outstanding contributions to the meeting were awarded to Zsolt Páles and Jacek Tabor.

After his resignation from the Scientific Committee at the end of this Symposium, Professor Aczél was named, and warmly applauded, Honorary Chairman of the Symposia. He is succeeded in the Committee by Professors Roman Ger and Ludwig Reich (Graz).

The European Conference on Iteration Theory 1998. Held from August 30 to September 5 at Złockie, it was organized by the Pedagogical University of Kraków. The Scientific Committee consisted of Professors Laura Gardini (Urbino), Christian Mira (Toulouse), Luigi Paganoni (Milan), Ludwig Reich (Graz), José Sousa Ramos (Lisbon), Jaroslav Smítal (Opava) and Marek C. Zdun (Kraków, local organizer). Dr. Z. Leśniak acted as secretary of the conference. There were 51 participants who came from 11 countries. Altogether 44 talks were presented, and 23 articles are printed in a special issue of the *Annales Mathematicae Silesianae* [B9].

The Conference was 12th in the series of biannual meetings initiated and chaired by the late Professor György Targonski. Organized for the first time in Poland, it was dedicated to the memory of the Founder. The commerative talk was given by B. Choczewski and the article by him and by Jolán Targonski *György Targonski's life and work* is published in [B9], 9-32. It contains the complete list of Professor Targonski's scientific legacy: 50 papers, 59 other contributions (mainly during conferences), 7 co-edited volumes containing Proceedings of ECITs since 1984.

The Thousandth Seminar on Functional Equations, 2002. This was a Festive Session, held on April 9 at Katowice of the *Seminar on Functional Equations* which was founded in 1964 by Professor Marek Kuczma in Katowice. There were 94 participants, among them 6 foreign guests who came from Austria (1), Czech Republic (2), Hungary (2) and Slovakia (1). From the booklet [B10], prepared by Professors Roman Ger and Maciej Sablik (with lists of participants and statistics compiled by Dr. Tomasz Powierża and Dr. Tomasz Szostok) one learns that the number of persons that participated in at least one of Seminars is equal to 239 (40 from abroad) and that of speakers equals 126 (25 from abroad), cf. [B10], 39-49. In R. Ger's historical article in [B10], 9-23, one finds a long list of subject matter of talks presented at the 999 sessions of the Seminar. To recall only numbers: 235 particular themes are listed in 13 groups of topics. Professor Kuczma when resignating in October, 1980, from his organizational duties at the University designated Professor Andrzej Smajdor to substitute him. In 1984 Professor Ger, being elected the chairman of the Chair of Functional Equations, conducted the Seminar for 16 years. After his resignation, since the elections in October, 2000, the Seminar is chaired by Professor Maciej Sablik.

In 1975, Marek Kuczma proposed to organize the *Contest for the Best Polish Paper on Functional Equations*, affiliated to the Seminar, cf. M. Sablik's article in [B10], 25-27. And from 1975, on the Jury of the Contest consisting of 13 members, being elected for two-years terms on special sessions of the Seminar, each year is estimating the papers published in the preceding year, authored or coauthored by a Polish mathematician. *Marek Kuczma Contest*, named after its founder who untimely died in 1991, belongs to those authorized and sponsored by the Rector of the Silesian University, and it is acknowledged by the Polish Mathematical Society. During 30 years of the existence of the Contest in its Jury acted 30 Polish mathematicians (see [B10], 29-30). More than 100 papers were awarded so far (for the complete list regarding the years 1974-2000 see [B10], 31-37).

The 40th International Symposium on Functional Equations, 2002. Held from August 25 to September 1 at Gronów (cf. [B11]), it was organized by the University of Zielona Góra. The Scientific Committee consisted of Professor János Aczél (Waterloo, ON, Canada) as Honorary Chairman, Professor Jürg Rätz (Bern) as Chairman, and Professors Zoltán Daróczy (Debrecen), Roman Ger (Katowice), Ludwig Reich (Graz) and Abe Sklar (Chicago). The Organizing Committee consisted of Professors Witold Jarczyk and Janusz Matkowski and Dr. Dorota Krassowska who also acted as secretary of the symposium. There were 10 sponsoring institutions. The 60 participants came from 11 countries (26 from Poland).

There were 56 talks and 25 contributions in problems-and-remarks sessions. Professor Karol Baron presented a survey talk *Recent results in the theory of functional equations in a single variable.* The special session *Mean Values* organized by Zoltán Daróczy gave, in reports of 9 authors, an up-to-date overview of the theory of means (in particular, concerning the recently arisen Matkowski–Sutô problems). The ISFE medals for outstanding contributions to the meeting were awarded to Janusz Brzdęk and Tomasz Szostok.

Katowice-Debrecen Winter Seminars on Functional Equations, 2001, 2003, 2005. These are annual meetings of two schools on functional equations: from the University of Debrecen and from the Silesian University of Katowice. The first Seminar was held in Cieszyn, from February 7 to 10, 2001, the third and the fifth took place at Będlewo, from January 29 to February 1, 2003, and from February 2 to February 5, 2005, respectively. The other two Seminars, in 2002 and 2004, were held in Hungary. On each meeting the same number of participants from each city were present, altogether 20 researchers both on the first and on the second meetings, and 24 on each of the others.

The leaders of the native groups of participants, and the main organizers of each of the five Seminars, were Professors Zsolt Páles (Debrecen) and Roman Ger (Katowice).

8. Complementary remarks

The 11th International Conference on Functional Equations and Inequalities is planned to be held at Będlewo in September 2006. There was constituted its Scientific Committee, the members of which are Professors: Nicole Brillouët-Belluot (Nantes, France), Dobiesław Brydak (Kraków, Honorary Chairman), Janusz Brzdęk (Kraków, Chairman), Bogdan Choczewski (Kraków), Roman Ger (Katowice), Hans-Heinrich Kairies (Clausthal-Zellerfeld, Germany), Lászlo Losonczi (Debrecen, Hungary), Marek Cezary Zdun (Kraków) and Dr. Jacek Chmieliński (Kraków, Secretary).

The Organizing Committee of the 11th ICFEI consists of: Professor Janusz Brzdęk (Chairman), Dr. Paweł Solarz, Miss Janina Wiercioch and Mr Władysław Wilk.

The Scientific Committee of the International Symposia on Functional Equations, now chaired by Professor Roman Ger, gratefully accepted the declaration of Professor Kazimierz Nikodem, so that the 45th ISFE in 2007 will be organized for the fourth time in Poland, by the colleagues from the Technical-Humanistic Academy of Bielsko-Biała.

On Tuesday, December 6, 2005, the 1100th session of the *Seminar on Functional Equations* was held at 4.15 pm., as usual, in the Institute of Mathematics of the Silesian University of Katowice. Noteworthy is the fact that at the Institute there are also working two other seminars on functional equations: *The Seminar on Functional Equations and Inequalities in Several Variables*, conducted by Professor Roman Ger, and *The Seminar on Real Analysis*, conducted by Professor Karol Baron. The first seminar already celebrated its 600th, the second one – 500th session.

To complete the picture, let us note that regular seminars on functional equations and related topics are acting in Poland (and are chaired by the Professor(s)) at the: Silesian University of Technology in Gliwice (Stefan Czerwik and Wilhelmina Smajdor), Technical-Humanistic Academy of Bielsko-Biała (Kazimierz Nikodem), University of Rzeszów (Stanisław Midura and Józef Tabor), University of Zielona Góra (Witold Jarczyk and Janusz Matkowski).

What concerns the Kraków school of functional equations, a "Sister Seminar" with Kuczma's one in Katowice started its work in 1970 at the AGH University of Science and Technology, under direction of Professors Bogdan Choczewski and Zenon Moszner. It reached about 750 sessions, is since 1994 affiliated to the Institute of Mathematics of the Pedagogical University of Kraków, and since October, 2005, it is managed by Professor Janusz Brzdęk. At the University two other relevant seminars are acting, *The Seminar on Multifunctions* (Andrzej Smajdor) and *The Seminar on Iteration Theory* (Marek Cezary Zdun).

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Aggregations in classes of fuzzy relations

Abstract. We consider aggregations of fuzzy relations using aggregation functions of n variables. After recalling fundamental properties of fuzzy relations we examine aggregation functions which preserve reflexivity, symmetry, connectedness and transitivity of fuzzy relations.

1. Introduction

Aggregations of relations are important in the group choice theory (cf. [8]) and multiple-criteria decision making (cf. [14]). Formally, instead of crisp relations we aggregate their characteristic functions. However, the aggregation results appear to be fuzzy relations. Therefore, the most fruitful approach to such aggregations begins with fuzzy relations (cf. [12], [9] or [13]).

Since fuzzy relations have values in $[0, 1]$, for their transformations we use real functions $F\colon [0, 1]^n \longrightarrow [0, 1]$. This leads to new functional equations and functional inequalities connected with the particular properties of fuzzy relations. Usually, the properties are checked each time for concrete assumptions on the form of aggregation functions (cf. e.g. [15]). We shall consider obtained equations without additional assumptions about expected aggregation functions.

We consider the fundamental properties of fuzzy relations during aggregations of finite families of these relations. Firstly, we describe the problem of aggregation of fuzzy relations (Section 2). Next, we describe solutions of functional equations and inequalities connected with: reflexivity (Section 3), symmetry (Section 4), connectedness (Section 5) and transitivity (Section 6) of fuzzy relations. All sections are preceded by suitable definitions of commonly used properties of fuzzy relations.

2. Fuzzy relations

The notion of fuzzy relations is a generalization of that of the characteristic function of crisp relations.

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Definition 1 (Zadeh [17])
Let $X \neq \emptyset$. A fuzzy relation in X is an arbitrary function $R: X \times X \longrightarrow [0,1]$. The family of all fuzzy relations in X is denoted by $FR(X)$.

Fuzzy relations form a lattice $(FR(X), \vee, \wedge)$ with the induced partial order

$$R \leqslant S \iff \forall\, x, y \in X \quad R(x,y) \leqslant S(x,y)$$

and with the lattice operations (cf. [17])

$$(R \vee S)(x,y) = \max(R(x,y), S(x,y)),$$
$$(R \wedge S)(x,y) = \min(R(x,y), S(x,y)), \qquad x, y \in X.$$

For $R, S \in FR(X)$ we also use the sup-$\star$ composition of fuzzy relations (cf. [10])

$$(R \circ S)(x,z) = \sup_{y \in X} [R(x,y) \star S(y,z)], \qquad x, z \in X,$$

where $\star : [0,1]^2 \longrightarrow [0,1]$ is a binary operation. Case $\star = \min$ is referred to as the standard fuzzy relation composition.

Definition 2 (Fodor [9])
Let $n \geqslant 2$, $F: [0,1]^n \longrightarrow [0,1]$, $R_1, \ldots, R_n \in FR(X)$. We define the aggregated fuzzy relation R_F by the formula

$$R_F(x,y) = F(R_1(x,y), \ldots, R_n(x,y)), \qquad x, y \in X. \tag{1}$$

We shall examine properties of the relation (1) under suitable assumptions on fuzzy relations $R_1, \ldots, R_n$. We look for such aggregation functions F which preserve some properties of aggregated fuzzy relations $R_1, \ldots, R_n$. Examples of such properties and appropriate aggregation functions can be found in the papers: [5]-[7] and [14]-[16]. In particular, any projection function

$$P_k(t_1, \ldots, t_n) = t_k\,, \qquad t_1, \ldots, t_n \in [0,1],\ k = 1, \ldots, n \tag{2}$$

preserves arbitrary properties of fuzzy relations, because $R = R_k$ in (1).

3. Reflexivity

At first, we examine the reflexivity properties of the relation (1). Presented definitions of fuzzy relation classes are based on [4], Chapter 5.

Definition 3
A fuzzy relation R is called

reflexive, if $\forall\, x \in X \quad R(x,x) = 1,$ (3)

irreflexive, if $\forall\, x \in X \quad R(x,x) = 0,$ (4)

weakly reflexive, if $\forall\, x \in X \quad R(x,x) > 0,$ (5)

weakly irreflexive, if $\forall\, x \in X \quad R(x,x) < 1.$ (6)

Theorem 1 (cf. [6], Theorem 1)
Let $R_1, \dots, R_n \in FR(X)$ *be reflexive (resp. irreflexive). The relation* R_F *is reflexive (resp. irreflexive), if and only if the function* F *satisfies the condition* (7) *(resp.* (8)*), where*

$$F(1, \dots, 1) = 1, \tag{7}$$

$$F(0, \dots, 0) = 0. \tag{8}$$

Proof. Let $x \in X$. If $F(1, \dots, 1) = 1$, then we get (3) for R_F whenever $R_1, \dots, R_n$ are reflexive. Conversely, if $F(1, \dots, 1) < 1$, then R_F does not fulfil (3). In the case of irreflexive fuzzy relations the proof is similar.

Example 1
Any idempotent function F,

$$F(t, \dots, t) = t \qquad \text{for } t \in [0, 1] \tag{9}$$

fulfils the conditions (7) and (8).

Theorem 2
The fuzzy relation (1) *is weakly reflexive (resp. weakly irreflexive) for every weakly reflexive (resp. weakly irreflexive)* $R_1, \dots, R_n \in FR(X)$*, if and only if the function* F *satisfies the condition* (10) *(resp.* (11)*), where*

$$t_1 > 0, \dots, t_n > 0 \Longrightarrow F(t_1, \dots, t_n) > 0, \qquad t_1, \dots, t_n \in [0, 1], \tag{10}$$

$$t_1 < 1, \dots, t_n < 1 \Longrightarrow F(t_1, \dots, t_n) < 1, \qquad t_1, \dots, t_n \in [0, 1]. \tag{11}$$

Proof. Let $x \in X$. If F fulfils (10), then we get (5) for R_F whenever $R_1, \dots, R_n$ are weakly reflexive. Conversely, if $t_1 > 0, \dots, t_n > 0$ in $[0, 1]$, then fuzzy relations $R_k \equiv t_k$, $k = 1, \dots, n$ are weakly reflexive and from the condition (5) for R_F we obtain (10). In the case of weakly irreflexive fuzzy relations the proof is similar.

Example 2
Any increasing, idempotent function F fulfils the conditions (10) and (11) (cf. [9], Proposition 5.1).

Directly from the definition of increasing bijections we get

Lemma 1
If $\varphi\colon [0, 1] \longrightarrow [0, 1]$ *is an increasing bijection, then for every* $s \in [0, 1]$ *we have*

$$\varphi(s) = 0 \Longleftrightarrow s = 0, \qquad \varphi(s) = 1 \Longleftrightarrow s = 1, \tag{12}$$

$$\varphi(s) > 0 \Longleftrightarrow s > 0, \qquad \varphi(s) < 1 \Longleftrightarrow s < 1. \tag{13}$$

Using the above lemma for operations F_φ isomorphic with a given one,

$$F_\varphi(t_1,\dots,t_n)=\varphi^{-1}F(\varphi(t_1),\dots,\varphi(t_n)),\qquad t_1,\dots,t_n\in[0,1], \tag{14}$$

we can generate new transformations fulfilling conditions from Theorems 1 and 2.

Theorem 3
The conditions (7), (8), (10) *and* (11) *are invariant with respect to all increasing bijections, i.e., with any function F fulfilling one of these conditions, also the functions* (14) *fulfil the respective condition.*

Now, we examine the symmetry properties of the relation (1).

Definition 4
A fuzzy relation R is called

symmetric, if $\forall\, x,y\in X \;\; R(y,x)=R(x,y),$ (15)

semi-symmetric, if $\forall\, x,y\in X \;\; R(x,y)=0 \Longleftrightarrow R(y,x)=0,$ (16)

asymmetric, if $\forall\, x,y\in X \;\; \min(R(x,y),R(y,x))=0,$ (17)

antisymmetric, if $\forall\, x,y\in X, x\neq y \;\; \min(R(x,y),R(y,x))=0,$ (18)

weakly symmetric, if $\forall\, x,y\in X \;\; R(x,y)=1 \Longleftrightarrow R(y,x)=1,$ (19)

weakly asymmetric, if $\forall\, x,y\in X \;\; \min(R(x,y),R(y,x))<1,$ (20)

weakly antisymmetric, if $\forall\, x,y\in X, x\neq y \;\; \min(R(x,y),R(y,x))<1.$ (21)

Symmetry appears to be the most stable property of fuzzy relations, because immediately we get

Theorem 4 (cf. [6], Theorem 2)
Let $R_1,\dots,R_n\in FR(X)$ be symmetric. For every function F the fuzzy relation R_F is also symmetric.

Definition 5
Let $p\in[0,1]$, $s=(s_1,\dots,s_n)\in[0,1]^n$, $t=(t_1,\dots,t_n)\in[0,1]^n$, $F(t)=F(t_1,\dots,t_n)$. We say that $s,t\in[0,1]^n$ are *p-equivalent* ($s\sim_p t$), if

$$\forall\, 1\leqslant k\leqslant n \;\; s_k=p \Longleftrightarrow t_k=p.$$

Theorem 5
Let $\operatorname{card} X\geqslant 2$. *The relation R_F is semi-symmetric (resp. weakly symmetric) for every semi-symmetric (resp. weakly symmetric) $R_1,\dots,R_n\in FR(X)$, if and only if the function F satisfies the condition* (22) *(resp.* (23)*), where*

$$s\sim_0 t \Longrightarrow (F(s)=0 \Leftrightarrow F(t)=0) \qquad \textit{for } s,t\in[0,1]^n, \tag{22}$$

$$s\sim_1 t \Longrightarrow (F(s)=1 \Leftrightarrow F(t)=1) \qquad \textit{for } s,t\in[0,1]^n. \tag{23}$$

Proof. Let F fulfil (22), $x, y \in X$. If $R_1, \ldots, R_n \in FR(X)$ are semi-symmetric, then putting

$$s_k = R_k(x,y), \quad t_k = R_k(y,x), \qquad k = 1, 2, \ldots, n \tag{24}$$

we see that $s \sim_0 t$. Thus,

$$\begin{aligned} F(R_1(x,y), \ldots, R_n(x,y)) = 0 &\Leftrightarrow F(s) = 0 \\ &\Leftrightarrow F(t) = 0 \\ &\Leftrightarrow F(R_1(y,x), \ldots, R_n(y,x)) = 0, \end{aligned}$$

which proves (16) for R_F.

Conversely, let $x, y \in X$, $s, t \in [0,1]^n$, $s \sim_0 t$. Since $\operatorname{card} X \geqslant 2$, then there exist $a, b \in X$, $a \neq b$. The fuzzy relations

$$R_k(x,y) = \begin{cases} s_k, & \text{if } (x,y) = (a,b) \\ t_k, & \text{if } (x,y) = (b,a) \\ 1, & \text{otherwise} \end{cases}, \qquad k = 1, \ldots, n,$$

are semi-symmetric. Thus, the relation R_F is also semi-symmetric and we get

$$\begin{aligned} F(s) = 0 &\Leftrightarrow F(R_1(a,b), \ldots, R_n(a,b)) = 0 \\ &\Leftrightarrow F(R_1(b,a), \ldots, R_n(b,a)) = 0 \\ &\Leftrightarrow F(t) = 0, \end{aligned}$$

which proves (22). In the case of weakly symmetric fuzzy relations the proof is similar.

Example 3
There are many operations fulfilling the conditions (22) and (23). For example n-ary $F = \min$, $F = \max$ or the weighted mean:

$$F(t_1, \ldots, t_n) = \sum_{k=1}^{n} w_k t_k, \qquad t, w \in [0,1]^n,\ \textstyle\sum_{k=1}^{n} w_k = 1. \tag{25}$$

In virtue of Lemma 1, also quasilinear means (cf. [1], p. 287):

$$F(t_1, \ldots, t_n) = \varphi^{-1}\left(\sum_{k=1}^{n} w_k \varphi(t_k)\right), \tag{26}$$

fulfil (22), where $\varphi\colon [0,1] \longrightarrow [0,1]$ is an increasing bijection.

Theorem 6

Let card $X \geqslant 2$. *The relation* R_F *is asymmetric (resp. antisymmetric) for every asymmetric (resp. antisymmetric)* $R_1, \ldots, R_n \in FR(X)$, *if and only if the function* F *satisfies the condition* (27), *where*

$$\forall\, s,t \in [0,1]^n \quad (\forall\, 1 \leqslant k \leqslant n \quad \min(s_k, t_k) = 0) \Longrightarrow \min(F(s), F(t)) = 0. \quad (27)$$

Proof. Let F fulfil (27), $x, y \in X$. If $R_1, \ldots, R_n \in FR(X)$ are asymmetric, then using (24) we see that

$$\forall\, 1 \leqslant k \leqslant n \quad \min(s_k, t_k) = 0 \tag{28}$$

and the relation R_F is asymmetric by (27).

Conversely, let $s, t \in [0,1]^n$ fulfil (28). Since card $X \geqslant 2$, then there exist $a, b \in X$, $a \neq b$. The fuzzy relations

$$R_k(x,y) = \begin{cases} s_k\,, & \text{if } (x,y) = (a,b) \\ t_k\,, & \text{if } (x,y) = (b,a)\,, \\ 0, & \text{otherwise} \end{cases} \qquad k = 1, \ldots, n \tag{29}$$

are asymmetric. Thus, the relation R_F is also asymmetric and we get

$$\begin{aligned} \min(F(s), F(t)) &= \min(F(R_1(a,b), \ldots, R_n(a,b)), F(R_1(b,a), \ldots, R_n(b,a))) \\ &= 0, \end{aligned}$$

which proves (27). In the case of antisymmetric fuzzy relations the proof is similar.

Example 4

As the first example of functions fulfilling (27) we can consider $F = \min$. A simple condition sufficient for (27) is connected with zero element $z = 0$ of operation F with respect to a certain coordinate:

$$\exists\, 1 \leqslant k \leqslant n \quad \forall\, i \neq k \quad \forall\, t_i \in [0,1] \quad F(t_1, \ldots, t_{k-1}, 0, t_{k+1}, \ldots, t_n) = 0.$$

In particular, the weighted geometric mean:

$$F(t_1, \ldots, t_n) = \prod_{k=1}^{n} t_k^{w_k}, \qquad t, w \in [0,1]^n,\ \textstyle\sum_{k=1}^n w_k = 1,$$

fulfils (27). As another example we consider the median (cf. [3], p. 21):

$$\operatorname{med}(t_1, \ldots, t_n) = \begin{cases} \dfrac{s_k + s_{k+1}}{2}, & \text{if } n = 2k \\ s_{k+1}\,, & \text{if } n = 2k+1 \end{cases}, \tag{30}$$

where $(s_1, \ldots, s_n)$ is an increasing permutation of $(t_1, \ldots, t_n)$, $(s_1 \leqslant \ldots \leqslant s_n)$.

If a function F fulfils the condition

$$\forall\, t \in [0,1]^n \quad \mathrm{card}\{k:\ t_k = 0\} > \frac{n}{2} \Longrightarrow F(t) = 0, \tag{31}$$

then we also get (27) (e.g. the median (30) fulfils (31)). However, the above condition is not necessary for (27), because it does not cover the projections (2).

Similarly as Theorem 6 we get

THEOREM 7
Let $\mathrm{card}\, X \geqslant 2$. *The fuzzy relation* R_F *is weakly asymmetric (resp. weakly antisymmetric) for every weakly asymmetric (resp. weakly antisymmetric)* $R_1, \ldots, R_n \in FR(X)$, *if and only if the function* F *satisfies the condition* (32), *where*

$$\forall\, s,t \in [0,1]^n \quad (\forall\, 1 \leqslant k \leqslant n \quad \min(s_k, t_k) < 1) \Longrightarrow \min(F(s), F(t)) < 1. \tag{32}$$

Proof. Let F fulfil (32), $x, y \in X$. If $R_1, \ldots, R_n \in FR(X)$ are weakly asymmetric, then using (24) we see that

$$\forall\, 1 \leqslant k \leqslant n \quad \min(s_k, t_k) < 1 \tag{33}$$

and the relation R_F is weakly asymmetric by (32).

Conversely, let $s, t \in [0,1]^n$ fulfil (33). Since $\mathrm{card}\, X \geqslant 2$, then there exist $a, b \in X$, $a \neq b$. Fuzzy relations (29) are weakly asymmetric. Thus, the relation R_F is also weakly asymmetric and we get

$$\begin{aligned} \min(F(s), F(t)) &= \min(F(R_1(a,b), \ldots, R_n(a,b)), F(R_1(b,a), \ldots, R_n(b,a))) \\ &< 1, \end{aligned}$$

which proves (32). In the case of weakly antisymmetric fuzzy relations the proof is similar.

EXAMPLE 5
As examples of n-ary operations fulfilling (32) we have $F = \min$ and the weighted mean (25).

In virtue of Lemma 1 we get

THEOREM 8
The conditions (22), (23), (27) *and* (32) *are invariant with respect to increasing bijections.*

In particular, every quasilinear mean (26) fulfils (32).

4. Connectedness

Next we examine connectedness properties of the relation (1).

Definition 6

A fuzzy relation R is called

$$\textit{connected}, \text{ if } \quad \forall\, x,y \in X, x \neq y \quad \max(R(x,y), R(y,x)) = 1, \tag{34}$$

$$\textit{totally connected}, \text{ if } \quad \forall\, x,y \in X \quad \max(R(x,y), R(y,x)) = 1, \tag{35}$$

$$\textit{weakly connected}, \text{ if } \quad \forall\, x,y \in X, x \neq y \quad \max(R(x,y), R(y,x)) > 0, \tag{36}$$

$$\textit{weakly totally connected}, \text{ if } \quad \forall\, x,y \in X \quad \max(R(x,y), R(y,x)) > 0. \tag{37}$$

The above definitions are very similar to those considered in Definition 4. This similarity can be described by the use of the complement R' of fuzzy relation R:

$$R'(x,y) = 1 - R(x,y), \qquad x,y \in [0,1].$$

Lemma 2

A fuzzy relation R is asymmetric (resp. antisymmetric, weakly asymmetric, weakly antisymmetric), if and only if its complement is totally connected (resp. connected, weakly totally connected, weakly connected).

In virtue of this lemma conditions for aggregated connected fuzzy relations can be obtained by negation of conditions considered above for aggregated asymmetric and antisymmetric fuzzy relations.

Similarly as Theorem 6 we get

Theorem 9

Let $\mathrm{card} X \geqslant 2$. *The relation R_F is connected (resp. totally connected) for every connected (resp. totally connected) $R_1, \ldots, R_n \in FR(X)$, if and only if the function F satisfies the condition* (38), *where*

$$\forall\, s,t \in [0,1]^n \ \ (\forall\, 1 \leqslant k \leqslant n \quad \max(s_k, t_k) = 1) \Longrightarrow \max(F(s), F(t)) = 1. \tag{38}$$

Example 6

As examples of functions fulfilling (38) we can consider $F = \max$, $F = \mathrm{med}$ or operations F with neutral element $z = 1$ with respect to a certain coordinate:

$$\exists\, 1 \leqslant k \leqslant n \ \ \forall\, i \neq k \ \ \forall\, t_i \in [0,1] \ \ F(t_1, \ldots, t_{k-1}, 1, t_{k+1}, \ldots, t_n) = 1.$$

Now a dual property for (31) have the form:

$$\forall\, t \in [0,1]^n \ \ \mathrm{card}\{k : \ t_k = 1\} > \frac{n}{2} \Longrightarrow F(t) = 1. \tag{39}$$

Similarly as Theorem 7 we get

THEOREM 10
Let $\operatorname{card} X \geqslant 2$. *The fuzzy relation* R_F *is weakly connected (resp. weakly totally connected) for every weakly connected (resp. weakly totally connected)* $R_1, \ldots, R_n \in FR(X)$, *if and only if the function* F *satisfies the condition* (40), *where*

$$\forall\, s,t \in [0,1]^n \;\; (\forall\, 1 \leqslant k \leqslant n \quad \max(s_k,t_k) > 0) \Longrightarrow \max(F(s),F(t)) > 0. \quad (40)$$

EXAMPLE 7
As examples of operations fulfilling (40) we have $F = \max$ and the weighted mean (25).

In virtue of Lemma 1 we get

THEOREM 11
The conditions (38), (40) *are invariant with respect to increasing bijections.*

In particular, every quasilinear mean (26) fulfils (40).

5. Transitivity

Finally, we examine transitivity properties of the relation (1).

DEFINITION 7 (cf. [2])
Let $\star\colon [0,1]^2 \longrightarrow [0,1]$ be a binary operation. A fuzzy relation R is called

$$\star\text{-}\textit{transitive}, \text{ if } \quad \forall\, x,y,z \in X \quad R(x,y) \star R(y,z) \leqslant R(x,z), \quad (41)$$
$$\textit{transitive}, \text{ if } \quad \forall\, x,y,z \in X \quad \min(R(x,y),R(y,z)) \leqslant R(x,z). \quad (42)$$

DEFINITION 8 (cf. [11])
Binary operation $\star$ in $[0,1]$ is said to be a *triangular norm*, if it is increasing, associative, commutative and with the neutral element $e = 1$.

In particular, the Łukasiewicz multivalued conjunction

$$T_L(u,v) = \max(u+v-1,0), \qquad u,v \in [0,1]$$

is a triangular norm. The case of transitivity was discussed in details in [16].

THEOREM 12 (Saminger et al. [16], Theorem 3.1)
Let $\operatorname{card} X \geqslant 3$, $\star$ *be a triangular norm and function* $F\colon [0,1]^n \longrightarrow [0,1]$ *be increasing with respect to the induced order in* $[0,1]^n$, *i.e.,*

$$s_k \leqslant t_k,\; k = 1,\ldots,n \Longrightarrow F(s_1,\ldots,s_n) \leqslant F(t_1,\ldots,t_n).$$

The relation R_F *is* $\star$*-transitive for every* $\star$*-transitive* $R_1, \ldots, R_n \in FR(X)$, *if and only if the function* F *dominates the operation* $\star$, *i.e.,*

$$\forall\, s,t \in [0,1]^n \quad F(s_1 \star t_1, \ldots, s_n \star t_n) \geqslant F(s_1,\ldots,s_n) \star F(t_1,\ldots,t_n). \quad (43)$$

Example 8
The main example of domination for $\star = \min$ is $F = \min$ (cf. [16], Proposition 5.1). Thus $F = \min$ preserves min-transitivity of fuzzy relations.

Example 9
Saminger et al. [16] presented some examples of aggregating functions preserving T_L-transitivity. In particular any weighted mean (25) preserves T_L-transitivity of fuzzy relations.

Let us observe that condition (43) is not invariant with respect to increasing bijections.

Example 10
Let $n = 2$, $\operatorname{card} X = 3$, $\varphi(x) = x^2$, $x \in [0,1]$. From the above example we know that the arithmetic mean $F(u,v) = \frac{u+v}{2}$, $u, v \in [0,1]$ dominates T_L. However, the operation $F_\varphi(u,v) = \sqrt{\frac{u^2+v^2}{2}}$, $u, v \in [0,1]$ does not dominate T_L. For $u = 0.9$, $v = 0.1$, $w = 0.8$, $z = 0.2$ it can be verified that

$$\sqrt{\frac{\max(u+v-1,0)^2 + \max(w+z-1,0)^2}{2}} < \max\left(\sqrt{\frac{u^2+w^2}{2}} + \sqrt{\frac{v^2+z^2}{2}} - 1, 0\right),$$

contrary to (43).

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A theorem on divergence of Fourier series

Abstract. The paper contains an extension of Calderon's work [2] on the optimality of the Dini test for Fourier series in a set of positive measure.

1. Introduction

Let us begin with recalling the classical Dini's result:

PROPOSITION
If $\delta \in (0,\pi)$ and $f \in L^1_{2\pi}$ satisfies at the point x the following condition

$$\int_{-\delta}^{\delta} \frac{|f(x)-f(x-t)|}{|t|}\,dt < \infty, \tag{1}$$

then the sequence of partial sums of the Fourier expansion of f at x is convergent to $f(x)$.

The aim of this paper is to supplement the idea of Calderon [2] in which the author proves that condition (1) is optimal for x belonging to some set E of positive measure: $|E| > 0$. He also studies a weaker condition than (1), which is now recalled.

CONDITION (W)
The function $w\colon [0,1) \longrightarrow R$ is continuous and increasing in $[0,\delta), w(0)=0$, and

$$\int_{0}^{\delta} \frac{w(t)}{t}\,dt = \infty. \tag{2}$$

We aim at proving the following

THEOREM
Suppose that Condition (W) *is satisfied. Then there exists a function $g \in L[0,2\pi)$ and a set of positive measure F such that*

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$$\int_{-\delta}^{\delta} \frac{|g(x+t)-g(x)|}{|t|} w(|t|)\,dt < \infty \qquad \text{for } x \in F \tag{3}$$

and the sequence of partial sums of the Fourier expansion of g is divergent almost everywhere in F.

2. Auxiliary results

Before giving the proof of the Theorem we recall some basic results.

Marcinkiewicz's Theorem ([2], p. 382)
Let φ be a continuous, increasing function, defined on $[0, 2\pi]$ such that $\varphi(0) = 0$ and

$$[\varphi(t)]^{-1} = o\left(\ln \frac{1}{t}\right), t \to 0^+. \tag{4}$$

Then there exists a function $f \in L^1[0, 2\pi]$ satisfying

$$\frac{1}{|h|} \int_{0}^{h} |f(x+t) - f(x)|\,dt = O(\varphi(|h|)), \quad |h| \to 0, \tag{5}$$

for almost every $x \in [0, 2\pi]$, and the sequence of partial sums of the Fourier expansion of f is divergent almost everywhere.

Lemma 1
If

$$\int_{0}^{2\pi} dx \int_{0}^{\pi} \frac{[g(x+t) - g(x-t)]^2}{t}\,dt < \infty,$$

then the sequence of partial sums of the Fourier expansion of g is convergent almost everywhere.

Lemma 2 ([1], p. 383)
Let φ be given in the form

$$[\varphi(t)]^{-1} = \int_{t}^{1} \frac{w(s)}{s}\,ds, \qquad 0 < t < 1,$$

where w obeys Condition (W). *If $f \in L[0, 2\pi]$ satisfies the asymptotic condition* (5) *almost everywhere in $[0, 2\pi]$, then for each $\varepsilon > 0$ there is a perfect subset F of $[0, 2\pi]$ and a constant C such that:*

$$|F| > 2\pi - \varepsilon,$$

$$|f(x_1) - f(x_2)| \le C\varphi(|x_1 - x_2|), \qquad x_1, x_2 \in F,\ 0 < |x_1 - x_2| < \frac{1}{2},$$

$$\frac{1}{|h|}\int_0^h |f(x+t) - f(x)|\, dt < C\varphi(|h|) \qquad \textit{for } x \in F.$$

3. Proof of the Theorem

Consider the function

$$\bar\varphi(t) = \left[\int_t^1 \frac{\bar w(s)}{s}\, ds\right]^{-1},$$

where $0 < t < \frac{1}{2}$, $\bar w(s) = \max\{|\ln\ s|^{-\delta}, w(s)\}$, $0 < \delta < \frac{1}{4}$, $0 < s < \frac{1}{2}$, and w satisfies Condition (W). From the definition of $\bar\varphi$ we have condition (4) for $\bar\varphi$. Hence, the assumptions of Marcinkiewicz's Theorem are fulfilled for $\bar\varphi$. Then there exists a function f satisfying (4).

Let $\varepsilon > 0$, F be a perfect subset of $[0, 2\pi]$ and C be a constant (see Lemma 2). Denote by $\bar f$ any continuous extension of f from F to $[0, 2\pi]$ such that

$$|\bar f(x_1) - \bar f(x_2)| \le C\bar\varphi(|x_1 - x_2|), \qquad x_1, x_2 \in [0, 2\pi],\ |x_1 - x_2| < \frac{1}{2}. \tag{6}$$

Let us define $g = f - \bar f$, hence $f = \bar f + g$.

Consider the double integral

$$J = \int_F \left(\int_0^{2\pi} |g(x) - g(y)| \frac{\bar w(|x-y|)}{|x-y|}\, dy \right) dx.$$

Since $g(u) = 0$ for any $u \in F$, then

$$J = \int_F \left(\int_G |g(y)| \frac{\bar w(|x-y|)}{|x-y|}\, dy \right) dx, \tag{7}$$

where $G = [0, 2\pi] \setminus F$.

We shall construct intervals I_k such that

$$G = \bigcup_{k=1}^{\infty} I_k\,,$$

where $\forall\, i, j \in N,\ i \ne j :\ I_i^0 \cap I_j^0 = \emptyset,\ I_i^0 = \operatorname{int} I_i$ and the distance $d(I_k, F)$ satisfies the inequalities

$$|I_k| \le d(I_k, F) \le 2|I_k| \qquad \text{for } k = 1, 2, \dots .$$

Let F_1, F_2 be subsets of F. For the simplicity, let us assume that the part of G lying between F_1 and F_2 is of length 1.

Let x be a point in G such that $d(x, F_1) = d(x, F_2)$. Let us define the intervals I_k as follows:

1) $I_1 = [x - \frac{1}{6}, x + \frac{1}{6}]$, $|I_1| = \frac{1}{3} = d(I_1, F) \le \frac{2}{3} = 2|I_1|$;

2) I_1, I_2 are closed intervals, symmetric with respect to x and such that the right end of I_2 equals the left end of I_1 and moreover $|I_2| = d(F, I_2) \le \frac{1}{6} = 2|I_2|$. I_3 has analogical properties.

n) At the n-th step we define I_{2n-2} , I_{2n-1} to be closed intervals, symmetric with respect to x and such that the right end of I_{2n-2} equals the left end of I_{2n-4} and moreover

$$|I_{2n-2}| = d(I_{2n-2}, F) = \left(\frac{1}{2}\right)^{n-2} \frac{1}{6} \le 2|I_{2n-2}| = \left(\frac{1}{2}\right)^{n-2} \frac{2}{6}.$$

I_{2n-1} has analogical properties.

It is clear that $\sum_{k=1}^{\infty} |I_k| = 1$. Since I_k are closed, $\bigcup_{k=1}^{\infty} I_k$ fils totally the gap between F_1 and F_2. Because there is a countable number of gaps between the particular parts of F so we obtain a countable number of intervals I_k . Let us arrange all the intervals I_k in a sequence.

From the construction above it is seen that actually we have

$$\forall\, i, j \in N,\ i \ne j :\ I_i^0 \cap I_j^0 = \emptyset, \qquad \bigcup_{k=1}^{\infty} I_k = G.$$

We can assume that $\bar{f}(c) = f(c)$ for c being the midpoint of I_k . Such possible modification will not change the properties of f. Then

$$\begin{aligned}
\int_{I_k} |g(y)|\, dy &= \int_{I_k} |f(y) - \bar{f}(y)|\, dy \\
&\le \int_{I_k} |\bar{f}(c) - \bar{f}(y)|\, dy + \int_{I_k} |f(y) - f(c)|\, dy + \int_{I_k} |f(c) - \bar{f}(y)|\, dy \\
&= 2 \int_{I_k} |\bar{f}(c) - \bar{f}(y)|\, dy + \int_{I_k} |f(c) - f(y)|\, dy.
\end{aligned}$$

Applying (6) we have

$$\int_{I_k} |\bar{f}(c) - \bar{f}(y)|\, dy \le \int_{I_k} C\varphi(|c - y|)\, dy \le C|I_k|\varphi(|I_k|),$$

and by using Marcinkiewicz's Theorem we can estimate

$$\int_{I_k} |f(c) - f(y)|\, dy = \int_0^{|I_k|} |f(c+t) - f(c)| dt = |I_k| \cdot O(\varphi(|I_k|)).$$

Finally,

$$\int_{I_k} |g(y)|\, dy \le 2C \cdot \varphi(|I_k|) \cdot |I_k| + |I_k| \cdot O(\varphi(|I_k|)) = K_0 \cdot \varphi(|I_k|) \cdot |I_k|, \qquad (8)$$

for any k, with a suitable constant K_0.

Using (8) in (7), we obtain

$$\begin{aligned} J &= \sum_{k=1}^{\infty} \int_{I_k} |g(y)| \left(\int_F \frac{\bar{w}(|x-y|)}{|x-y|}\, dx \right) dy \\ &< K_1 \sum_{k=1}^{\infty} \left(\int_{I_k} |g(y)|\, dy \right) [\varphi(|I_k|)]^{-1} < K_2 \sum_{k=1}^{\infty} |I_k|, \end{aligned}$$

provided that

$$\int_F \frac{\bar{w}(|x-y|)}{|x-y|}\, dx \le K_3 \cdot [\varphi(|I_k|)]^{-1}, \qquad (9)$$

where K_1, K_2, K_3 are some constants. To prove (9) take $y \in I_k$, $x \in F$, such that $|x-y| \ge |I_k|$. Then

$$\int_F \frac{\bar{w}(|x-y|)}{|x-y|}\, dx \le |I_k|^{-1} \int_F \bar{w}(|x-y|)\, dx.$$

Let $|x-y| = s$. Since $\bar{w}(s) = 0$ for $s \in (1, 2\pi]$, we obtain

$$\begin{aligned} |I_k|^{-1} \int_F \bar{w}(|x-y|)\, dx &\le \int_{|I_k|}^{2\pi} \frac{\bar{w}(s)}{|I_k|}\, ds \le \int_{|I_k|}^{2\pi} \frac{\bar{w}(s)}{s}\, ds \\ &= \int_{|I_k|}^{1} \frac{\bar{w}(s)}{s}\, ds = [\varphi(|I_k|)]^{-1}. \end{aligned}$$

From Fubini's theorem it follows

$$\int_{-\delta}^{\delta} |g(x) - g(x+t)| \frac{\bar{w}(|t|)}{|t|}\, dt < \infty \qquad (10)$$

for almost each x of F. The condition (10) is also satisfied for function w because $w(t) \leq \bar{w}(t)$. The Fourier series of f is divergent almost everywhere in F (see Marcinkiewicz's Theorem and the above construction of f). The Fourier series of $\bar{f}$ is convergent almost everywhere because

$$\int_0^{2\pi}\int_0^{2\pi} |\bar{f}(x) - \bar{f}(y)|^2 \frac{1}{|x-y|}\, dxdy < \infty$$

(see Lemma 1). If $g = f - \bar{f}$, so the Fourier series of g is divergent almost everywhere. This proves the theorem.

Remark
The apparent inconsistency of convergence of the Fourier series $\bar{f}$ almost everywhere and divergence of the Fourier series f almost everywhere in F ($|F| > 0$) results from the fact that set F contains no intervals.

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On continuous and residual spectra of operators connected with iterative functional equations

Abstract. The sum type operator F, given by

$$F[\varphi](x) := \sum_{\nu=0}^{\infty} 2^{-\nu}\varphi(2^{\nu}x),$$

will be considered on the space D of bounded real functions, equipped with the supremum norm and on its three proper closed subspaces. All the according restrictions are Banach space automorphisms. In their spectral theory some iterative functional equations arise in a natural way. We determine in all four cases the resolvent set, the point spectrum, the continuous spectrum and the residual spectrum.

1. Introduction

All the sets

$$\begin{aligned}
D_{11} &:= \left\{\varphi\colon \mathbb{R} \longrightarrow \mathbb{R};\ \sum_{\nu=0}^{\infty} 2^{-\nu}\varphi(2^{\nu}x) \text{ converges for every } x\in\mathbb{R}\right\},\\
D_{21} &:= \{\varphi \in D_{11};\ \varphi \text{ bounded in a vicinity of } -\infty \text{ and of } +\infty\},\\
D_{31} &:= \{\varphi \in D_{11};\ \varphi \text{ bounded}\} = D,\\
D_{41} &:= \{\varphi \in D_{11};\ \varphi \text{ bounded and continuous}\},
\end{aligned}$$

as well as

$$\begin{aligned}
D_{k2} &:= \{\varphi \in D_{k1};\ \varphi \text{ 1-periodic}\},\ 1 \le k \le 4,\\
D_{k3} &:= \{\varphi \in D_{k1};\ \varphi \text{ even}\},\ 1 \le k \le 4,\\
D_{k4} &:= \{\varphi \in D_{k1};\ \varphi \text{ 1-periodic and even}\},\ 1 \le k \le 4,
\end{aligned}$$

are real vector spaces and, in particular, the sets D_{3m} and D_{4m}, $1 \le m \le 4$, are real Banach spaces (equipped with the supremum norm).

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The sum type operators $F_{km}\colon D_{km} \longrightarrow F_{km}(D_{km})$, given by

$$F_{km}[\varphi](x) := \sum_{\nu=0}^{\infty} 2^{-\nu}\varphi(2^{\nu}x), \tag{1}$$

are vector space isomorphisms and, in particular, the operators F_{3m} and F_{4m} , $1 \le m \le 4$, are Banach space automorphisms.

Motivations to study the operators F_{km} and further references are given in [1], [2] and [3]. The structure of the basic domain D_{11} is described in [3]. A first connection of our operators to iterative functional equations is given in

Proposition 1
Assume that $\varphi \in D_{km}$. *Then* $f = F_{km}[\varphi]$ *satisfies*

$$\forall\, x \in \mathbb{R}:\ f(x) - \frac{1}{2}f(2x) = \varphi(x). \tag{2}$$

A second connection appears when describing the eigenspace $E(F_{km},\lambda)$ of F_{km} with respect to the eigenvalue $\lambda \in \sigma_p(F_{km})$:

Proposition 2
$\varphi \in E(F_{km},\lambda)$ *iff* $\varphi \in D_{km}$ *and*

$$\forall\, x \in \mathbb{R}:\ \varphi(x) = \gamma\varphi(2x), \quad \gamma := \frac{1}{2}\frac{\lambda}{\lambda - 1}. \tag{3}$$

A proof of the de Rham type equation (2) for $F_{km}[\varphi]$ is straightforward, a proof of Proposition 2 is given in [1]. The Schröder equation (3) and more iterative functional equations will appear later again when we consider the surjectivity of the operator $\lambda I - F_{km}$, $I = \mathrm{id}\,|_{D_{km}}$.

The point spectra $\sigma_p(F_{k1})$, $1 \le k \le 4$, and the corresponding eigenspaces can be found in [2], as well as the continuous spectra $\sigma_c(F_{31})$, $\sigma_c(F_{41})$ and the residual spectra $\sigma_r(F_{31})$, $\sigma_r(F_{41})$. The full set of point spectra $\sigma_p(F_{km})$, $1 \le k$, $m \le 4$, and a description of the corresponding eigenspaces is given in the recent paper [1].

In Chapter 2. we shall describe the resolvent $\rho(F_{3m})$ and the continuous and the residual spectra of F_{3m} for the remaining values $2 \le m \le 4$.

So this note can be considered as an extension of [2] and as well as an extension of [1].

2. The spectra of F_{km}

In this chapter we are interested in the case $k \geqslant 3$. Then $F_{km}\colon D_{km} \longrightarrow D_{km}$ is a Banach space automorphism. Let us repeat that

$$\rho(F_{km}) = \{\lambda \in \mathbb{R};\ (\lambda I - F_{km})^{-1} \in L(D_{km}, D_{km})\}.$$

$$\sigma_c(F_{km}) = \{\lambda \in \mathbb{R};\ \lambda I - F_{km} \text{ injective, not surjective,}$$
$$\text{cl}(\lambda I - F_{km})[D_{km}] = D_{km}\},$$

$$\sigma_r(F_{km}) = \{\lambda \in \mathbb{R};\ \lambda I - F_{km} \text{ injective, not surjective,}$$
$$\text{cl}(\lambda I - F_{km})[D_{km}] \neq D_{km}\}.$$

So, to compute the continuous and the residual spectra of F_{km} we first have to determine, for which $\lambda \in \mathbb{R}$ the operator $(\lambda I - F_{km})$ is injective and not surjective. The injectivity information is provided by the point spectrum $\sigma_p(F_{km})$, which is known from [1] in all cases $3 \leq k \leq 4$, $1 \leq m \leq 4$. The surjectivity information has to be discussed for the individual F_{km}, but some general remarks are possible. The following first statement is easily verified.

Proposition 3
The operator $\lambda I - F_{km}$ is surjective if and only if

$$\forall\, f \in D_{km}\, \exists \varphi \in D_{km}\, \forall\, x \in \mathbb{R}:\ (\lambda - 1)\varphi(x) - \frac{1}{2}\lambda\varphi(2x) = f(x) - \frac{1}{2}f(2x). \quad (4)$$

Now we show that equation (4) has, for any given $\lambda \in \mathbb{R}\setminus\{\frac{2}{3}, 2\}$ and bounded $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, exactly one bounded solution φ, which we shall denote by $\Phi_{\lambda,f}$.

For $\lambda = 1$ we get $-\frac{1}{2}\varphi(2x) = f(x) - \frac{1}{2}f(2x)$, hence $\Phi_{1,f}(x) = f(x) - 2f(\frac{x}{2})$.

For $\lambda \neq 1$, equation (4) can be written in the equivalent form

$$\varphi(x) - \gamma\varphi(2x) = (\lambda - 1)^{-1}\tilde{f}(x), \quad (5)$$

where γ is as in (3), $\tilde{f}(x) := f(x) - \frac{1}{2}f(2x)$. Note that $\tilde{f}(x)$ is the left hand side of the de Rham equation (2) and that the associated homogeneous equation of (5) is just the Schröder equation (3).

For $\mid \gamma \mid < 1$, i.e., $\lambda \in (-\infty, \frac{2}{3}) \cup (2, \infty)$, iteration of (5) gives

$$(\lambda - 1)\Phi_{\lambda,f}(x) = \sum_{\nu=0}^{\infty} \gamma^{\nu}\tilde{f}(2^{\nu}x). \quad (6)$$

For $\mid \gamma \mid > 1$, i.e., $\lambda \in (\frac{2}{3}, 2) \setminus \{1\}$, iteration of

$$\varphi(x) = \frac{1}{\gamma}\varphi\left(\frac{x}{2}\right) - \frac{1}{\gamma(\lambda - 1)}\tilde{f}\left(\frac{x}{2}\right)$$

gives

$$(1 - \lambda)\Phi_{\lambda,f}(x) = \sum_{\nu=1}^{\infty} \gamma^{-\nu}\tilde{f}(2^{-\nu}x). \quad (7)$$

It is straightforward to check that in each case $\Phi_{\lambda,f}$ is in fact a bounded solution of (4).

After these preparations we are ready to supply the announced information on $\rho(F_{km})$, $\sigma_c(F_{km})$ and $\sigma_r(F_{km})$ for $k = 3$ in the following

THEOREM
We have

a) $\rho(F_{31}) = \mathbb{R} \setminus \{\frac{2}{3}, 2\}$, $\sigma_c(F_{31}) = \emptyset$, $\sigma_r(F_{31}) = \emptyset$,

b) $\rho(F_{32}) = (-\infty, \frac{2}{3}) \cup (2, \infty)$, $\sigma_c(F_{32}) = \emptyset$, $\sigma_r(F_{32}) = (\frac{2}{3}, 2)$,

c) $\rho(F_{33}) = \mathbb{R} \setminus \{\frac{2}{3}, 2\}$, $\sigma_c(F_{33}) = \emptyset$, $\sigma_r(F_{33}) = \emptyset$,

d) $\rho(F_{34}) = (-\infty, \frac{2}{3}) \cup (2, \infty)$, $\sigma_c(F_{34}) = \emptyset$, $\sigma_r(F_{34}) = (\frac{2}{3}, 2)$.

Proof. In [1] it is shown that $\sigma_p(F_{3m}) = \{\frac{2}{3}, 2\}$, $1 \leq m \leq 4$. So the operator $\lambda I - F_{3m}$ is injective for every $\lambda \in \mathbb{R} \setminus \{\frac{2}{3}, 2\}$.

a) This was already proved in [2].

b) Let $\lambda \in \mathbb{R} \setminus \{\frac{2}{3}, 2\}$. Then the operator $\lambda I - F_{32} \colon D_{32} \longrightarrow D_{32}$ is surjective iff (4) holds with $k = 3$, $m = 2$.

For $f \in D_{32}$ (i.e. f bounded and 1-periodic) and $\lambda \in (-\infty, \frac{2}{3}) \cup (2, \infty)$ we get $\Phi_{\lambda,f} \in D_{32}$, because then also $\tilde{f} \in D_{32}$ and therefore also $\Phi_{\lambda,f}$, given by (6), is bounded and of period 1 (note that $|\gamma| < 1$).

On the other hand, in case $\lambda \in (\frac{2}{3}, 2)$ there are $f \in D_{32}$ such that neither of the functions $\Phi_{1,f}$ nor $\Phi_{\lambda,f}$, given by (7), are 1-periodic. Explicit examples are given below.

Therefore, $\lambda I - F_{32} \colon D_{32} \longrightarrow D_{32}$ is bijective exactly for $\lambda \in (-\infty, \frac{2}{3}) \cup (2, \infty)$, and by the inverse operator theorem, this set coincides with the resolvent set.

So far we have shown that $\sigma_c(F_{32}) \cup \sigma_r(F_{32}) = (\frac{2}{3}, 2)$.

We shall prove now that $\sigma_c(F_{32}) = \emptyset$. To this end, take first the case $\lambda = 1$. Consider the particular element $d \in D_{32}$ given by

$$d(x) := \operatorname{dist}(x, \mathbb{Z}).$$

Then $\Phi_{1,d}(x) = d(x) - 2d(\frac{x}{2})$ is not of period 1, hence $\Phi_{1,d} \notin D_{32}$. Therefore there is no $\varphi \in D_{32}$ such that $(I - F_{32})[\varphi] = d$.

Now let $h \in D_{32}$ such that $\|h - d\| < \frac{1}{10}$. We get

$$\Phi_{1,h}(0) = h(0) - 2h(0) \in [-\tfrac{1}{10},\ \tfrac{1}{10}]\ ,$$
$$\Phi_{1,h}(1) = h(1) - 2h\left(\frac{1}{2}\right) \in [-\tfrac{13}{10}, -\tfrac{7}{10}],$$

which means that also $\Phi_{1,h} \notin D_{32}$. So $(I - F_{32})[D_{32}] \cap U_{\frac{1}{10}}(d) = \emptyset$, i.e., $\mathrm{cl}(I - F_{32})[D_{32}] \neq D_{32}$ and $1 \in \sigma_r(F_{32})$.

Finally consider the case $\lambda \in (\frac{2}{3}, 2) \setminus \{1\}$ (i.e., $|\gamma| > 1$). The surjectivity of $\lambda I - F_{32}$ would imply in particular that $\Phi_{\lambda,d}$, given by (7), is 1-periodic. Here we have

$$(1-\lambda)\Phi_{\lambda,d}(x) = \sum_{\nu=1}^{\infty} \gamma^{-\nu} \tilde{d}(2^{-\nu}x)$$

with $\tilde{d}(x) = d(x) - \frac{1}{2}d(2x)$. The function $\tilde{d}$ is piecewise affine with vertices through $(0,0)$, $(\frac{1}{4},0)$, $(\frac{1}{2},\frac{1}{2})$, $(\frac{3}{4},0)$, $(1,0)$ and is of period one. However $\Phi_{\lambda,d}$ is not 1-periodic, because

$$\begin{aligned}(1-\lambda)\Phi_{\lambda,d}(0) &= 0,\\ (1-\lambda)\Phi_{\lambda,d}(1) &= \frac{1}{\gamma}\cdot\frac{1}{2} + 0 \neq 0.\end{aligned}$$

Now let $h \in D_{32}$ such that $\|h - d\| < \varepsilon$. Then a short calculation shows that $\|\tilde{h} - \tilde{d}\| < 2\varepsilon$. We want to prove that $\Phi_{\lambda,h}$, given by

$$(1-\lambda)\Phi_{\lambda,h}(x) = \sum_{\nu=1}^{\infty} \gamma^{-\nu} \tilde{h}(2^{-\nu}x),$$

is not 1-periodic, if

$$\varepsilon < \frac{|\gamma| - 1}{4|\gamma|}.$$

Consider first the case $\gamma > 1$ (i.e. $1 < \lambda < 2$). We have $\tilde{d}(t) - 2\varepsilon \leq \tilde{h}(t) \leq \tilde{d}(t) + 2\varepsilon$ for every $t \in \mathbb{R}$. Therefore

$$\begin{aligned}\sum_{\nu=1}^{\infty} \gamma^{-\nu}[\tilde{d}(2^{-\nu}x) - 2\varepsilon] &= (1-\lambda)\Phi_{\lambda,d}(x) - \frac{2\varepsilon}{\gamma - 1}\\ &\leq (1-\lambda)\Phi_{\lambda,h}(x) \leq \sum_{\nu=1}^{\infty} \gamma^{-\nu}[\tilde{d}(2^{-\nu}x) + 2\varepsilon]\\ &= (1-\lambda)\Phi_{\lambda,d}(x) + \frac{2\varepsilon}{\gamma - 1}.\end{aligned}$$

In particular, we get

$$-\frac{2\varepsilon}{\gamma - 1} \leq (1-\lambda)\Phi_{\lambda,h}(0) \leq \frac{2\varepsilon}{\gamma - 1}$$

and

$$-\frac{2\varepsilon}{\gamma - 1} + \frac{1}{2\gamma} \leq (1-\lambda)\Phi_{\lambda,h}(1) \leq \frac{2\varepsilon}{\gamma - 1} + \frac{1}{2\gamma}.$$

So we obtain $\Phi_{\lambda,h}(0) \neq \Phi_{\lambda,h}(1)$ as $4\varepsilon\gamma < \gamma - 1$.

Now we consider the case $\gamma < -1$ (i.e. $\frac{2}{3} < \lambda < 1$). As γ is negative, we get

$$\begin{aligned}\sum_{\nu=1}^{\infty} \gamma^{-\nu}[\tilde{d}(2^{-\nu}x) + (-1)^{\nu-1}2\varepsilon] &= (1-\lambda)\Phi_{\lambda,d}(x) + \frac{2\varepsilon}{\gamma+1} \\ &\leq (1-\lambda)\Phi_{\lambda,h}(x) \\ &\leq \sum_{\nu=1}^{\infty} \gamma^{-\nu}[\tilde{d}(2^{\nu}x) + (-1)^{\nu}2\varepsilon] \\ &= (1-\lambda)\Phi_{\lambda,d}(x) - \frac{2\varepsilon}{\gamma+1},\end{aligned}$$

hence

$$(1-\lambda)\Phi_{\lambda,d}(x) - \frac{2\varepsilon}{\mid \gamma \mid -1} \leq (1-\lambda)\Phi_{\lambda,h}(x) \leq (1-\lambda)\Phi_{\lambda,d}(x) + \frac{2\varepsilon}{\mid \gamma \mid -1}.$$

So we obtain as above $\Phi_{\lambda,h}(0) \neq \Phi_{\lambda,h}(1)$ as $4\varepsilon \mid \gamma \mid < \mid \gamma \mid -1$.

Consequently $(\lambda I - F_{32})[D_{32}] \cap U_\varepsilon(d) = \emptyset$, i.e., $\mathrm{cl}(\lambda I - F_{32})[D_{32}] \neq D_{32}$ and $\lambda \in \sigma_r(F_{32})$.

c) Let $\lambda \in \mathbb{R} \setminus \{\frac{2}{3}, 2\}$. Then the operator $\lambda I - F_{33}\colon D_{33} \longrightarrow D_{33}$ is surjective iff (4) holds with $k = m = 3$.

If f is bounded and even, then $\tilde{f}$ and the function $x \mapsto f(x) - 2f(\frac{x}{2})$ are bounded and even, hence the unique bounded solutions $\Phi_{\lambda,f}$ of (4) are bounded and even as well.

Therefore $\lambda I - F_{33}$ is bijective for every $\lambda \in \mathbb{R} \setminus \{\frac{2}{3}, 2\}$, and again by the inverse operator theorem, $\rho(F_{33}) = \mathbb{R} \setminus \{\frac{2}{3}, 2\}$.

d) Let $\lambda \in \mathbb{R} \setminus \{\frac{2}{3}, 2\}$. Then the operator $\lambda I - F_{34}\colon D_{34} \longrightarrow D_{34}$ is surjective iff (4) holds with $k = 3$, $m = 4$.

If $f \in D_{34}$ and $\lambda \in (-\infty, \frac{2}{3}) \cup (2, \infty)$, the representation (6) shows that also $\Phi_{\lambda,f} \in D_{34}$.

On the other hand, for $\lambda \in (\frac{2}{3}, 2)$, the counterexample $d \in D_{32}$ from b) can be used here as well, because we have also $d \in D_{34}$ (d is 1-periodic and even). With the same argument as in b) we see that $(\lambda I - F_{34})[D_{34}] \cap U_\varepsilon(d) = \emptyset$, provided that $\varepsilon < \frac{1}{10}$ in case $\lambda = 1$, $\varepsilon < \frac{|\gamma|-1}{4|\gamma|}$ in case $\lambda \in (\frac{2}{3}, 2) \setminus \{1\}$. So $\mathrm{cl}(\lambda I - F_{34})[D_{34}] \neq D_{34}$ and $\lambda \in \sigma_r(F_{34})$.

Determining the corresponding statements about $\rho(F_{4k})$, $\sigma_c(F_{4k})$ and $\sigma_r(F_{4k})$ remains the subject of future research.

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Partial difference equations arising from the Cauchy-Riemann equations

Abstract. We consider some functional equations arising from the Cauchy–Riemann equations, and certain related functional equations. First we propose a new functional equation (E.1) below, over a 2-divisible Abelian group, which is a discrete version of the Cauchy-Riemann equations, and give the general solutions of (E.1). Next we study a functional equation which is equivalent to (E.1). Further we propose and solve partial difference-differential functional equations and nonsymmetric partial difference equations which are also arising from the Cauchy–Riemann equations.

1. Introduction

Let $(G, +)$ be an additive Abelian group in which it is possible to divide by 2. Let $\mathbb{C}$ be the field of complex numbers. The main aim of this note is to determine the general solution of the following new functional equation

$$f(x+t, y) - f(x-t, y) = -i[f(x, y+t) - f(x, y-t)] \tag{E.1}$$

for all $x, y, t \in G$, where $f\colon G \times G \longrightarrow \mathbb{C}$ and i is the imaginary unit.

Let $\mathbb{R}$ be the field of real numbers. For a function $f\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ we define the divided partial difference operators $\triangle_{x,t}$ and $\triangle_{y,t}$ by

$$(\triangle_{x,t} f)(x, y) = \frac{f(x+t, y) - f(x, y)}{t}$$

and

$$(\triangle_{y,t} f)(x, y) = \frac{f(x, y+t) - f(x, y)}{t},$$

respectively. Then the partial difference equation

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$$\triangle_{x,t} f = -i \triangle_{y,t} f$$

may be considered as a discrete analogue of the Cauchy–Riemann equation

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}.$$

This equation may be rewritten in the form

$$f(x+t,y) - f(x,y) = -i[f(x,y+t) - f(x,y)] \tag{E.2}$$

for all $x, y, t \in \mathbb{R}$, which has a simple geometric interpretation on the plane.

Equation (E.2) is considered in the papers of J. Aczél and S. Haruki 1981 [3], and S. Haruki 1986 [6]. The authors show, among others, that (E.2) does not lead essentially beyond a linear function in the case when $\mathbb{R}$ is replaced by an arbitrary monoid M. Further, in the paper of S. Haruki and C.T. Ng 1994 [7] the general solution of (E.2) is obtained in more general algebraic structures than M and $\mathbb{C}$.

It is natural to ask what happens if, instead of operators $\triangle_{x,t}$ and $\triangle_{y,t}$, we impose the divided partial mean difference operators $\bigtriangledown_{x,t}$ and $\bigtriangledown_{y,t}$ defined by

$$(\bigtriangledown_{x,t} f)(x,y) = \frac{f(x+t,y) - f(x-t,y)}{2t}$$

and

$$(\bigtriangledown_{y,t} f)(x,y) = \frac{f(x,y+t) - f(x,y-t)}{2t}.$$

In this case we have the partial difference equation

$$\bigtriangledown_{x,t} f = -i \bigtriangledown_{y,t} f,$$

which is also a discrete analogue of the Cauchy–Riemann equation. This leads to the above functional equation (E.1) which also has a simple geometric interpretation on the plane. As a main result of this note we show that equation (E.1) for $f: G \times G \longrightarrow \mathbb{C}$ does not lead essentially beyond a quadratic function.

In Section 2 we determine the general and the regular solutions (when G is replaced by $\mathbb{R}$) of equation (E.1).

We also show in Section 3 that similar results hold for certain related functional equations. In Section 3.1 we consider the functional equation

$$f(x+t,y+t) - f(x-t,y-t) = -i[f(x-t,y+t) - f(x+t,y-t)]. \tag{E.3}$$

In Section 3.2 we study the partial difference-differential equations

$$\frac{\partial f(x,y)}{\partial x} = -i \left[\frac{f(x,y+t) - f(x,y-t)}{2t} \right],$$

$$\frac{f(x+t,y) - f(x-t,y)}{2t} = -i \frac{\partial f(x,y)}{\partial y}.$$

Finally in Section 3.3 we propose and solve several functional equations of nonsymmetric type, which are also analogous to the Cauchy–Riemann equation.

2. Functional equation (E.1)

2.1. The general solution

A function $A^1 \colon G \longrightarrow \mathbb{C}$ is said to be *additive* if A^1 satisfies

$$A^1(x+y) = A^1(x) + A^1(y) \qquad \text{for all } x, y \in G.$$

A function $A_2 \colon G \times G \longrightarrow \mathbb{C}$ is said to be *bi-additive* if A_2 satisfies both equations

$$A_2(x+y, z) = A_2(x, z) + A_2(y, z)$$

and

$$A_2(x, y+z) = A_2(x, y) + A_2(x, z)$$

for all $x, y, z \in G$. A function $A^2 \colon G \longrightarrow \mathbb{C}$ is *the diagonalization of the* A_2 if

$$A^2(x) = A_2(x, x),$$

whenever $A_2 \colon G \times G \longrightarrow \mathbb{C}$ is symmetric and additive in each argument.

Our main result of this note is as follows.

THEOREM 2.1
A function $f \colon G \times G \longrightarrow \mathbb{C}$ satisfies equation (E.1) *for all $x, y, z \in G$ if and only if there exist*

(i) *a complex constant A^0,*

(ii) *an additive function $A^1 \colon G \longrightarrow \mathbb{C}$,*

(iii) *a symmetric bi-additive function $A_2 \colon G \times G \longrightarrow \mathbb{C}$*

such that

$$f(x, y) = A^0 + A^1(x) + iA^1(y) + A^2(x) - A^2(y) + 2iA_2(x, y) \tag{S.1}$$

for all $x, y \in G$, where $A^2 \colon G \longrightarrow \mathbb{C}$ is the diagonalization of A_2.

We impose the following notations. Define the shift operators X^t and Y^t by

$$(X^t f)(x, y) = f(x+t, y) \quad \text{and} \quad (Y^t f)(x, y) = f(x, y+t) \qquad \text{for all } x, y, t \in G.$$

In particular $1 = X^0 = Y^0$ denote the identity operator. Further, define the partial mean difference operators $\delta_{x,t}$ and $\delta_{y,t}$ by

$$\delta_{x,t} = X^t - X^{-t} \quad \text{and} \quad \delta_{y,t} = Y^t - Y^{-t} \qquad \text{for all } x, y, t \in G.$$

Notice that the ring of operators generated by this family of operators is commutative and distributive.

In order to prove Theorem 2.1 we need the following two lemmas. One of them is:

Lemma 2.1
If a function $f: G \times G \longrightarrow \mathbb{C}$ *satisfies equation* (E.1) *for all* $x, y, t \in G$, *then* f *also satisfies each one of the following three functional equations*

$$(\delta^3_{x,t} f)(x,y) = 0 \qquad and \qquad (\delta^3_{y,t} f)(x,y) = 0 \tag{2.1}$$

$$((\delta^2_{x,t} + \delta^2_{y,t}) f)(x,y) = 0 \tag{2.2}$$

or as the expanded form (2t replaced by t)

$$f(x+t,y) + f(x-t,y) + f(x,y+t) + f(x,y-t) = 4f(x,y) \tag{2.3}$$

for all $x, y \in G$.

The above Lemma 2.1 shows that equation (E.1) yields equation (2.3). On the other hand, J. Aczél, H. Haruki, M.A. McKiernan and G.N. Sakovič 1968 [2, p. 43, Lemma 3] proved that equation (2.3) is equivalent to the Haruki functional equation (M.A. McKiernan [11], H. Światak [13], among others)

$$f(x+t,y+t) + f(x+t,y-t) + f(x-t,y+t) + f(x-t,y-t) = 4f(x,y). \tag{2.4}$$

Hence, if we directly apply a general theorem of M.A. McKiernan 1970 [12, p.32, Theorem 2] to equation (2.4), then we obtain

$$(\delta^k_{x,t} f)(x,y) = 0 \qquad \text{and} \qquad (\delta^k_{y,t} f)(x,y) = 0 \tag{2.5}$$

with $k = 11$. On the other hand, it is also known, cf. [2, p. 43, Lemma 3], that if an arbitrary f satisfies (2.4), then f also satisfies difference equations (2.5) for $k = 4$. However, equations (2.1), that is, (2.5) serve as a better tool to prove the *'only if'* part of Theorem 2.1, since if $k > 3$ in (2.5), then the solution of (2.1) contains more symmetric multiadditive functions of higher order (cf. S. Mazur and W. Orlicz 1934 [9], and M.A. McKiernan 1967 [10], among others).

The other is a lemma which is a particular case of Lemma 6 in [2, p. 49-50]. We note that if we replace $f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by $f: G \times G \longrightarrow \mathbb{C}$ in Lemma 6 of [2], then it follows from a general theorem of S. Mazur and W. Orlicz [9] that the result of Lemma 6 in [2] still holds for the case $\delta_{x,t} = X^t - X^{-t}$ and $\delta_{y,t} = Y^t - Y^{-t}$ instead of $\Delta_{x,t} = X^{\frac{t}{2}} - X^{-\frac{t}{2}}$ and $\Delta_{y,t} = Y^{\frac{t}{2}} - Y^{-\frac{t}{2}}$ defined in [2, p. 43].

Lemma 2.2
A function $f: G \times G \longrightarrow \mathbb{C}$ *satisfies both equations* (2.1) *for all* $x, y, t \in G$ *if and only if* f *is given by*

$$\begin{aligned} f(x,y) = A^0 + A^1(x) + A^2(x) + B^1(y) + B^2(y) \\ + A^{1,1}(x;y) + A^{2,1}(x;y) + A^{1,2}(x;y) + A^{2,2}(x;y) \end{aligned} \tag{2.6}$$

for all $x, y \in G$, *where* $A^0, A^1, A^2 : G \longrightarrow \mathbb{C}$ *are defined in Theorem 2.1,* $B^1 : G \longrightarrow \mathbb{C}$ *is an additive function, and* $B^2 : G \longrightarrow \mathbb{C}$ *is the diagonalization of a symmetric bi-additive function. Further, the functions* $A^{1,1}, A^{2,1}, A^{1,2}, A^{2,2}$: $G \times G \longrightarrow \mathbb{C}$ *are defined as follows:*

$$A^{1,1}(x;y) = A_{1,1}(x;y), \qquad A^{2,1}(x;y) = A_{2,1}(x,x;y),$$
$$A^{1,2}(x;y) = A_{1,2}(x;y,y), \qquad A^{2,2}(x;y) = A_{2,2}(x,x;y,y),$$

where $A_{i,j}$, $i, j = 1, 2$ *are additive functions in each of their variables.*

By Lemma 6 in [2, p. 49-50] we have

$$f(x,y) = \sum_{n,m=0}^{2} A^{n,m}(x;y),$$

that is, $f(x,y)$ is a generalized quadratic polynomial in x and y and can be written as expression (2.6).

Proof of Lemma 2.1. We multiply (E.1) by i and then write equation (E.1) in the operator form

$$[(iX^t - iX^{-t})f](x,y) = [(Y^t - Y^{-t})f](x,y)$$

which may be written briefly as

$$iX^t - iX^{-t} = Y^t - Y^{-t} \tag{2.7}$$

for f. We will omit the f whenever no confusion can rise. Now, cube the operators on both sides of (2.7) to obtain

$$-iX^{3t} + 3iX^t - 3iX^{-t} + iX^{-3t} = Y^{3t} - 3Y^t + 3Y^{-t} - Y^{-3t}. \tag{2.8}$$

By multiplying (2.7) by 3

$$3iX^t - 3iX^{-t} = 3Y^t - 3Y^{-t}, \tag{2.9}$$

while by multiplying the both sides by -1 and by replacing t by $3t$ in (2.7) we have

$$-iX^{3t} + iX^{-3t} = -Y^{3t} + Y^{-3t}. \tag{2.10}$$

If we substitute (2.9) and (2.10) in (2.8) in order to eliminate the operators $3iX^t$, $-3X^{-t}$, $-X^{3t}$, and iX^{-3t} from (2.8), then

$$Y^{3t} - 3Y^t = Y^{-3t} - 3Y^{-t}, \qquad \text{and} \qquad (Y^t - Y^{-t})^3 = 0, \tag{2.11}$$

which is the second equation of (2.1).

Similarly, substitute (2.9) and (2.10) in (2.8) to eliminate the operators Y^{3t}, $-3Y^t$, $3Y^{-t}$, and $-Y^{-3t}$ from (2.8). Then we obtain the first equation of (2.1).

Next, square both sides of (2.7) to obtain (2.2), while replace $2t$ by t in (2.2) to obtain (2.3). This completes the proof of Lemma 2.1.

Proof of Theorem 2.1. Since f satisfies (2.1), by Lemma 2.2 f is given by (2.6). If we substitute (2.6) into equation (2.3), then

$$\begin{aligned} A_2(t,t) + A_{2,1}(t,t;y) + B_2(t;t) + A_{1,2}(x;t,t) \\ + A_{2,2}(x,x;t,t) + A_{2,2}(t,t;y,y) \\ = 0. \end{aligned} \tag{2.12}$$

Set $x = y = 0$ in (2.12) to obtain

$$B^2(t) = -A^2(t), \tag{2.13}$$

which, with (2.12), implies

$$A_{2,1}(t,t;y) + A_{1,2}(x;t,t) + A_{2,2}(x,x;t,t) + A_{2,2}(t,t;y,y) = 0. \tag{2.14}$$

Further, set $x = 0$, $y = 0$, respectively, in (2.14). Then we have

$$A^{2,1}(t;y) + A^{2,2}(t;y) = 0 \qquad \text{and} \qquad A^{1,2}(x;t) + A^{2,2}(x;t) = 0$$

for all $x, y, t \in G$, which show that

$$A^{2,1}(x;y) + A^{2,2}(x;y) = 0 \tag{2.15}$$

and

$$A^{1,2}(x;y) + A^{2,2}(x;y) = 0 \tag{2.16}$$

for all $x, y \in G$. Subtract (2.16) from (2.15) to obtain

$$A^{2,1}(x;y) - A^{1,2}(x;y) = 0. \tag{2.17}$$

Thus it follows from (2.6), (2.13), and (2.15) that

$$f(x,y) = A^0 + A^1(x) + A^2(x) + B^1(y) - A^2(y) + A^{1,1}(x;y) + A^{1,2}(x;y). \tag{2.18}$$

Next, substitute (2.18) into equation (2.7), that is,

$$i[f(x+t,y) - f(x-t,y)] = f(x,y+t) - f(x,y-t)],$$

to obtain

$$\begin{aligned} iA^1(t) + 2iA_2(x,t) + iA^{1,1}(t,y) + iA_{1,2}(t;y,y) \\ = B^1(t) + A^{1,1}(x,t) - 2A_2(y,t) + 2A_{1,2}(x;y,t). \end{aligned} \tag{2.19}$$

Set $x = 0$, $y = 0$, and $x = y = 0$, respectively in (2.19). Then we have the following three equations

$$iA^1(t) + iA^{1,1}(t,y) + iA_{1,2}(t;y,y) = B^1(t) - 2A_2(y,t) \tag{2.20}$$

$$iA^1(t) + 2iA_2(x,t) = B^1(t) + A^{1,1}(x,t) \tag{2.21}$$

$$iA^1(t) = B^1(t). \tag{2.22}$$

Equations (2.20) and (2.22) yield

$$iA^{1,1}(t;y) + iA_{1,2}(t;y,y) = \, -\, 2A_2(y,t)$$

which can be rewritten in the form

$$iA^{1,1}(x,y) + iA_{1,2}(x;y,y) = \, -\, 2A_2(y,x) \tag{2.23}$$

while (2.21) and (2.22) imply $2iA_2(x,t) = A^{1,1}(x,t)$ and

$$2iA_2(x,y) = A^{1,1}(x;y). \tag{2.24}$$

Further, it follows from (2.24) and (2.23) that $-2A_2(x,y) + iA_{1,2}(x;y,y) = -2A_2(y,x)$, and, since A_2 is symmetric,

$$A^{1,2}(x;y) = 0. \tag{2.25}$$

Thus equation (2.18) with (2.22), (2.24) and (2.25) implies (S.1) for all $x, y \in G$.

Conversely, (S.1) satisfies equation (E.1). This completes the proof of Theorem 2.1.

2.2. Regular solutions

In addition, as soon as some suitable regularity assumptions are imposed on f for the case $G = \mathbb{R}$ in the above Theorem 2.1, it can be readily shown by the following lemma that f is an ordinary complex polynomial of degree at most two. The following lemma is a consequence of Theorem 2.1.

LEMMA 2.3

Let $(F,+)$ be an additive group. If $f\colon F \times F \longrightarrow \mathbb{C}$ is given by (S.1) *for all $x, y \in F$, then all functions $A^1, A^2\colon F \longrightarrow \mathbb{C}$ and $A_2\colon F \times F \longrightarrow \mathbb{C}$ can be represented in terms of f and a constant A^0 by*

$$A^1(x) = \frac{f(x,y) - f(-x,-y) - f(-x,y) + f(x,-y)}{4}, \tag{2.26}$$

$$A^1(y) = \frac{f(x,y) - f(-x,-y) + f(-x,y) - f(x,-y)}{4i}, \tag{2.27}$$

$$A_2(x,y) = \frac{f(x,y) + f(-x,-y) - f(-x,y) - f(x,-y)}{8i}, \tag{2.28}$$

$$A^2(x) = f(x,0) - A^0 - \frac{f(x,y) - f(-x,-y) - f(-x,y) + f(x,-y)}{4}, \tag{2.29}$$

$$A^2(y) = \, -\, f(0,y) + A^0 + \frac{f(x,y) - f(-x,-y) + f(-x,y) - f(x,-y)}{4} \tag{2.30}$$

for all $x, y \in F$.

Proof. It follows from

$$f(x,y) = A^0 + A^1(x) + iA^1(y) + A^2(x) - A^2(y) + 2iA_2(x,y) \qquad \text{(S.1)}$$

that

$$f(-x,-y) = A^0 - A^1(x) - iA^1(y) + A^2(x) - A^2(y) + 2iA_2(x,y), \qquad (2.31)$$

$$f(-x,y) = A^0 - A^1(x) + iA^1(y) + A^2(x) - A^2(y) - 2iA_2(x,y), \qquad (2.32)$$

$$f(x,-y) = A^0 + A^1(x) - iA^1(y) + A^2(x) - A^2(y) - 2iA_2(x,y). \qquad (2.33)$$

Subtract (2.31) from (S.1) and (2.33) from (2.32), respectively, to obtain

$$f(x,y) - f(-x,-y) = 2A^1(x) + 2iA^1(y), \qquad (2.34)$$

$$f(-x,y) - f(x,-y) = -2A^1(x) + 2iA^1(y). \qquad (2.35)$$

Add (2.34) and (2.35) to obtain (2.27). Subtract (2.35) from (2.34) to obtain (2.26). Next by adding (S.1) and (2.31) we have

$$f(x,y) + f(-x,-y) = 2A^0 + 2A^2(x) - 2A^2(y) + 4iA_2(x,y). \qquad (2.36)$$

Further, add (2.32) and (2.33) to obtain

$$f(-x,y) + f(x,-y) = 2A^0 + 2A^2(x) - 2A^2(y) - 4iA_2(x,y). \qquad (2.37)$$

If we subtract (2.37) from (2.36), then we have (2.28). By setting $y = 0$ in (S.1) and then by using (2.26) we have (2.29). Set $x = 0$ in (S.1) and then use (2.27) to obtain (2.30). This completes the proof of Lemma 2.3.

If we assume that, for example, $f\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ is continuous everywhere, then by applying the above Lemma 2.3 we have the following result.

Theorem 2.2
A continuous function $f\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ satisfies (E.1) *for all $x, y, t \in \mathbb{R}$ if and only if f is given by*

$$f(x,y) = a(x^2 - y^2 + 2ixy) + b(x + iy) + c \qquad (2.38)$$

for all $x, y \in \mathbb{R}$, where a, b, and c are complex constants.

Proof. Lemma 2.3 clearly holds for the case $F = \mathbb{R}$. If f is continuous everywhere, then it readily follows from (2.26) and (2.28) of Lemma 2.3 that $A^1(x)$ and $A_2(x,y)$ are also continuous for all $x, y \in \mathbb{R}$. It is well-known that a continuous additive function $A^1(x)\colon \mathbb{R} \longrightarrow \mathbb{C}$ is given by $A^1(x) = bx$ [1, p. 36] for all $x \in \mathbb{R}$, where b is a complex constant. It readily follows from this result that a continuous symmetric bi-additive function $A_2\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ is given by $A_2(x,y) = axy$ for all $x, y \in \mathbb{R}$, where a is a complex constant. Hence, (2.38) follows from (S.1) with $A^0 = c$. Conversely, (2.38) satisfies (E.1). This completes the proof of Theorem 2.2.

Equation (E.1) can also be rewritten in the complex form

$$f(z+t)-f(z-t) = -i[f(z+it)-f(z-it)] \tag{2.39}$$

for all $z \in \mathbb{C}$ and $t \in \mathbb{R}$, where $f(z) := f(x,y)$ for all $x, y \in \mathbb{R}$ and $f\colon \mathbb{C} \longrightarrow \mathbb{C}$. In this case the continuous solution (2.38) is given by a complex polynomial of degree at most two such that

$$f(z) = az^2 + bz + c$$

for all $z \in \mathbb{C}$.

3. Certain related functional equations

3.1. Equations (E.3) and its variations

Here we mainly consider the functional equation

$$f(x+t,y+t)-f(x-t,y-t) = -i[f(x-t,y+t)-f(x+t,y-t)] \tag{E.3}$$

for all $x, y, t \in G$, where $f\colon G\times G \longrightarrow \mathbb{C}$, and determine the general and regular solutions of (E.3).

One of applications of functional equations is that to a geometric characterization of complex polynomials from the standpoint of conformal mapping properties. In particular, H. Haruki 1971 [4] obtains the functional equation

$$f(z+te^{\frac{\pi i}{4}})-f(z-te^{\frac{\pi i}{4}}) = i[f(z+te^{-\frac{\pi i}{4}})-f(z-te^{-\frac{\pi i}{4}})] \tag{3.1}$$

for all $z \in \mathbb{C}$ and $t \in \mathbb{R}$, where $f\colon \mathbb{C} \longrightarrow \mathbb{C}$, from two geometric properties on f. Equation (3.1) yields (E.3) for all $x, y, t \in \mathbb{R}$, where $f(x,y) := f(z)$ for $z = x+iy$ and $f\colon \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{C}$. The continuous solution $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ of equation (3.1) is obtained in [4] by using the regularity of solutions of Haruki's functional equation (2.4). We note that the continuity assumption in order to consider equation (3.1) is natural from the point of view of geometric properties of f yielding (3.1) in [4]. Further, it is possible to obtain the general solution of equations (E.3) and (3.1) for $f\colon \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{C}$ and $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ when no regularity assumptions are imposed on f, since it is shown in [4] that (3.1) implies (2.4), and the general solution of (2.4) is obtained in [2, p. 50-51, Theorem 5]. However, in this section we first show that equation (E.3) is equivalent to (E.1) when no regularity assumptions are imposed on f so that by Theorem 2.1 in §2.1 we immediately obtain the general solution of equation (E.3) under no regularity assumptions on f. We emphasize that we do not apply the general solution of equation (2.4). We will be able to generalize our results from $\mathbb{R}$ to G.

THEOREM 3.1
The function $f: G \times G \longrightarrow \mathbb{C}$ *satisfies equation* (E.1) *for all* $x, y, t \in G$, *if and only if it satisfies equation* (E.3) *for all* $x, y, t \in G$.

Proof. As before equation (E.1) can be rewritten in the simple operator form

$$X^t - X^{-t} + iY^t - iY^{-t} = 0 \qquad \text{or} \qquad iX^t - iX^{-t} - Y^t + Y^{-t} = 0. \quad (3.2)$$

Multiply the second equation of (3.2) by the operator $(1-i)(X^t + X^{-t} + Y^t + Y^{-t})$ to obtain

$$\begin{aligned} 2(iX^tY^t - iX^{-t}Y^{-t} - X^{-t}Y^t + X^tY^{-t}) \\ + (X^{2t} - X^{-2t} + iY^{2t} - iY^{-2t}) \\ + (iX^{2t} - iX^{-2t} - Y^{2t} + Y^{-2t}) \\ = 0. \end{aligned}$$

By replacing t by $2t$ in (3.2) we have

$$X^{2t} - X^{-2t} + iY^{2t} - iY^{-2t} = 0 \qquad \text{or} \qquad iX^{2t} - iX^{-2t} - Y^{2t} + Y^{-2t} = 0.$$

Hence, it follows from these operator equations that

$$iX^tY^t - iX^{-t}Y^{-t} - X^{-t}Y^t + X^tY^{-t} = 0 \quad (3.3)$$

and

$$iX^tY^t - iX^{-t}Y^{-t} = X^{-t}Y^t - X^tY^{-t}.$$

Further, multiply the both sides of this equation by $-i$ to obtain

$$X^tY^t - X^{-t}Y^{-t} = -i(X^{-t}Y^t - X^tY^{-t}) \quad (3.4)$$

which is the operator form of equation (E.3). Thus equation (E.1) implies (E.3).

Conversely, multiply (3.4) by i to obtain equation (3.3). Next, by multiplying (3.3) by the operator $(1+i)(X^tY^t + X^{-t}Y^{-t} + X^{-t}Y^t + X^tY^{-t})$ we have

$$\begin{aligned} (iX^{2t}Y^{2t} - iX^{-2t}Y^{-2t} - X^{-2t}Y^{2t} + X^{2t}Y^{-2t}) \\ + (-X^{2t}Y^{2t} + X^{-2t}Y^{-2t} - iX^{-2t}Y^{2t} + iX^{2t}Y^{-2t}) \\ + 2(iX^{2t} - iX^{-2t} - Y^{2t} + Y^{-2t}) \\ = 0. \end{aligned}$$

It follows from (3.3) and (3.4) that

$$iX^{2t}Y^{2t} - iX^{-2t}Y^{-2t} - X^{-2t}Y^{2t} + X^{2t}Y^{-2t} = 0$$

and

$$-X^{2t}Y^{2t} + X^{-2t}Y^{-2t} - iX^{-2t}Y^{2t} + iX^{2t}Y^{-2t} = 0.$$

Therefore we obtain

$$iX^{2t} - iX^{-2t} = Y^{2t} - Y^{-2t}. \tag{3.5}$$

If we multiply both sides of (3.5) by $-i$ and replace $2t$ by t, then we obtain equation (E.1). Hence, equation (E.3) implies equation (E.1). Therefore (E.1) and (E.3) are equivalent.

The following result is an immediate consequence of Theorems 2.1 and 3.1.

Corollary 3.1
A function $f\colon G \times G \longrightarrow \mathbb{C}$ *satisfies equation* (E.3) *for all* $x, y, t \in G$ *if and only if* f *is given by expression* (S.1) *for all* $x, y \in G$.

We also readily obtain from Theorems 3.1 and 2.2 in the case of $G = \mathbb{R}$ that a continuous function $f\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ satisfies (E.3) for all $x, y, t \in \mathbb{R}$ if and only if f is given by (2.38) for all $x, y \in \mathbb{R}$.

H. Haruki in [4, p. 37] proved the following theorem on regular solutions of functional equation (2.4).

Theorem 3.2
A continuous function $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ *satisfies equation* (3.1) *for all* $z \in \mathbb{C}$ *and* $t \in \mathbb{R}$ *if and only if* f *is given by a quadratic polynomial of* z.

The following proof of Theorem 3.2 is an alternative one, without applying a regularity of functional equation (2.4).

Proof. Equation (3.1) yields the functional equation

$$f(z + t + it) - f(z - t - it) = -i[f(z - t + it) - f(z + t - it)] \tag{3.6}$$

for all $z \in \mathbb{C}$ and $t \in \mathbb{R}$. Define $f(z) = f(x, y)$ for $z = x + iy$. Then equation (3.6) implies equation (E.3). Hence, by Theorem 2.2 and Theorem 3.1 f is a quadratic polynomial. Conversely, a quadratic polynomial satisfies (3.6). This proves Theorem 3.2.

Theorem 3.3
Assume that $f\colon G \times G \longrightarrow \mathbb{C}$ *satisfies one of the following three functional equations*

$$f(x + t, y) - f(x - t, y) = -i[f(x, y + t) - f(x, y - t)] \tag{E.36}$$

$$f(x + t, y) - f(x, y) = -i[f(x, y + t) - f(x, y)] \tag{E.37}$$

$$f(x + t, y + t) - f(x - t, y - t) = -i[f(x - t, y + t) - f(x + t, y - t)] \tag{E.38}$$

for all $x, y, t \in G$. Then f also satisfies the Haruki functional equation

$$f(x+t,y+t)+f(x-t,y+t)+f(x+t,y-t)+f(x-t,y-t) = 4f(x,y) \quad (2.4)$$

for all $x, y, t \in G$.

Proof. Replace t by $-t$ in (E.2) and then substract the result from (E.2) to obtain equation (E.1). Theorem 3.1 shows that (E.1) is equivalent to (E.3). By Lemma 2.1, (E.1) implies (2.3) which is equivalent to (2.4) ([2, Lemma 3, p. 43]). We note that if we replace $\mathbb{R}$ by G in Lemma 3 of [2, p. 43], then the proof of equivalency of (2.3) and (2.4) still holds.

3.2. Differential-difference equations

If one side of the Cauchy–Riemann equation $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$ is replaced by the operators $\bigtriangledown_{x,t} f$ or $\bigtriangledown_{y,t} f$ defined in §1, then we have the following two partial difference-differential equations

$$\frac{\partial f(x,y)}{\partial x} = -i\left[\frac{f(x,y+t)-f(x,y-t)}{2t}\right] \quad (3.7)$$

$$\frac{f(x+t,y)-f(x-t,y)}{2t} = -i\frac{\partial f(x,y)}{\partial y}. \quad (3.8)$$

We determine the general solutions of equations (3.7) and (3.8).

Theorem 3.4
A function $f\colon \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{C}$ satisfies equation (3.7) for all $x, y \in \mathbb{R}$ and $t \in \mathbb{R}\setminus\{0\}$ if and only if f is given by

$$f(x,y) = \frac{1}{2}a(y^2 - x^2 - 2ixy) + b(y - ix) + c, \quad (3.9)$$

where $a, b,$ and c are complex constants.

Proof. We replace (3.7) by

$$\frac{f(x,y+t)-f(x,y-t)}{2t} = \phi(x,y), \quad (3.10)$$

where $\phi(x,y) := i\frac{\partial f(x,y)}{\partial x}$, or, since x is the same parameter in each term of (3.10), by

$$\frac{g(y+t)-g(y-t)}{2t} = \psi(y). \quad (3.11)$$

It follows from [5, p. 577] that the general solution of (3.11) is given by

$$\psi(y) = \alpha y + \beta, \qquad g(y) = \frac{1}{2}\alpha y^2 + \beta y + \gamma,$$

where α, β, and γ are complex constants. Hence, ϕ and f are represented by

$$\phi(x,y) = i\frac{\partial f(x,y)}{\partial x} = \alpha(x)y + \beta(x), \tag{3.12}$$

$$f(x,y) = \frac{1}{2}\alpha(x)y^2 + \beta(x)y + \gamma(x), \tag{3.13}$$

where $\alpha, \beta, \gamma\colon \mathbb{R} \longrightarrow \mathbb{C}$. Now, substitute (3.13) into (3.12) to obtain

$$i\left[\frac{1}{2}\alpha'(x)y^2 + \beta'(x)y + \gamma'(x)\right] = \alpha(x)y + \beta(x).$$

Therefore $\alpha'(x) = 0$, $i\beta'(x) = \alpha(x)$, and

$$\beta(x) = \gamma'(x)i. \tag{3.14}$$

Consequently, $\alpha(x) = a$, where a is a complex constant, $\beta'(x) = -ai$. Therefore $\beta(x) = -axi + b$, which with (3.14) implies $\gamma'(x) = -ax - bi$ and $\gamma(x) = -\frac{1}{2}ax^2 - bxi + c$, where b and c are complex constants. These $\alpha(x)$, $\beta(x)$, and $\gamma(x)$ with (3.13) yield (3.9). Conversely, (3.9) satisfies differential functional equation (3.7). This completes the proof of Theorem 3.4.

In view of the similarity of equations (3.7) and (3.8) the following theorem readily follows from a proof similar to the above proof of Theorem 3.4.

Theorem 3.5
A function $f\colon \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{C}$ satisfies equation (3.8) *for all $x, y \in \mathbb{R}$ and $t \in \mathbb{R}\backslash\{0\}$ if and only if f is given by*

$$f(x,y) = \frac{1}{2}a(x^2 - y^2 + 2ixy) + b(x + iy) + c,$$

where a, b, and c are complex constants.

3.3. Nonsymmetric partial difference equations

In Sections 2 and 3.1, as well as in the papers [3], [6], [7] as a discrete analogue of the Cauchy–Riemann equation $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$, the symmetric equations

$$\triangle_{x,t} f = -i \triangle_{y,t} f \qquad \text{and} \qquad \nabla_{x,t} f = -i \nabla_{y,t} f$$

were considered, and the general solution of functional equations (E.1) and (E.2) was determined when no regularity assumptions are imposed on f.

In this final subsection, we determine the general solutions of the following two nonsymmetric partial difference equations which are also discrete analogue of the Cauchy–Riemann equations:

$$\triangle_{x,t} f = -i \triangle_{y,s} f \qquad \text{and} \qquad \nabla_{x,t} f = -i \nabla_{y,s} f.$$

These equations are given by

$$\frac{f(x+t,y)-f(x,y)}{t} = -i\left[\frac{f(x,y+s)-f(x,y)}{s}\right] \tag{3.15}$$

$$\frac{f(x+t,y)-f(x-t,y)}{2t} = -i\left[\frac{f(x,y+s)-f(x,y-s)}{2s}\right] \tag{3.16}$$

for all $x, y \in \mathbb{R}$ and $s, t \in \mathbb{R}\backslash\{0\}$. By applying the general solutions of functional equations (E.1) and (E.2) we obtain the general solutions of nonsymmetric functional equations (3.15) and (3.16) under no regularity assumptions.

As a closely related to (E.3) we can also derive the following functional equation, and we obtain its general solution without any regularity assumptions as well:

$$\begin{aligned}\frac{f(x+t,y+t)-f(x-t,y-t)}{t}& \\ = -i\left[\frac{f(x-s,y+s)-f(x+s,y-s)}{s}\right]&\end{aligned} \tag{3.17}$$

for all $x, y \in \mathbb{R}$ and $s, t \in \mathbb{R} \setminus \{0\}$.

THEOREM 3.6
A function $f\colon \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{C}$ *satisfies equation* (3.15) *for all* $x, y \in \mathbb{R}$ *and* $s, t \in \mathbb{R}\setminus\{0\}$ *if and only if* f *is given by*

$$f(x,y) = a(x+iy)+b \tag{3.18}$$

for all $x, y \in \mathbb{R}$*, where* a *and* b *are complex constants.*

Proof. Set $t = s$ in (3.15) to obtain equation (E.2). From [3, Theorem 2, p. 99] it is known that the general solution of (E.2) is given by

$$f(x,y) = A(x) + iA(y) + b, \tag{3.19}$$

where $A\colon \mathbb{R} \longrightarrow \mathbb{C}$ is additive and b is a complex constant. Substitute (3.19) into (3.15). Then we have $\frac{A(t)}{t} = \frac{A(s)}{s}$ which implies $A(t) = at$ for fixed $s = s_0 \neq 0$ where $a = \frac{A(s_0)}{s_0}$ is a complex constant. Hence, we obtain (3.18) from (3.19). Conversely, (3.18) satisfies equation (3.15).

THEOREM 3.7
A function $f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ *satisfies equation* (3.16) *for all* $x, y \in \mathbb{R}$ *and* $s, t \in \mathbb{R} \setminus \{0\}$ *if and only if* f *is given by* (2.38) *for all* $x, y \in \mathbb{R}$, *where* a, b, *and* c *are complex constants.*

Proof. Set $t = s$ in equation (3.16) to obtain equation (E.1). Hence, it follows from Theorem 2.1 that f is given by (S.1). Substitute (S.1) into equation (3.16) to obtain

$$\begin{aligned} &\frac{2A^1(t) + 4A_2(x,t) + 4iA_2(t,y)}{2t} \\ &\quad = -\frac{[2iA^1(s) - 4A_2(y,s) + 4iA_2(x,s)]i}{2s}. \end{aligned} \tag{3.20}$$

Now, set $x = y = 0$ in (3.20). Then we have $\frac{A^1(t)}{t} = \frac{A^1(s)}{s}$, since $A_2(0,t) = A_2(t,0) = 0$, which implies

$$A^1(t) = bt \tag{3.21}$$

where $b = \frac{A^1(s_0)}{s_0}$, $s_0 \neq 0$, is a complex constant. Next, by setting $y = 0$ and $t = 1$ in (3.20) with (3.21) we obtain $A_2(x,s) = sA_2(x,1) = sA(x)$, where $A(x) := A_2(x,1)$ is additive for all $x \in \mathbb{R}$. But A_2 is symmetric. Hence, we obtain $sA(x) = xA(s)$ and $A(x) = ax$ where $a = \frac{A(s_0)}{s_0}$, $s_0 \neq 0$, is a complex constant. Hence, we obtain

$$A_2(x,y) = axy. \tag{3.22}$$

Then it follows from (3.21), (3,22), and (S.1) with $A^0 = c$ that f is given by (2.38). Conversely, (2.38) satisfies equation (3.16).

THEOREM 3.8
A function $f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ *satisfies equation* (3.17) *for all* $x, y \in \mathbb{R}$ *and* $s, t \in \mathbb{R} \setminus \{0\}$ *if and only if* f *is given by* (2.38) *for all* $x, y \in \mathbb{R}$, *where* a, b, *and* c *are complex constants.*

Proof. Set $t = s$ in equation (3.17). Then we have equation (E.3). Hence, by Corollary 3.1, f is given by (S.1). Substitute (S.1) in equation (3.17). Then we have

$$\begin{aligned} &\frac{A^1(t) + iA^1(t) + 2A_2(x,t) - 2A_2(t,y) + 2iA_2(t,y) + 2iA_2(x,t)}{t} \\ &= \frac{iA^1(s) + A^1(s) + 2iA_2(s,x) + 2iA_2(s,y) + 2A_2(x,s) - 2A_2(s,y)}{s}. \end{aligned} \tag{3.23}$$

Next, set $x = y = 0$ in (3.23) to obtain

$$\frac{A^1(t) + iA^1(t)}{t} = \frac{iA^1(s) + A^1(s)}{s}, \tag{3.24}$$

since $A_2(0,x) = A_2(x,0) = 0$ for all $x \in \mathbb{R}$. So, it follows from (3.23) and (3.24) that

$$\begin{aligned} &\frac{A_2(x,t) - A_2(y,t) + iA_2(x,t) + iA_2(y,t)}{t} \\ &\qquad = \frac{iA_2(x,s) + iA_2(y,s) + A_2(x,s) - A_2(y,s)}{s}. \end{aligned} \tag{3.25}$$

Further, on putting $x = 0$ and $y = 0$ in (3.25) independently, we have the following two equations

$$\frac{-A_2(y,t) + iA_2(y,t)}{t} = \frac{iA_2(y,s) - A_2(y,s)}{s} \tag{3.26}$$

$$\frac{A_2(x,t) + iA_2(x,t)}{t} = \frac{iA_2(x,s) + A_2(x,s)}{s}. \tag{3.27}$$

Replace x by y in (3.27) to obtain

$$\frac{A_2(y,t) + iA_2(y,t)}{t} = \frac{iA_2(y,s) + A_2(y,s)}{s}. \tag{3.28}$$

Add both sides of (3.26) and (3.28) to obtain $\frac{A_2(y,t)}{t} = \frac{A_2(y,s)}{s}$, which yields $A_2(y,t) = tA(y)$, where $A(y) = \frac{A_2(y,s_0)}{s_0}$ for fixed $s = s_0 \neq 0$ is additive, and $A_2(x,y) = xA(y)$. Since A_2 is symmetric, as before,

$$A_2(x,y) = axy \tag{3.29}$$

where a is a complex constant. On the other hand, (3.24) implies $\frac{A^1(t)}{t} = \frac{A^1(s)}{s}$. Hence

$$A^1(t) = bt \tag{3.30}$$

where $b = \frac{A^1(s_0)}{s_0}$, $s_0 \neq 0$, is a complex constant. Hence, it follows from (3.29), (3.30), and (S.1) with $A^0 = c$ that f is given by (2.38). Conversely, (2.38) satisfies (3.17). This completes the proof of Theorem 3.8.

Functional equations (3.15), (3.16), and (3.17) can also be rewritten in the complex forms

$$\frac{f(z+t) - f(z)}{t} = -i\left[\frac{f(z+is) - f(z)}{s}\right] \tag{3.31}$$

$$\frac{f(z+t) - f(z-t)}{2t} = -i\left[\frac{f(z+is) - f(z-is)}{2s}\right] \tag{3.32}$$

$$\frac{f(z+t+it) - f(z-t-it)}{t} = -i\left[\frac{f(z-s+is) - f(z+s-is)}{s}\right] \tag{3.33}$$

for all $z \in \mathbb{C}$ and $s, t \in \mathbb{R} \setminus \{0\}$, where $f(z) := f(x, y)$ for all $x, y \in \mathbb{R}$ and $f: \mathbb{C} \longrightarrow \mathbb{C}$. In this case, it follows from Theorem 3.6 and (3.18) that the general solution of (3.31) is given by $f(z) = az + b$. On the other hand, by Theorems 3.7 and 3.8 and (2.38) the general solution of (3.32) and (3.33) are given by $f(z) = az^2 + bz + c$. Thus, it is remarkable that the only solutions of the above three functional equations are certain complex polynomials of bounded degree when no regularity assumptions are imposed on f.

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Some properties of convex and $*$-concave multifunctions

Abstract. We investigate some properties of $*$-concave and convex multifunctions on the real line with convex bounded closed values. In particularly we consider the Hadamard inequality and the Hardy–Littlewood–Pólya majorization theory in the case of multifunctions.

1. Basic definitions

Let X be a real Banach space. Denote by $clb(X)$ the set of all nonempty bounded closed convex subsets of X. For given $A, B \in clb(X)$ and $\lambda \geq 0$ we define $A + B = \{a + b : a \in A, b \in B\}$, $\lambda A = \{\lambda a : a \in A\}$,

$$A \overset{*}{+} B = cl(A + B) = cl(clA + clB).$$

The structure $(clb(X), \overset{*}{+})$ is an Abelian semigroup with the neutral element $\{0\}$. It is clear that

$$\lambda(A \overset{*}{+} B) = \lambda A \overset{*}{+} \lambda B, \quad (\lambda + \mu)A = \lambda A \overset{*}{+} \mu A, \quad \lambda(\mu A) = \lambda\mu A, \quad 1 \cdot A = A$$

for all $\lambda, \mu \geq 0$ and $A, B \in clb(X)$. Thus the triple $(clb(X), \overset{*}{+}, \cdot)$ is also an abstract convex cone (for definition see e.g. [11]). Since

$$A \overset{*}{+} C = B \overset{*}{+} C \implies A = B$$

(cf. [11]), the cancellation law is satisfied.

Let d be the Hausdorff metric in $clb(X)$ derived from the norm $||\cdot||$ in X, i.e. $d(A, B) = \max\{e(A, B), e(B, A)\}$, where $e(A, B) = \sup_{a\in A} \rho(a, B)$ and $\rho(a, B) = \inf_{b\in B} ||a - b||$ for $A, B \in clb(X)$. For given $A \in clb(X)$ we define $||A|| = \sup\{||a|| : a \in A\} = d(A, \{0\})$. The metric space $(clb(X), d)$ is complete (see e.g. [1, Theorem II-3, p. 40]). Moreover d is translation invariant since

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$$d(A \overset{*}{+} C, B \overset{*}{+} C) = d(A + C, B + C) = d(A, B)$$

and positively homogeneous

$$d(\lambda A, \lambda B) = \lambda d(A, B)$$

for all $A, B, C \in clb(X)$ (cf. [2, Lemma 2.2]).

A multifunction $F\colon [a, b] \longrightarrow clb(X)$ is said to be $*$-*concave ($*$-convex)* if

$$F\left(\lambda x + (1-\lambda)y\right) \subset \lambda F(x) \overset{*}{+} (1-\lambda)F(y),$$
$$\left(\ \lambda F(x) \overset{*}{+} (1-\lambda)F(y) \subset F\left(\lambda x + (1-\lambda)y\right)\ \right)$$

for all $x, y \in [a, b]$ and $\lambda \in (0, 1)$.

Remark 1

The concavity of multifunctions, defined as follows,

$$F(\lambda x + (1-\lambda)y) \subset \lambda F(x) + (1-\lambda)F(y), \qquad x, y \in [a, b],\ \lambda \in (0, 1)$$

implies the $*$-concavity, but not conversely. To see this we consider two sets $A, B \in clb(X)$ such that $A + B \neq cl(A + B)$ (an example could be found in [10, pp. 712-713]) and the multifunction $F\colon [0, 1] \longrightarrow clb(X)$ given by the formula $F(t) = tA \overset{*}{+} (1-t)B$. It is easy to check that $F(\lambda t + (1-\lambda)s) \subset \lambda F(t) \overset{*}{+} (1-\lambda)F(s)$ for all $t, s \in [0, 1]$ and $\lambda \in (0, 1)$ but

$$F\left(\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1\right) = F\left(\frac{1}{2}\right) = \frac{1}{2}(A \overset{*}{+} B) \not\subset \frac{1}{2}\left(A + B\right) = \frac{1}{2}[F(0) + F(1)].$$

Remark 2

A multifunction $F\colon [a, b] \longrightarrow clb(X)$ is $*$-convex if and only if it is convex i.e.,

$$\lambda F(x) + (1-\lambda)F(y) \subset F\left(\lambda x + (1-\lambda)y\right), \qquad x, y \in [a, b],\ \lambda \in (0, 1).$$

We note that every convex multifunction with non-empty values has convex values. Indeed, $\lambda F(x) + (1-\lambda)F(x) \subset F(x)$ for all $\lambda \geq 0$ and $x \in [a, b]$.

A multifunction $F\colon [a, b] \longrightarrow clb(X)$ is said to be *increasing* if $F(x) \subset F(y)$ whenever $x, y \in [a, b]$ and $x < y$.

A set $\Delta = \{y_0, y_1, \ldots, y_n\}$, where $a = y_0 < y_1 < \ldots < y_n = b$, is said to be a *partition* of $[a, b]$. For given partition Δ we set $\delta(\Delta) = \max\{y_i - y_{i-1}\colon\ i = 1, \ldots, n\}$. For the partition Δ and for a system $\tau = (\tau_1, \ldots, \tau_n)$ of intermediate points $\tau_i \in [y_{i-1}, y_i]$ we create the *Riemann sum*

$$S(\Delta, \tau) = (y_1 - y_0)F(\tau_1) \overset{*}{+} \cdots \overset{*}{+} (y_n - y_{n-1})F(\tau_n).$$

If for every sequence (Δ^ν) of partitions $\Delta^\nu = \{y_0^\nu, y_1^\nu, \ldots, y_{n_\nu}^\nu\}$ of $[a,b]$ such that $\lim_{\nu\to\infty} \delta(\Delta^\nu) = 0$, and for every sequence (τ^ν) of systems of intermediate points, the sequence of the Riemann sums $(S(\Delta^\nu, \tau^\nu))$ tends to the same limit $I \in clb(X)$, then F is said to be *Riemann integrable* over $[a,b]$ and $\int_a^b F(y)\,dy := I$.

The Riemann integral for multifunction with compact convex values was investigated by A. Dinghas [3] and M. Hukuhara [4]. Some properties of Riemann integral of multifunctions with convex closed bounded values may be found in paper [8].

2. Hadamard inequality in case of multifunctions

We believe that the following theorem is known. Nevertheless we prove it for convenience of the reader.

THEOREM 1
Every $*$-concave multifunction $F\colon [a,b] \longrightarrow clb(X)$ is continuous on (a,b) with respect to the Hausdorff metric.

Proof. Since all values of F are bounded we may find a constant $M > 0$ such that $||\lambda F(a) \overset{*}{+} (1-\lambda)F(b)|| \le M$ for $\lambda \in [0,1]$. Thus by $*$-concavity of F we have $||F(x)|| \le M$, $x \in [a,b]$.

Let us fix $x_0 \in (a,b)$ and let x be a point belonging to the interval (x_0, b). There exist $\lambda, \mu \in (0,1)$ such that $x = \lambda x_0 + (1-\lambda)b$ and $x_0 = \mu x + (1-\mu)a$. Hence $\lambda = \frac{x-b}{x_0-b} \to 1^-$ and $\mu = \frac{x_0-a}{x-a} \to 1^-$ as $x \to x_0^+$. Then by the $*$-concavity we obtain

$$\begin{aligned} e(F(x), F(x_0)) &\le e(\lambda F(x_0) \overset{*}{+} (1-\lambda)F(b), F(x_0)) \\ &\le d(\lambda F(x_0) \overset{*}{+} (1-\lambda)F(b), \lambda F(x_0) \overset{*}{+} (1-\lambda)F(x_0)) \\ &= (1-\lambda)d(F(b), F(x_0)) \end{aligned}$$

and

$$\begin{aligned} e(F(x_0), F(x)) &\le e(\mu F(x) \overset{*}{+} (1-\mu)F(a), F(x)) \\ &\le d(\mu F(x) \overset{*}{+} (1-\mu)F(a), \mu F(x) \overset{*}{+} (1-\mu)F(x)) \\ &= (1-\mu)d(F(a), F(x)) \\ &\le 2M(1-\mu), \end{aligned}$$

whence $d(F(x), F(x_0)) \to 0$ as $x \to x_0^+$. We have shown that F is right-hand side continuous at x_0. The similar argument can be used to get the left-hand side continuity of F at x_0.

Remark 3
A $*$-concave multifunction on $[a,b]$ need not be continuous. To see this it is enough to take $F\colon [0,1] \longrightarrow clb(\mathbb{R})$ defined by

$$F(x) = \begin{cases} \{0\}, & x > 0, \\ [0,1], & x = 0. \end{cases}$$

The continuity of convex multifunctions can be obtained from Theorem 3.7 in [7]. We give here an independent, straightforward proof similar to that of Theorem 1.

Theorem 1′
Every convex multifunction $F\colon [a,b] \longrightarrow clb(X)$ is continuous on (a,b) with respect to the Hausdorff metric and bounded on $[a,b]$.

Proof. At first we will prove that F is bounded on $[a,b]$. We observe that for every $x \in [a, \frac{a+b}{2}]$ there exists $\lambda \in [\frac{1}{2}, 1]$ such that $\lambda x + (1-\lambda)b = \frac{a+b}{2}$. Let us fix $u \in F(b)$. The convexity of F yields

$$\lambda F(x) + (1-\lambda)u \subset F\left(\frac{a+b}{2}\right),$$

whence

$$F(x) \subset \frac{1}{\lambda} F\left(\frac{a+b}{2}\right) - \left(\frac{1}{\lambda} - 1\right) u.$$

Thus F is bounded on $[a, \frac{a+b}{2}]$. In similar manner we show that F is bounded on $[\frac{a+b}{2}, b]$. Consequently there is a constant M such that $||F(x)|| \le M$ for $x \in [a,b]$.

Let us fix x_0 belonging to (a,b) and let $x_0 < x < b$. We can find $\lambda, \mu \in (0,1)$ such that $x = \lambda x_0 + (1-\lambda)b$, $x_0 = \mu x + (1-\mu)a$. Clearly $\lambda F(x_0) + (1-\lambda)F(b) \subset F(x)$ and $\mu F(x) + (1-\mu)F(a) \subset F(x_0)$. We note that $\lambda, \mu \to 1^-$ as $x \to x_0^+$. By the convexity of F and properties of e we obtain two inequalities

$$\begin{aligned} \lambda e(F(x_0), F(x)) &= e(\lambda F(x_0), \lambda F(x)) \\ &= e(\lambda F(x_0) + (1-\lambda)F(b), \lambda F(x) + (1-\lambda)F(b)) \\ &\le e(F(x), \lambda F(x) + (1-\lambda)F(b)) \\ &\le d(F(x), \lambda F(x) + (1-\lambda)F(b)) \\ &= (1-\lambda) d(F(x), F(b)) \\ &\le 2M(1-\lambda), \end{aligned}$$

$$\begin{aligned} e(F(x), F(x_0)) &= \sup_{v \in F(x)} \rho(v, F(x_0)) \\ &\le \sup_{v \in F(x)} \rho(v, \mu F(x) + (1-\mu)F(a)) \end{aligned}$$

$$\begin{aligned}
&= e(F(x), \mu F(x) + (1-\mu)F(a)) \\
&\le d(F(x), \mu F(x) + (1-\mu)F(a)) \\
&= (1-\mu) d(F(x), F(a)) \\
&\le 2M(1-\mu).
\end{aligned}$$

Consequently

$$\lim_{x \to x_0^+} d(F(x), F(x_0)) = 0.$$

The left continuity of F at x_0 may be shown analogously.

A continuous multifunction $F\colon [a,b] \longrightarrow clb(X)$ is Riemann integrable on $[a,b]$ (cf. [8]). A $*$-concave multifunction on $[a,b]$ is commonly bounded on this interval. Therefore it is not difficult to see that a $*$-concave multifunction has to be Riemann integrable on each $[c,d] \subset [a,b]$ (cf. [8]).

In the case of convex functions on $[a,b]$ the following Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}$$

is well known (cf. [5, pp. 196-197]). We are going to deal with suitable inclusion for convex and $*$-concave multifunctions.

Theorem 2

If $F\colon [a,b] \longrightarrow clb(X)$ is $*$-concave multifunction, then

$$F\left(\frac{x+y}{2}\right) \subset \frac{1}{y-x}\int_x^y F(t)\,dt \subset \frac{F(x) \overset{*}{+} F(y)}{2} \tag{1}$$

for each x, y such that $x < y$ and $[x,y] \subset [a,b]$.

Proof. Let us fix $n \in \mathbb{N}$ and let $x_i = x + i\frac{y-x}{n}$ and $\tau_i = x + \frac{2i-1}{2n}(y-x)$ for $i \in \{1, \dots, n\}$. These points create the partition $\Delta_n = \{x, x_1, \dots, x_{n-1}, y\}$ of the interval $[x,y]$ and $\tau = (\tau_1, \dots, \tau_n)$ is a system of intermediate points. We note that

$$\tau_i = \frac{x_{i-1}+x_i}{2} = \frac{[2n-(2i-1)]x + (2i-1)y}{2n}.$$

Using the $*$-concavity of F we obtain

$$F(\tau_i) \subset \frac{2n-(2i-1)}{2n}F(x) \overset{*}{+} \frac{2i-1}{2n}F(y)$$

for $i \in \{1, \dots, n\}$. Summing up over i we get

$$\begin{aligned}
&F(\tau_1) \overset{*}{+} \dots \overset{*}{+} F(\tau_n) \\
&\quad \subset \left(\frac{2n-1}{2n} + \frac{2n-3}{2n} + \dots + \frac{1}{2n}\right)F(x) \overset{*}{+} \left(\frac{1}{2n} + \frac{3}{2n} + \dots + \frac{2n-1}{2n}\right)F(y).
\end{aligned}$$

Since $1+3+\cdots+(2n-1)=n^2$, we obtain

$$\frac{1}{y-x}\big[F(\tau_1)\overset{*}{+}\cdots\overset{*}{+}F(\tau_n)\big]\frac{y-x}{n}\subset\frac{F(x)\overset{*}{+}F(y)}{2}. \tag{2}$$

Now we let $n\to\infty$. Then $\delta(\Delta_n)\to 0$ and with respect to the definition of the integral, by (2) and by the closedness of the set $\frac{1}{2}(F(x)\overset{*}{+}F(y))$ we have

$$\frac{1}{y-x}\int_x^y F(t)\,dt\subset\frac{F(x)\overset{*}{+}F(y)}{2}.$$

To obtain the first inclusion of (1) we take an even positive integer n. Let $k=n/2$ and let us choose $i\in\{1,\dots,k\}$. We note that $\frac{1}{2}(\tau_i+\tau_j)=\frac{1}{2}(x+y)$ for $j=n+1-i$. Again by the $*$-concavity of F we infer

$$F\left(\frac{x+y}{2}\right)\subset\frac{1}{2}(F(\tau_i)\overset{*}{+}F(\tau_j)).$$

Summing up over $i\in\{1,\dots,k\}$ leads to

$$kF\left(\frac{x+y}{2}\right)\subset k\big[F(\tau_1)\overset{*}{+}\cdots\overset{*}{+}F(\tau_k)\overset{*}{+}F(\tau_{k+1})\overset{*}{+}\cdots\overset{*}{+}F(\tau_n)\big]$$

or

$$F\left(\frac{x+y}{2}\right)\subset\frac{1}{y-x}\big[F(\tau_1)\overset{*}{+}\cdots\overset{*}{+}F(\tau_n)\big]\cdot\frac{y-x}{n} \tag{3}$$

for all even n. The right-hand side of inclusion (3) tends to $\frac{1}{y-x}\int_x^y F(t)\,dt$ as $n\to\infty$. Hence

$$F\left(\frac{x+y}{2}\right)\subset\frac{1}{y-x}\int_x^y F(t)\,dt.$$

The proof of the next theorem runs similarly.

Theorem $2'$

If $F\colon[a,b]\longrightarrow clb(X)$ is a convex multifunction, then the following inclusions hold

$$\frac{F(x)\overset{*}{+}F(y)}{2}\subset\frac{1}{y-x}\int_x^y F(t)\,dt\subset F\left(\frac{x+y}{2}\right) \tag{4}$$

for all intervals $[x,y]\subset[a,b]$.

Inclusions (4) for the Aumann integral may be found in the paper of E. Sadowska [9, Theorem 1], where the integral Jensen inequality is applied (see the paper of J. Matkowski and K. Nikodem [6]). The assumptions of Theorem $2'$ differ somewhat from that of Theorem 1 in [9].

3. Hardy-Littlewood-Pólya majorization theorem for multifunctions

In this part of the note we are going to transfer the Hardy–Littlewood–Pólya majorization principle for convex functions (cf. [5, ch. 8, § 5]) to convex and $*$-concave multifunctions.

Theorem 3
Let x_1, x_2, y_1, y_2 be real numbers such that $x_2 \leq x_1$, $y_2 \leq y_1$, $x_1 \leq y_1$, $x_1 + x_2 = y_1 + y_2$. If $F\colon \mathbb{R} \longrightarrow clb(X)$ is $*$-concave, then

$$F(x_1) \overset{*}{+} F(x_2) \subset F(y_1) \overset{*}{+} F(y_2). \tag{5}$$

Proof. The assumptions of the theorem imply the inequality $y_2 \leq x_2 \leq x_1 \leq y_1$. At first we assume that $y_1 \neq y_2$. Setting $\lambda = \frac{y_1 - x_2}{y_1 - y_2}$, $\mu = \frac{y_1 - x_1}{y_1 - y_2}$ by the $*$-concavity we have

$$F(x_2) = F(\lambda y_2 + (1-\lambda)y_1) \subset \lambda F(y_2) \overset{*}{+} (1-\lambda)F(y_1),$$
$$F(x_1) = F(\mu y_2 + (1-\mu)y_1) \subset \mu F(y_2) \overset{*}{+} (1-\mu)F(y_1).$$

Multiplying the above inclusions by $y_1 - y_2$ and summing them up together we obtain

$$(y_1 - y_2)(F(x_1) \overset{*}{+} F(x_2)) \subset (x_2 - y_2 + x_1 - y_2)F(y_1) \overset{*}{+} (y_1 - x_2 + y_1 - x_1)F(y_2).$$

The equality $x_1 + x_2 = y_1 + y_2$ and the above inclusions lead to

$$F(x_1) \overset{*}{+} F(x_2) \subset F(y_1) \overset{*}{+} F(y_2).$$

If $y_1 = y_2$, then $y_1 = x_1 = x_2 = y_2$ and condition (5) holds true.

Theorem 3 for concave multifunctions can be found in [7, Theorem 2.14] in another formulation. The same concerns the next theorem. Its proof is similar to the previous one.

Theorem 3′
Let x_1, x_2, y_1, y_2 be real numbers such that $x_2 \leq x_1$, $y_2 \leq y_1$, $x_1 \leq y_1$ and, $x_1 + x_2 = y_1 + y_2$. If $F\colon \mathbb{R} \longrightarrow clb(X)$ is convex, then

$$F(y_1) \overset{*}{+} F(y_2) \subset F(x_1) \overset{*}{+} F(x_2).$$

Corollary 1
Let a, b, c be non-negative numbers and let $a + b \leq c$. If $F\colon [0,\infty) \longrightarrow clb(X)$ is a $*$-concave multifunction, then

$$F(a+b) \overset{*}{+} F(c) \subset F(a) \overset{*}{+} F(b+c).$$

Proof. To obtain the Corollary from Theorem 3 it is enough to set $x_1 = c$, $x_2 = a + b$, $y_1 = b + c$, $y_2 = a$ (see [5, pp. 194-195]).

COROLLARY 2
Let x_1, x_2, y_1, y_2 be real numbers satisfying the conditions: $x_2 \le x_1$, $y_2 \le y_1$, $x_1 \le y_1$ and $x_1 + x_2 \le y_1 + y_2$. If $F\colon \mathbb{R} \longrightarrow clb(X)$ is an increasing $*$-concave multifunction, then

$$F(x_1) \overset{*}{+} F(x_2) \subset F(y_1) \overset{*}{+} F(y_2)$$

holds true.

Proof. Taking $z_1 = y_1$ and $z_2 = x_1 + x_2 - y_1$ we can easily check that the numbers x_1, x_2, z_1, z_2 satisfy the assumption of Theorem 3. Hence

$$F(x_1) \overset{*}{+} F(x_2) \subset F(z_1) \overset{*}{+} F(z_2).$$

Moreover, F is increasing and $z_2 \le y_2$, so

$$F(x_1) \overset{*}{+} F(x_2) \subset F(y_1) \overset{*}{+} F(y_2).$$

THEOREM 4
Assume that x_i, y_i, $i \in \{1, \dots, n\}$ are real numbers such that

$$x_n \le x_{n-1} \le \dots \le x_1, \qquad y_n \le y_{n-1} \le \dots \le y_1, \tag{6}$$

$$\sum_{i=1}^{k} x_i \le \sum_{i=1}^{k} y_i, \quad k \in \{1, \dots, n-1\}, \qquad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \tag{7}$$

and

$$x_{k+1} \le y_k, \qquad k \in \{2, \dots, n-1\}. \tag{8}$$

If $F\colon \mathbb{R} \longrightarrow clb(X)$ is a $*$-concave multifunction, then

$$F(x_1) \overset{*}{+} \cdots \overset{*}{+} F(x_n) \subset F(y_1) \overset{*}{+} \cdots \overset{*}{+} F(y_n). \tag{9}$$

Proof. The theorem is valid for $n = 2$ thanks to Theorem 3.

Now we assume (9) true for an $n \in \mathbb{N}$, $n \ge 2$ and take arbitrary numbers x_i, y_i, $i \in \{1, \dots, n, n+1\}$ satisfying

$$x_{n+1} \le x_n \le \dots \le x_1, \qquad y_{n+1} \le y_n \le \dots \le y_1, \tag{6_{n+1}}$$

$$\sum_{i=1}^{k} x_i \le \sum_{i=1}^{k} y_i, \quad k \in \{1, \dots, n\}, \qquad \sum_{i=1}^{n+1} x_i = \sum_{i=1}^{n+1} y_i \tag{7_{n+1}}$$

and

$$x_{k+1} \le y_k, \qquad k \in \{2, \dots, n\}. \tag{8_{n+1}}$$

By (7_{n+1}) we have

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n-1} y_i + (y_n + y_{n+1} - x_{n+1}).$$

According to the induction hypothesis

$$F(x_1) \overset{*}{+} \cdots \overset{*}{+} F(x_n) \subset F(y_1) \overset{*}{+} \cdots \overset{*}{+} F(y_{n-1}) \overset{*}{+} F(y_n + y_{n+1} - x_{n+1})$$

since $y_n + y_{n+1} - x_{n+1} \leq y_{n-1}$ (see (6_{n+1}) and (7_{n+1})). If we show that

$$F(y_n + y_{n+1} - x_{n+1}) \overset{*}{+} F(x_{n+1}) \subset F(y_n) \overset{*}{+} F(y_{n+1}) \tag{10}$$

holds, the proof will be complete.

Consider two cases: (a) $x_{n+1} \leq y_n + y_{n+1} - x_{n+1}$ and (b) $x_{n+1} > y_n + y_{n+1} - x_{n+1}$. In case (b) $(y_n + y_{n+1} - x_{n+1}) + x_{n+1} = y_n + y_{n+1}$, $y_{n+1} \leq y_n$, $y_n + y_{n+1} - x_{n+1} < x_{n+1}$ and $x_{n+1} \leq y_n$ according to (8_{n+1}). By Theorem 3 condition (10) holds. In case (a), $x_{n+1} + (y_n + y_{n+1} - x_{n+1}) = y_n + y_{n+1}$, $y_{n+1} \leq y_n$, $x_{n+1} \leq y_n + y_{n+1} - x_{n+1}$ and $y_n + y_{n+1} - x_{n+1} = y_n + (y_{n+1} - x_{n+1}) \leq y_n$ because $y_{n+1} \leq x_{n+1}$. By Theorem 3 condition (10) holds.

Theorem 4′
Assume that x_i, y_i, $i \in \{1, \ldots, n\}$ are real numbers such that

$$x_n \leq x_{n-1} \leq \ldots \leq x_1\,, \qquad y_n \leq y_{n-1} \leq \ldots \leq y_1\,,$$

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i\,, \quad k \in \{1, \ldots, n-1\}\,, \qquad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

and

$$x_{k+1} \leq y_k\,, \qquad k \in \{2, \ldots, n-1\}\,.$$

If $F\colon \mathbb{R} \longrightarrow clb(X)$ is a convex multifunction, then

$$F(y_1) \overset{*}{+} \cdots \overset{*}{+} F(y_n) \subset F(x_1) \overset{*}{+} \cdots \overset{*}{+} F(x_n).$$

Results of the same kind as Theorem 4 and 4′, formulated in some other language, were obtained by K. Nikodem (cf. [7, Theorem 2.14]).

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Second Hukuhara derivative and cosine family of linear set-valued functions

Abstract. Let K be a closed convex cone with the nonempty interior in a real Banach space and let $cc(K)$ denote the family of all nonempty convex compact subsets of K. If $\{F_t : t \geq 0\}$ is a regular cosine family of continuous linear set-valued functions $F_t : K \longrightarrow cc(K)$, $x \in F_t(x)$ for $t \geq 0$, $x \in K$ and $F_t \circ F_s = F_s \circ F_t$ for $s, t \geq 0$, then

$$D^2 F_t(x) = F_t(H(x))$$

for $x \in K$ and $t \geq 0$, where $D^2 F_t(x)$ denotes the second Hukuhara derivative of $F_t(x)$ with respect to t and $H(x)$ is the second Hukuhara derivative of this multifunction at $t = 0$.

Let X be a vector space. Through this paper all vector spaces are supposed to be real. We introduce the notations

$$A + B := \{a + b : a \in A, \ b \in B\},$$
$$\lambda A := \{\lambda a : a \in A\}$$

for $A, B \subset X$ and $\lambda \in \mathbb{R}$.

A subset K of X is called a *cone* if $tK \subset K$ for all $t \in (0, +\infty)$. A cone is said to be *convex* if it is a convex set.

Let X and Y be two vector spaces and let $K \subset X$ be a convex cone. A set-valued function $F : K \longrightarrow n(Y)$, where $n(Y)$ denotes the family of all nonempty subsets of Y, is called *additive* if

$$F(x + y) = F(x) + F(y)$$

for all $x, y \in K$. If additionally F satisfies

$$F(\lambda x) = \lambda F(x)$$

for all $x \in K$ and $\lambda \geq 0$, then F is called *linear*.

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A set-valued function $F\colon [0,+\infty) \longrightarrow n(Y)$ is said to be *concave* if

$$F(\lambda t + (1-\lambda)s) \subset \lambda F(t) + (1-\lambda)F(s)$$

for all $s,t \in [0,+\infty)$ and $\lambda \in (0,1)$.

From now on we assume that X is a normed vector space, $c(X)$ denotes the family of all compact members of $n(X)$ and $cc(X)$ stands for the family of all convex sets of $c(X)$.

Let A, B, C be sets of $cc(X)$. We say that the set C is the *Hukuhara difference* of A and B when $C = A - B$ if $B + C = A$. By Rådström Cancellation Lemma [7] it follows that if this difference exists, then it is unique.

Let A, A_1, $A_2, \ldots$ be elements of the space $cc(X)$. We say that the *sequence* $(A_n)_{n\in\mathbb{N}}$ *is convergent to* A and we write $A_n \to A$ if $d(A, A_n) \to 0$, where d denotes the Hausdorff metric induced by the norm in X.

Lemma 1
Let X be a Banach space, $A, A_1, A_2, \ldots, B, B_1, B_2, \ldots \in cc(X)$. If $A_n \to A$, $B_n \to B$ and there exist the Hukuhara differences $A_n - B_n$ in $cc(X)$ for $n \in \mathbb{N}$, then there exists the Hukuhara difference $A - B$ and $A_n - B_n \to A - B$.

Proof. Let $C_n = A_n - B_n$ for $n \in \mathbb{N}$. By the definition of the Hukuhara difference $A_n = B_n + C_n$ for $n \in \mathbb{N}$. By properties of the Hausdorff metric for $m, n \in \mathbb{N}$ we have

$$\begin{aligned} d(C_m, C_n) &= d(B_n + B_m + C_m, B_m + B_n + C_n) \\ &= d(B_n + A_m, B_m + A_n) \\ &\le d(B_n, B_m) + d(A_m, A_n). \end{aligned}$$

Sequences $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ are Cauchy sequences thus by the last inequality $(C_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, too. By the completness of $cc(X)$ (see Theorem II.3 in [1]) there exists $C \in cc(X)$ such that $C_n \to C$. Moreover, $B_n + C_n \to B + C$ since

$$\begin{aligned} d(B_n + C_n, B + C) &\le d(B_n + C_n, B_n + C) + d(B_n + C, B + C) \\ &= d(C_n, C) + d(B_n, B). \end{aligned}$$

On the other hand $A_n \to A$ and $A_n = B_n + C_n$ so $A = B + C$, i.e., there exists the Hukuhara difference $A - B = C$.

Let $F, G\colon K \longrightarrow cc(K)$. We can define the multifunctions $F + G$ and $F - G$ on K as follows

$$(F+G)(x) := F(x) + G(x) \qquad \text{for } x \in K$$

and

$$(F-G)(x) := F(x) - G(x)$$

if the Hukuhara differences $F(x) - G(x)$ exist for all $x \in K$.

Lemma 2
For each set $A \subset K$ the inclusion

$$(F+G)(A) \subset F(A)+G(A) \tag{1}$$

holds. Moreover, if there exist the Hukuhara difference $F(A)-G(A)$ and the multifunction $F-G$, then

$$F(A)-G(A) \subset (F-G)(A). \tag{2}$$

Proof. Inclusion (1) is obvious. To prove (2) we observe that $(F-G)+G = F$. Hence by (1) we obtain

$$F(A) \subset (F-G)(A)+G(A). \tag{3}$$

Since $F(A) = G(A)+(F(A)-G(A))$, (3) and Rådström Cancellation Lemma yield inclusion (2).

Lemma 3 (Lemma 3 in [8])
Let X and Y be two normed vector spaces and let K be a closed convex cone in X. Assume that $F\colon K \longrightarrow cc(K)$ is continuous additive set-valued function and $A, B \in cc(K)$. If there exists the difference $A-B$, then there exists $F(A)-F(B)$ and $F(A)-F(B) = F(A-B)$.

Lemma 4 (Lemma 3 in [5])
If $(A_n)_{n\in\mathbb{N}}$ is a sequence of elements of the set $c(X)$ such that $A_{n+1} \subset A_n$ for $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n\,.$$

Lemma 5 (Lemma 3 in [9])
Let K be a closed convex cone such that $\operatorname{int} K \neq \emptyset$ in a Banach space X and let Y be a normed space. If $(F_n)_{n\in\mathbb{N}}$ is a sequence of continuous additive set-valued functions $F_n\colon K \longrightarrow cc(Y)$ such that $F_{n+1}(x) \subset F_n(x)$ for all $x \in K$ and $n \in \mathbb{N}$, then the formula

$$F_0(x) := \bigcap_{n=1}^{\infty} F_n(x), \qquad x \in K,$$

defines a continuous additive set-valued function $F_0\colon K \longrightarrow cc(Y)$. Moreover,

$$\lim_{n\to\infty} F_n(x) = F_0(x), \qquad x \in K, \tag{4}$$

and the convergence in (4) is uniform on every nonempty compact subset of K.

Lemma 6 (Lemma 4 in [5])
If $(A_n)_{n\in\mathbb{N}}$ is a sequence of elements of $c(X)$ satisfying $A_n \subset A_{n+1} \subset B$ for $n \in \mathbb{N}$ and a compact set B, then

$$\lim_{n\to\infty} A_n = \operatorname{cl}\left(\bigcup_{n=1}^{\infty} A_n\right).$$

Lemma 7
Let K be a closed convex cone such that $\operatorname{int} K \neq \emptyset$ in a Banach space X and let Y be a normed space. If $(F_n)_{n\in\mathbb{N}}$ is a sequence of continuous additive set-valued functions $F_n\colon K \longrightarrow cc(Y)$ such that

1) *$F_n(x) \subset F_{n+1}(x)$ for all $x \in K$ and $n \in \mathbb{N}$,*
2) *$F_n(x) \subset G(x)$ for all $x \in K$, $n \in \mathbb{N}$ and a set-valued function $G\colon K \longrightarrow c(Y)$,*

then the formula

$$F_0(x) := \operatorname{cl}\left(\bigcup_{n=1}^{\infty} F_n(x)\right), \qquad x \in K, \tag{5}$$

defines a continuous additive set-valued function $F_0\colon K \longrightarrow cc(Y)$. Moreover,

$$\lim_{n\to\infty} F_n(x) = F_0(x), \qquad x \in K, \tag{6}$$

and the convergence in (6) is uniform on every nonempty compact subset of K.

Proof. The sets $F_0(x)$ defined by the formula (5) are obviously closed and convex. Since $F_0(x) \subset G(x)$ and $G(x)$ are compact, they belong to $cc(Y)$ for every $x \in K$. Equality (6) holds according to Lemma 6. By Lemma 5.6 in [4] we have

$$F_0(x+y) = \lim_{n\to\infty} F_n(x+y) = \lim_{n\to\infty} \left(F_n(x) + F_n(y)\right) = F_0(x) + F_0(y)$$

for all $x, y \in K$. Thus the set-valued function F_0 is additive. Since $F_1(x) \subset F_0(x)$ for $x \in K$ and F_1 is continuous, the set-valued function F_0 is continuous on $\operatorname{int} K$ (see Theorem 5.2 in [4]). Fix $y \in \operatorname{int} K$ and $x_0 \in K$, then $\frac{x_0+y}{2} \in \operatorname{int} K$ (see Chapter V, §1, Lemma 8 in [3]). Let (x_n) be a sequence of elements of K convergent to x_0. Then

$$\begin{aligned} d\left(F_0(x_n), F_0(x_0)\right) &= d\left(F_0(x_n) + F_0(y), F_0(x_0) + F_0(y)\right) \\ &= 2d\left(F_0\left(\frac{x_n+y}{2}\right), F_0\left(\frac{x_0+y}{2}\right)\right). \end{aligned}$$

The continuity of F_0 at $\frac{x_0+y}{2}$ implies that

$$\lim_{n\to\infty} F_0(x_n) = F_0(x_0).$$

This means that F_0 is continuous on K. The sequence $(d(F_n(x), F_0(x)))\ n \in \mathbb{N}$ is a decreasing sequence of continuous functions convergent to the zero function and according to Dini Theorem this function is the uniform limit of this sequence on every nonempty compact subset of K.

Let $F\colon [0, +\infty) \longrightarrow cc(X)$ be a set-valued function such that there exist the Hukuhara differences $F(t) - F(s)$ for $0 \le s \le t$. The *Hukuhara derivative* of F at $t > 0$ is defined by the formula

$$DF(t) = \lim_{h\to 0^+} \frac{F(t+h) - F(t)}{h} = \lim_{h\to 0^+} \frac{F(t) - F(t-h)}{h},$$

whenever both these limits exist (see [2]). Moreover,

$$DF(0) = \lim_{h\to 0^+} \frac{F(h) - F(0)}{h}.$$

Let $(K, +)$ be a semigroup. A one-parameter family $\{F_t :\ t \ge 0\}$ of set-valued functions $F_t\colon K \longrightarrow n(K)$ is said to be a *cosine family* if

$$F_0(x) = \{x\} \qquad \text{for } x \in K$$

and

$$F_{t+s}(x) + F_{t-s}(x) = 2F_t(F_s(x)) := 2\bigcup\{F_t(y) :\ y \in F_s(x)\} \tag{7}$$

for $x \in K$ and $0 \le s \le t$.

Let X be a normed space. A cosine family $\{F_t :\ t \ge 0\}$ is said to be *regular* if

$$\lim_{t\to 0^+} d(F_t(x), \{x\}) = 0.$$

Lemma 8
Let X be a Banach space and let K be a closed convex cone in X such that $\operatorname{int} K \ne \emptyset$. *Assume that $\{F_t :\ t \ge 0\}$ is a regular cosine family of continuous additive set-valued functions $F_t\colon K \longrightarrow cc(K)$ and $x \in F_t(x)$ for all $x \in K$ and $t \ge 0$. Then there exist the Hukuhara differences $F_t(x) - F_s(x)$ for all $0 \le s \le t$ and $x \in K$.*

Proof. We first prove, by induction on n, that there exist the Hukuhara differences

$$F_{ns}(x) - F_{(n-1)s}(x) \tag{8}$$

for all $s \ge 0$, $x \in K$, $n \in \mathbb{N}$.

For $n = 1$ it suffices to show that

$$F_s(x) - x \subset K$$

for $x \in K$ and $s \geq 0$. Let $x \in K$ and $s \geq 0$. Putting $t = s$ in (7) we have

$$F_{2s}(x) + x = 2F_s(F_s(x)). \tag{9}$$

Hence and by the assumption $x \in F_t(x)$ we get

$$F_s(x) \subset \frac{1}{2}F_{2s}(x) + \frac{1}{2}x.$$

Replacing s by $2s$ in the last inclusion we obtain

$$F_{2s}(x) \subset \frac{1}{2}F_{4s}(x) + \frac{1}{2}x,$$

whence

$$F_s(x) \subset \frac{1}{4}F_{4s}(x) + \frac{1}{4}x + \frac{1}{2}x.$$

By induction we can prove that

$$F_s(x) \subset \frac{1}{2^p}F_{2^p s}(x) + \frac{1}{2^p}x + \cdots + \frac{1}{2}x$$

for all $p \in \mathbb{N}$. Therefore

$$F_s(x) \subset K + \left(1 - 2^{-p}\right) x.$$

Let $y \in F_s(x)$. Then $y - (1 - 2^{-p})\, x \in K$ and letting $p \to \infty$ we have $y - x \in K$. Thus $F_s(x) - x \subset K$.

By (9) and by the additivity of F_s we obtain

$$F_{2s}(x) + x = 2F_s(F_s(x) - x) + 2F_s(x)$$

and

$$F_{2s}(x) - F_s(x) = 2F_s(F_s(x) - x) + F_s(x) - x.$$

Let $k \in \mathbb{N}$. Assuming (8) to hold for $n = k$, we will prove it for $n = k + 1$. Putting $t = ks$ in (7) we get

$$F_{(k+1)s}(x) + F_{(k-1)s}(x) = 2F_{ks}(F_s(x)),$$

whence and by the additivity of F_s

$$F_{(k+1)s}(x) + F_{(k-1)s}(x) = 2F_{ks}(F_s(x) - x) + 2F_{ks}(x).$$

By the induction assumption we obtain

$$F_{(k+1)s}(x) = 2F_{ks}(F_s(x) - x) + \big(F_{ks}(x) - F_{(k-1)s}(x)\big) + F_{ks}(x).$$

Thus

$$F_{(k+1)s}(x) - F_{ks}(x) = 2F_{ks}(F_s(x) - x) + \big(F_{ks}(x) - F_{(k-1)s}(x)\big)\,.$$

From this we see that there exist the Hukuhara differences

$$F_{ns}(x) - F_{ms}(x) \tag{10}$$

for all $m, n \in \mathbb{N}$, $m \le n$, $s \ge 0$. Suppose that $0 \le s \le t$. Replacing s by $\frac{t}{n}$ in (10) we can assert that there exist the Hukuhara differences

$$F_t(x) - F_{\frac{m}{n}t}(x).$$

There exists a sequence $a_n \in \mathbb{Q} \cap [0,1]$ such that $a_n t$ is convergent to s. By the continuity of $t \mapsto F_t(x)$ (Theorem 2 in [10]), $F_{a_n t}(x) \to F_s(x)$ and by Lemma 1, there exists the difference

$$F_t(x) - F_s(x) = \lim_{n\to\infty} \left(F_t(x) - F_{a_n t}(x)\right).$$

A cosine family $\{F_t : \ t \ge 0\}$ of set-valued functions $F_t \colon K \longrightarrow cc(K)$ is said to be *differentiable* if all set-valued functions $t \mapsto F_t(x)$, $x \in K$, have Hukuhara derivative on $[0, +\infty)$.

Lemma 9

Let X be a Banach space and let K be a closed convex cone in X such that $\operatorname{int} K \ne \emptyset$. Suppose that $\{F_t : \ t \ge 0\}$ is a regular cosine family of continuous additive set-valued functions $F_t \colon K \longrightarrow cc(K)$ and $x \in F_t(x)$ for all $x \in K$ and $t \ge 0$. Then multifunctions $t \mapsto F_t(x)$ $(x \in K)$ are concave, there exist set-valued functions $G_t^+ \colon K \longrightarrow cc(K)$ and $G_t^- \colon K \longrightarrow cc(K)$ such that

$$G_t^+(x) = \lim_{h\to 0^+} \frac{F_{t+h}(x) - F_t(x)}{h}, \qquad G_t^-(x) = \lim_{h\to 0^+} \frac{F_t(x) - F_{t-h}(x)}{h}$$

for all $t > 0$, $x \in K$ and the convergence is uniform on every nonempty compact subset of K. Moreover, G_t^+ and G_t^- are additive, continuous,

$$G_t^+(x) = \bigcap_{h>0} \frac{F_{t+h}(x) - F_t(x)}{h}, \qquad G_t^-(x) = \operatorname{cl}\left(\bigcup_{t\ge h>0} \frac{F_t(x) - F_{t-h}(x)}{h}\right)$$

and $G_t^-(x) \subset G_t^+(x)$ for $x \in K$.

Proof. Let us fix $x \in K$. We consider the mutlifunction $t \mapsto F_t(x)$ for $t \geq 0$. Setting $t = \frac{v+u}{2}$, $s = \frac{v-u}{2}$, $0 \leq u \leq v$ in (7) we get

$$F_v(x) + F_u(x) = 2F_{\frac{v+u}{2}}(F_{\frac{v-u}{2}}(x).$$

Since $x \in F_t(x)$ for all $t \geq 0$, we have

$$F_{\frac{v+u}{2}}(x) \subset \frac{F_v(x) + F_u(x)}{2}.$$

Hence, by the continuity (Theorem 2 in [10]) and by Theorem 4.1 in [4] the multifunction $t \mapsto F_t(x)$ is concave. Moreover, by Lemma 8 there exist the Hukuhara differences

$$F_{t+h}(x) - F_t(x), \qquad F_t(x) - F_{t-h}(x)$$

for all $0 \leq h \leq t$. Thus (Theorem 3.2 in [6]) there exist limits

$$G_t^+(x) = \lim_{h \to 0^+} \frac{F_{t+h}(x) - F_t(x)}{h}, \quad G_t^-(x) = \lim_{h \to 0^+} \frac{F_t(x) - F_{t-h}(x)}{h} \tag{11}$$

for all $t > 0$. As $t \mapsto F_t(x)$ is concave we see that $h \mapsto \frac{F_{t+h}(x) - F_t(x)}{h}$ is increasing, $h \mapsto \frac{F_t(x) - F_{t-h}(x)}{h}$ is decreasing in $(0, t)$ and $\frac{F_t(x) - F_{t-h}(x)}{h} \subset G_t^+(x)$.

Lemmas 5 and 7 respectively imply that the convergence in (11) is uniform on every nonempty compact subset of K and G_t^+, G_t^- are additive and continuous.

Theorem

Let X be a Banach space and let K be a closed convex cone with the nonempty interior. Suppose that $\{F_t : t \geq 0\}$ is a regular cosine family of continuous linear set-valued functions $F_t\colon K \longrightarrow cc(K)$, $x \in F_t(x)$ for all $x \in K$ and $t > 0$ and $F_t \circ F_s = F_s \circ F_t$ for all $s, t > 0$. Then this cosine family is twice differentiable and

$$D^2 F_t(x) = F_t(H(x))$$

for $x \in K$, $t \geq 0$, where $D^2 F_t(x)$ denotes the second Hukuhara derivative of $F_t(x)$ with respect to t and $H(x)$ is the second Hukuhara derivative of this multifunction at $t = 0$.

Proof. Let us fix $x \in K$. Consider the multifunction $t \mapsto F_t(x)$ for $t \geq 0$. By Lemma 8 there exist the Hukuhara differences $F_t(x) - F_s(x)$ for $0 \leq s \leq t$. By Lemma 9 the multifunction $t \mapsto F_t(x)$ is concave and there exist

$$G_t^+(x) = \lim_{h \to 0^+} \frac{F_{t+h}(x) - F_t(x)}{h} \qquad \text{and} \qquad G_t^-(x) = \lim_{h \to 0^+} \frac{F_t(x) - F_{t-h}(x)}{h}$$

for $t > 0$ and $G_t^-(x) \subset G_t^+(x)$. The same argument may be used to prove that there exists

$$\lim_{t \to 0^+} \frac{F_t(x) - x}{t}.$$

It follows from (7) that

$$\frac{F_{2t}(x) - x}{2t} = F_t\left(\frac{F_t(x) - x}{t}\right) + \frac{F_t(x) - x}{t}.$$

Letting $t \to 0^+$ we get

$$\lim_{t \to 0^+} F_t\left(\frac{F_t(x) - x}{t}\right) = \{0\}$$

and since

$$0 \in \frac{F_t(x) - x}{t} \subset F_t\left(\frac{F_t(x) - x}{t}\right)$$

we have

$$DF_0(x) = \lim_{t \to 0^+} \frac{F_t(x) - x}{t} = \{0\}. \tag{12}$$

Let $0 < h \le t$. By (7) and the additivity of F_t we obtain

$$F_{t+h}(x) - F_t(x) = 2F_t(F_h(x) - x) + F_t(x) - F_{t-h}(x).$$

Dividing the last equality by h we get

$$\frac{F_{t+h}(x) - F_t(x)}{h} = 2F_t\left(\frac{F_h(x) - x}{h}\right) + \frac{F_t(x) - F_{t-h}(x)}{h}.$$

Letting $h \to 0^+$, by Lemma 9 and (12) we have

$$G_t^+(x) = G_t^-(x) =: G_t(x) \qquad \text{for } t > 0.$$

This and (12) imply that the family $\{F_t : \ t \ge 0\}$ is differentiable.

Next we will show that there exist the Hukuhara differences $G_t(x) - G_s(x)$ for $0 \le s \le t$. It is enough to consider the case $0 < s < t$. Let $h > 0$ be such that $t - s - h \ge 0$. By Lemma 8 there exist the differences

$$F_{\frac{1}{2}t-\frac{1}{2}s+\frac{1}{2}h}(x) - F_{\frac{1}{2}t-\frac{1}{2}s-\frac{1}{2}h}(x), \quad F_{t+h}(x) - F_t(x) \quad \text{and} \quad F_{s+h}(x) - F_s(x)$$

in $cc(K)$. Since $F_{\frac{1}{2}t+\frac{1}{2}s+\frac{1}{2}h}$ is linear and continuous with respect to Lemma 3 there exists the difference

$$F_{\frac{1}{2}t+\frac{1}{2}s+\frac{1}{2}h}(F_{\frac{1}{2}t-\frac{1}{2}s+\frac{1}{2}h}(x)) - F_{\frac{1}{2}t+\frac{1}{2}s+\frac{1}{2}h}(F_{\frac{1}{2}t-\frac{1}{2}s-\frac{1}{2}h}(x)).$$

By (7) we have

$$\begin{aligned}2F_{\frac{1}{2}t+\frac{1}{2}s+\frac{1}{2}h}(F_{\frac{1}{2}t-\frac{1}{2}s+\frac{1}{2}h}(x)) - 2F_{\frac{1}{2}t+\frac{1}{2}s+\frac{1}{2}h}(F_{\frac{1}{2}t-\frac{1}{2}s-\frac{1}{2}h}(x))\\ = F_{t+h}(x) + F_s(x) - (F_t(x) + F_{s+h}(x))\\ = (F_{t+h}(x) - F_t(x)) - (F_{s+h}(x) - F_s(x)).\end{aligned}$$

Because of Lemma 1 there exists

$$G_t(x) - G_s(x) = \lim_{h\to 0^+}\left(\frac{F_{t+h}(x) - F_t(x)}{h} - \frac{F_{s+h}(x) - F_s(x)}{h}\right).$$

Our next claim is that the multifunction $t \mapsto G_t(x)$ is concave and differentiable. Replacing in (7) t by $t+h$, $h > 0$ and substracting $F_{t+s}(x) + F_{t-s}(x)$ from both the sides of this equality we get

$$F_{t+s+h}(x) - F_{t+s}(x) + F_{t-s+h}(x) - F_{t-s}(x) = 2F_{t+h}(F_s(x)) - 2F_t(F_s(x)).$$

The equality $F_t \circ F_s = F_s \circ F_t$, $s, t \geq 0$ leads to

$$\frac{F_{t+s+h}(x) - F_{t+s}(x)}{h} + \frac{F_{t-s+h}(x) - F_{t-s}(x)}{h} = 2F_s\left(\frac{F_{t+h}(x) - F_t(x)}{h}\right),$$

whence, as $h \to 0^+$,

$$G_{t+s}(x) + G_{t-s}(x) = 2F_s(G_t(x)). \tag{13}$$

Setting $t = \frac{v+u}{2}$, $s = \frac{v-u}{2}$, where $0 \leq u \leq v$ in (13) yields

$$G_v(x) + G_u(x) = 2F_{\frac{v-u}{2}}(G_{\frac{v+u}{2}}(x)).$$

By the assumption $x \in F_t(x)$ we get

$$G_{\frac{v+u}{2}}(x) \subset \frac{G_v(x) + G_u(x)}{2}.$$

Fix an interval $[a,b] \subset [0,\infty)$ and let $t \in [a,b]$. Since the multifunctions $t \mapsto F_t(x)$, $x \in K$, are concave and differences $F_t(x) - F_s(x)$ exist for $x \in K$ and $0 \leq s \leq t$, the multifunctions $t \mapsto G_t(x)$ are increasing (Theorem 3.2 in [6]) and we have $G_t(x) \subset G_b(x)$. Therefore the multifunctions $t \mapsto G_t(x)$ are bounded on $[a,b]$. By Theorem 4.4 in [4] the multifunction $t \mapsto G_t(x)$ is continuous in $(0,\infty)$ and by Theorem 4.1 in [4] it is concave. In virtue of Theorem 3.2 in [6] there exist

$$H_t^+(x) = \lim_{h\to 0^+}\frac{G_{t+h}(x) - G_t(x)}{h} \qquad \text{and} \qquad H_t^-(x) = \lim_{h\to 0^+}\frac{G_t(x) - G_{t-h}(x)}{h}$$

for $t > 0$ and $H_t^-(x) \subset H_t^+(x)$. Since $\frac{G_{\lambda t}(x)}{\lambda t} \subset \frac{G_t(x)}{t}$ for $t > 0$ and $\lambda \in (0,1)$, there also exists

$$\lim_{t \to 0^+} \frac{G_t(x)}{t} =: H(x)$$

and $H(x) \in cc(K)$.

Let $0 < s \le t$. The relation $F_t \circ F_s = F_s \circ F_t$ and Lemmas 2, 3 and 9 yield

$$\begin{aligned} &F_s(G_t(x)) \\ &\quad = F_s\left(\lim_{h \to 0^+} \frac{F_{t+h}(x) - F_t(x)}{h}\right) = \lim_{h \to 0^+} \frac{F_s(F_{t+h}(x)) - F_s(F_t(x))}{h} \\ &\quad = \lim_{h \to 0^+} \frac{F_{t+h}(F_s(x)) - F_t(F_s(x))}{h} \\ &\quad \subset \lim_{h \to 0^+} \frac{(F_{t+h} - F_t)(F_s(x))}{h} \\ &\quad = G_t(F_s(x)) \end{aligned}$$

which together with (13) lead to

$$G_{t+s}(x) + G_{t-s}(x) \subset 2G_t(F_s(x)).$$

By the additivity of G_t we get

$$G_{t+s}(x) + G_{t-s}(x) \subset 2G_t(F_s(x) - x) + 2G_t(x),$$

whence

$$G_{t+s}(x) - G_t(x) \subset 2G_t(F_s(x) - x) + G_t(x) - G_{t-s}(x).$$

Dividing the last inclusion by s and letting $s \to 0^+$ we obtain

$$H_t^+(x) \subset H_t^-(x).$$

Therefore

$$H_t^+(x) = H_t^-(x) =: H_t(x)$$

for $t > 0$ and the family $\{F_t : \ t \ge 0\}$ is twice differentiable.

It remains to prove the equality in the assertion. Let $0 < s < t$. Lemmas 1, 3 and (7) lead to

$$\begin{aligned} 2F_t(G_s(x)) &= 2F_t\left(\lim_{h \to 0^+} \frac{F_{s+h}(x) - F_s(x)}{h}\right) \\ &= \lim_{h \to 0^+} \frac{2F_t(F_{s+h}(x)) - 2F_t(F_s(x))}{h} \\ &= \lim_{h \to 0^+} \frac{F_{t+s+h}(x) + F_{t-s-h}(x) - (F_{t+s}(x) + F_{t-s}(x))}{h} \end{aligned}$$

$$= \lim_{h\to 0^+}\left[\frac{F_{t+s+h}(x)-F_{t+s}(x)}{h}-\frac{F_{t-s}(x)-F_{t-s-h}(x)}{h}\right]$$
$$= G_{t+s}(x)-G_{t-s}(x)$$
$$= G_{t+s}(x)-G_t(x)+G_t(x)-G_{t-s}(x).$$

Dividing the last equality by s we get

$$2F_t\left(\frac{G_s(x)}{s}\right)=\frac{G_{t+s}(x)-G_t(x)}{s}+\frac{G_t(x)-G_{t-s}(x)}{s},$$

letting $s\to 0^+$ and dividing by 2 we have

$$F_t(H(x))=H_t(x).$$

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Report of Meeting

9th International Conference on Functional Equations and Inequalities, Złockie, September 7-13, 2003

The Ninth International Conference on Functional Equations and Inequalities, in the series of those organized by the Institute of Mathematics of the Pedagogical University of Cracow since 1984 (biannual since 1991), was held from September 7 to September 13, 2003, for the fifth time in the hotel "Geovita" at Złockie.

A support of the State Committee for Scientific Research (KBN) as well as the Bank Przemysłowo-Handlowy PBK SA is acknowledged with gratitude.

The Conference was opened by the address of Prof. Dr. Eugeniusz Wachnicki, Deputy Rector of the Pedagogical University of Cracow, who spoke on behalf of Prof. Dr. Michał Śliwa, Rector Magnificus. He greeted the participants, thanked the organizers and wished a fruitful and nice stay in this beatiful region of Poland.

After honorary doctorates conferred to Professor János Aczél by the Universities in Karlsruhe, Graz, Katowice and Miskolc, him was promoted to the Degree of Doctor Honoris Causa by the University of Debrecen in June 2003. The participants signed an address on this occasion with congratulations saying, among others: *Thanks to Your valuable and commemorable presence here in September 1999 at the 7th Conference we feel authorized to call ourselves students of the created by You World School on Functional Equations.*

Best greetings were also sent to Mrs Irena Gołąb, wife of Professor Stanisław Gołąb, on the occassion of her 100th birthday, and to Professor Stanisław Midura on his 70th birthday.

There were 57 participants who came from 8 countries: Austria (1, Innsbruck), France (1, Nantes), Germany (4, Clausthal-Zellerfeld, Karlsruhe, Munich), Greece (1, Athens), Hungary (12, Debrecen, Gödöllő, Gyöngyös, Miskolc), Italy (2, Torino), Japan (1, Kobe) and from Poland (35, Bielsko-Biała, Gdańsk, Katowice, Kraków, Rzeszów, Zielona Góra).

During 17 sessions 51 talks were delivered. The papers presented may be divided into four groups: A) functional equations in several variables,B) functional equations in a single variable and iteration theory, C) functional inequalities, D) related topics.

Here is a more detailed list of topics:

A) Properties of additive functions and derivations; the generalized Gołąb-Schinzel, Hosszú and Wilson equations; conditional and alternative equations (d'Alembert's, Cauchy's, Wigner's, of isometries); regularity type theorems with applications; equations stemming from the Cauchy–Riemann ones and from Mean Value Theorems; analytic solutions of H. Haruki's type equations; stability problems for various equations; the translation and cocycle equations; iteration semigroups, also of multifunctions; determining translative and quasi-commutative operations; comparing of utility representations.

B) Schröder equation, also in iteration theory and in Banach spaces; flows on the plane; iterative roots; regular iteration groups on a circle and in Banach spaces; sum type operators in Banach spaces; dilation equation; linear equation with iterates of the unknown function; k-difference equation and application to image processing.

C) Convexity: approximate, Jensen, Wright, t- and λ-convexity; Prékopa–Leindler inequality; simultaneous difference inequalities; monotonicity of sequences in the sense of Leja; characterization of quasi-monotonicity.

D) Reciprocal polynomials and number theory; Hermite–Hadamard inequality; recursive sequence; decision functions; inequalities between Gini and Stolarsky means; characterization of continuous functions; Rădulescu problem; metric space of multimeasures.

There were presented several remarks and open problems in 4 sessions.

Prof. Dr. Roman Ger exhibited a draft of his article entitled *Functional Equations and Inequalities*, to appear (in Polish) in a jubilee volume *50 Years of Matematics in Upper Silesia* asking for any comments. There has been stressed the significant role and merits of Professor Marek Kuczma and his School of Functional Equations in the development of research during the last 40 years.

The participants enjoyed a picnic on Tuesday and a banquet on Thursday. On Wednesday afternoon (which was free of sessions), due to a very nice weather, several groups of participants hiked along trails of Beskid Sądecki, some others visited Krynica Zdrój, and also Stará L'ubovňa and L'ubotín in Slovakia.

The organizing Committee was chaired by Prof. Dobiesław Brydak in co-operation with Prof. Bogdan Choczewski from the AGH University of Science and Technology in Kraków. Dr. Jacek Chmieliński acted as scientific secretary. Miss Janina Wiercioch and Mr Władysław Wilk (technical assistant) worked in the course of preparation of the meeting and in the Conference Office at Złockie, with a help of other academics from the Institute of Mathematics of the Pedagogical University of Cracow, in particular of Mr Paweł Solarz and Dr. Joanna Szczawińska.

The Conference was closed by Professor Dobiesław Brydak. Cordial thanks were addressed to the participants who all presented valuable contributions and created the unique friendly, both scientific and social, atmosphere. Thanks were extended to the members of the whole office staff at Złockie for their effective and dedicated work and helpful assistance, and to the managers of the hotel "Geovita" for their hospitality and quality of services.

The 10th ICFEI is planned to be held in September, 2005.

The abstracts of talks are printed in the alphabetical order, whereas the problems and remarks are presented chronologically. The careful and efficient work of Dr. J. Chmieliński on completing the material and preparing (together with Mr W. Wilk) the present report for printing is acknowledged with thanks.

Bogdan Choczewski

Abstracts of Talks

Mirosław Adamek *A characterization of λ-convex functions*

Let $I \subseteq \mathbb{R}$ be an interval and $\lambda\colon I^2 \longrightarrow (0,1)$ be a fixed function. A real-valued function $f\colon I \longrightarrow \mathbb{R}$ is called *λ-convex* if

$$f(\lambda(x,y)x + (1-\lambda(x,y))y) \leq \lambda(x,y)f(x) + (1-\lambda(x,y))f(y) \quad \text{for } x,y \in I.$$

The main result shows that λ-convex functions can be characterized in terms of a lower second-order generalized derivative.

Anna Bahyrycz *On the conditional equation of the exponential function*

We consider the conditional equation of the exponential function:

$$\forall\, x,y \in \mathbb{R}(n) \quad f(x)\cdot f(y) \neq 0_m \implies f(x+y) = f(x)\cdot f(y), \tag{1}$$

where

$$\begin{aligned}
&f\colon \mathbb{R}(n) := [0,+\infty)^n \setminus \{0_n\} \longrightarrow \mathbb{R}(m), \quad n,m \in \mathbb{N},\\
&0_m := (0,\dots,0) \in \mathbb{R}^m,\\
&x+y := (x_1+y_1,\dots,x_k+y_k) \quad \text{and} \quad x\cdot y := (x_1\cdot y_1,\dots,x_k\cdot y_k)\\
&\qquad\qquad\qquad \text{for } x=(x_1,\dots,x_k),\ y=(y_1,\dots,y_k) \in \mathbb{R}(k).
\end{aligned}$$

We study the general solution of the equation (1); exactly, we find a description and properties of the system of the cones over $\mathbb{Q}$ giving this solution.

Karol Baron *Dense sets of additive functions*

Joint work with Peter Volkmann.

We consider the topological vector space $\mathcal{A}$ of all additive functions from $\mathbb{R}$ to $\mathbb{R}$ with the Tychonoff topology induced by $\mathbb{R}^{\mathbb{R}}$ and we prove that the following subsets of $\mathcal{A}$ and their complements (with respect to $\mathcal{A}$) are dense: the set of all additive injections, surjections, bijections, involutions, additive functions with countable image, additive functions with big graph. We are using a lemma which characterizes the density of subsets of $\mathcal{A}$.

Lech Bartłomiejczyk *Derivations with big graph*

A function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is called a *derivation* (cf. [1; Ch. XIV]) iff it is additive and satisfies the equation

$$f(xy) = xf(y) + yf(x)$$

for all $x, y \in \mathbb{R}$.

We say that a function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ has a *big graph* iff $B \cap \mathrm{Graph}\,(f) \neq \emptyset$ for every Borel subset B of $\mathbb{R}^2$ such that

$$\mathrm{card}\,\{x \in \mathbb{R} :\ (x,y) \in B\} = \mathfrak{c}.$$

The well known theorem of F.B. Jones [1; Theorem 3, p. 287] says that there exist additive functions with big graph; we prove that there exist derivations with big graph. This answers the question of Professor Ludwig Reich.

[1] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, Prace Naukowe Uniwersytetu Śląskiego w Katowicach 489, Państwowe Wydawnictwo Naukowe & Uniwersytet Śląski, Warszawa – Kraków – Katowice, 1985.

Mihály Bessenyei *On generalized Hermite–Hadamard inequality*

Joint work with Zsolt Páles.

The classical Hermite–Hadamard inequality provides the following lower and upper estimates for the integral average of a convex function $f\colon [a,b] \longrightarrow \mathbb{R}$:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

The aim of our talk is to present Hermite–Hadamard inequalities under more general assumption than ordinary convexity. These kind of inequalities offer a lower and upper estimate for the integral average of a function involving certain base points of the domain.

Dobiesław Brydak *On the nonlinear iterative functional inequality*

We shall present a comparison theorem concerning the inequality

$$\psi[f(x)] \leq F[x, \psi(x)]$$

where ψ is an unknown function.

Janusz Brzdęk *On measurable solutions of some functional equations connected with multiplicative symmetry*

Let $(X, \cdot)$ be a group endowed with a topology, $F\colon \mathbb{C} \longrightarrow X$ be continuous at 0 and -1, and $F(-1) = F(1)^m$, $F(0) = F(1)^k$ with some $k, m \in \mathbb{Z}$. Under suitable assumptions on X, we describe the solutions $f\colon X \longrightarrow \mathbb{C}$ of the functional equation

$$f(F(f(y))\cdot x) = f(y)f(x)$$

that are continuous at a point or (universally, Baire, Christensen or Haar) measurable.

Jacek Chmieliński *Stability of angle-preserving mappings on the plane*

We say that a mapping $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is *angle-preserving* iff it satisfies:

$$|\cos(f(x), f(y))| = |\cos(x, y)|, \qquad x, y \in \mathbb{R}^2 \setminus \{0\}$$

and $f(x) = 0 \ \Leftrightarrow \ x = 0$.

We prove that this property is stable and we apply this result to prove some kind of stability of the Wigner equation on the plane.

Bogdan Choczewski *k-difference equations and image processing*

The equations of the form

$$\varphi(kx) = g(\varphi(x))$$

are useful in a procedure of image processing proposed by S. Mann (*Intelligent Image Processing*, Willey and Sons, 2002). An example of such equation will be discussed.

Krzysztof Ciepliński *On rational iteration groups on the circle*

Denote by $\mathbb{S}^1$ the unit circle and let V be a linear space over $\mathbb{Q}$ with $\dim V \geq 1$. We recall that family $\{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ of homeomorphisms such that

$$F^{v_1} \circ F^{v_2} = F^{v_1+v_2}, \qquad v_1, v_2 \in V$$

is called an *iteration group*. An iteration group is said to be *rational*, if $V = \mathbb{Q}$.

In this talk we deal with rational iteration groups. We also give a general construction of all iteration groups $\{F^v\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ such that $\emptyset \neq \{z \in \mathbb{S}^1\colon\ F^v(z) = z,\ v \in V\} \neq \mathbb{S}^1$.

Péter Czinder *Comparison inequalities for Gini and Stolarsky means*

We investigate some inequalities concerning the two variable Gini and Stolarsky means, defined (in the most general case) by the formulae

$$G_{a,b}(x, y) = \left(\frac{x^a + y^a}{x^b + y^b}\right)^{\frac{1}{a-b}}; \qquad S_{a,b}(x, y) = \left(\frac{x^a - y^a}{a}\frac{b}{x^b - y^b}\right)^{\frac{1}{a-b}}.$$

After giving the summary of preliminary results (comparison theorems for means of the same kind), we present our new results regarding the comparison of Gini and Stolarsky means.

Zoltán Daróczy *On translative and quasi-commutative operations*

We determine all the continuous operations $\circ\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ that are translative $((x+z)\circ(y+z) = x\circ y + z)$ and quasi-commutative $(x\circ(y\circ z) = y\circ(x\circ z))$.

Joachim Domsta *On a linear equation in two variables*

For a bijective $\varphi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $\mathbb{R}_+ := (0,\infty)$, and two constants $a, b > 0$ let $g\colon \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be defined as follows,

$$g(x,y) = \varphi^{-1}(a\cdot\varphi(x) + b\cdot\varphi(y)), \qquad \text{for } x, y \in \mathbb{R}_+. \tag{$*$}$$

This function appears in problems considered by J. Matkowski et al., see e.g. [1] and the references therein, cf. also the proceedings of the 40$^{\text{th}}$ ISFE. Here we are proving the following fact: For every pair of a and b, function g determines φ uniquely, up to a multiplicative constant. Also a construction of the solution is presented, which goes through a special family of simultaneous Schröder equations in one variable derived from the equation $(*)$.

[1] J. Matkowski, *On a characterization of L^p-norm and a converse of Minkowski's inequality*, Hiroshima Math. J. **26**;2 (1996), 277-287.

Borbála Fazekas *Decision functions and their properties*

Our main aim is to characterize the relation between the properties of the so called decision functions and the properties of the decision generating functions. A function $D\colon \bigcup_{i=1}^{\infty} I^i \longrightarrow I$ is called a *decision function*, if it is symmetric, reflexive, regular and internal. We can generate a decision function D_d with a generalization of the least squares method using a decision generating function $d\colon I \times I \longrightarrow \mathbb{R}$. The reverse statement is also true, for every decision function D there exists a decision generating function d, that generates it. The main result, that characterizes the monotonicity property, is the following

THEOREM
A decision function $D_d\colon \bigcup_{i=1}^{\infty} I^i \longrightarrow I$, generated by the decision generating function $d\colon I\times I \longrightarrow \mathbb{R}$, is monotonic if and only if

$$d(x_1, y_1) + d(x_2, y_2) \le d(x_1, y_2) + d(x_2, y_1)$$

holds for every $x_1, x_2, y_1, y_2 \in I$, $x_1 \le x_2$, $y_1 \le y_2$.

Margherita Fochi *On a conditional-alternative functional equation*

Let X be a real inner product space with $\dim X \geq 3$ and let $f\colon X \longrightarrow \mathbb{R}$.

Taking into account known results about functional equations on orthogonal vectors, we investigate the relations between the class of the solutions of the three following equations: the alternative equation on the whole space

$$f(x+y)^2 = (f(x)+f(y))^2 \qquad \text{for all } x, y \in X, \tag{1}$$

the corresponding equation restricted to the pairs of orthogonal vectors

$$f(x+y)^2 = (f(x)+f(y))^2 \qquad \text{for all } x, y \in X \text{ with } x \perp y \tag{2}$$

and the conditional Cauchy equation

$$f(x+y) = f(x)+f(y) \qquad \text{for all } x, y \in X \text{ with } x \perp y. \tag{3}$$

First of all we prove that (1), (2) and (3) are not equivalent, afterwards we shall characterize the common solutions introducing suitable auxiliary conditions.

Roman Ger *Logarithmic concavity and the Prékopa–Leindler inequality*

The Prékopa–Leindler inequality states that:

Given a $\lambda \in (0,1)$ and functions $f, g, h\colon \mathbb{R}^n \longrightarrow [0,\infty)$ that are Lebesgue integrable and satisfying the functional inequality

$$h(\lambda x + (1-\lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda}$$

for all $x, y \in \mathbb{R}^n$, one has

$$\int_{\mathbb{R}^n} h\, d\ell_n \geq \left(\int_{\mathbb{R}^n} f\, d\ell_n\right)^\lambda \left(\int_{\mathbb{R}^n} g\, d\ell_n\right)^{1-\lambda}.$$

We discuss the pexiderized functional inequality of logarithmic concavity occurring here as the assumption. Among the corollaries issued from that studies we note down yet another proof of the logarithmic concavity of the Lebesgue measure.

Attila Gilányi *On a functional equation arising from the comparison of utility representations*

Joint work with János Aczél and Che Tat Ng.

In this talk the functional equation $F_1(t) - F_1(t+s) = F_2[F_3(t) + F_4(s)]$ is solved for real valued functions defined on intervals, assuming that F_2 is positive valued and strictly monotonic, and that F_3 is continuous. The equation with these assumptions arises from the comparison of utility representations characterized under the assumptions of separability, homogeneity and segregation (cf. [3]). It has been encountered before by A. Lundberg [2] and by J. Aczél, Gy. Maksa, C.T. Ng and Zs. Páles [1] under various conditions.

[1] J. Aczél, Gy. Maksa, C.T. Ng, Zs. Páles, *A functional equation arising from ranked additive and separable utility*, Proc. Amer. Math. Soc. **129** (2000), 989-998.

[2] A. Lundberg, *On the functional equation* $f(\lambda((x)+g(y))=\mu(x)+h(x+y)$, Aequationes Math. **16** (1977), 21-30.

[3] C.T. Ng, R.D. Luce, J. Aczél, *Functional characterization of basic properties of utility representations*, Monatsh. Math. **135** (2002), 305-319.

Roland Girgensohn *A recursive sequence*

Joint work with Jonathan Borwein.

The following problem appeared in the American Mathematical Monthly in 2002. Let

$$\begin{aligned} a_1 &= 1, \\ a_2 &= \frac{1}{2}+\frac{1}{3}, \\ a_3 &= \frac{1}{3}+\frac{1}{7}+\frac{1}{4}+\frac{1}{13}, \\ a_4 &= \frac{1}{4}+\frac{1}{13}+\frac{1}{8}+\frac{1}{57}+\frac{1}{5}+\frac{1}{21}+\frac{1}{14}+\frac{1}{183}, \end{aligned}$$

and continue the sequence, constructing a_{n+1} by replacing each fraction $\frac{1}{d}$ in the expression for a_n with $\frac{1}{(d+1)}+\frac{1}{(d^2+d+1)}$. Compute $\lim_{n\to\infty} a_n$.

In the talk we will show how this problem can be solved using functional equations, and we will give generalizations.

Grzegorz Guzik *Cocycles and continuous iteration semigroups of triangular mappings*

In the comprehensive paper [2] M.C. Zdun found a form of all continuous iteration semigroups of continuous selfmappings of a compact interval. We use some results of W. Jarczyk, J. Matkowski and the present author on continuous solutions of the so called cocycle equation (see [1]) for construction of some continuous iteration semigroups of triangular functions mapping the product of two compact intervals into itself.

[1] G. Guzik, W. Jarczyk, J. Matkowski, *Cocycles of continuous iteration semigroups*, Bull. Pol. Acad. Sci. **51**(2)(2003), 185-197.

[2] M.C. Zdun, *Continuous and differentiable iteration semigroups*, Prace Nauk. Uniw. Śl. w Katowicach **308**, Katowice 1979.

Gabriella Hajdu *An extension theorem for a Matkowski–Sutô type problem for weighted quasi-arithmetic means*

Joint work with Zoltán Daróczy.

Let $I \subset \mathbb{R}$ be an interval, $0 < \lambda < 1$, $\mu \neq 0, 1$. We consider the following generalized Matkowski–Sutô type problem.

$$\mu A_\varphi(x, y; \lambda) + (1 - \mu) A_\psi(x, y; \lambda) = \lambda x + (1 - \lambda) y \qquad (x, y \in I),$$

where φ, ψ are continuous strictly monotone real functions on I and

$$A_\varphi(x, y; \lambda) := \varphi^{-1}(\lambda\varphi(x) + (1 - \lambda)\varphi(y))$$

denotes the weighted quasi-arithmetic mean generated by φ.

The solutions of the equation, under some regularity conditions, were given by Daróczy and Páles.

Our aim is to prove that if K is a proper subinterval of I then the solutions of the equation can be uniquely extended from K to the whole interval I.

Attila Házy *On approximately t-convexity*

Joint work with Zsolt Páles.

A real valued function f defined on an open convex set D is called (ε, p, t)-convex if it satisfies

$$f\left(tx + (1 - t)y\right) \leq tf(x) + (1 - t)f(y) + \sum_{i=0}^{k} \varepsilon_i |x - y|^{p_i} \qquad \text{for } x, y \in D,$$

where $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_k) \in [0, \infty[^{k+1}$, $p = (p_0, \ldots, p_k) \in [0, 1[^{k+1}$ and $t \in]0, 1[$ are fixed parameters. The main result of the paper states that if f is locally bounded from above at a point of D and (ε, p, t)-convex then it satisfies the convexity-type inequality

$$f\big(sx + (1 - s)y\big) \leq sf(x) + (1 - s)f(y) + \sum_{i=0}^{k} \varepsilon_i \phi_{p_i}(s) |x - y|^{p_i}$$

for $x, y \in D$, $s \in [0, 1]$, where $\phi_{p_i} \colon [0, 1] \longrightarrow \mathbb{R}$ is defined by

$$\phi_{p_i}(s) = \max\left\{ \frac{1}{(1 - t)^{p_i} - (1 - t)}; \frac{1}{t^{p_i} - t} \right\} \big(s(1 - s)\big)^{p_i}.$$

The particular case $k = 0$, $p = 0$ of this result is due to Páles [4], the case $k = 0$, $p = 0$ and $t = \frac{1}{2}$ was investigated by Nikodem and Ng [3], the specialization $k = 0$, $\varepsilon_0 = 0$ yields the celebrated theorem of Bernstein and Doetsch [1]. The case $k = 1$, $\varepsilon = (\varepsilon_0, \varepsilon_1)$, $p = (1, 0)$ and $t = \frac{1}{2}$ was investigated in Házy and Páles [2].

[1] F. Bernstein, G. Doetsch, *Zur Theorie der konvexen Funktionen*, Math. Annalen **76** (1915), 514-526.

[2] A. Házy and Zs. Páles, *Approximately midconvex functions*, Bull. London Math. Soc. (2002), to appear.

[3] C.T. Ng, K. Nikodem, *On approximately convex functions*, Proc. Amer. Math. Soc. **118** (1993), no. 1, 103-108.

[4] Zs. Páles, *Bernstein–Doetsch-type results for general functional inequalities*, Rocznik Nauk.-Dydakt. **204** Prace Mat. **17** (2000), Dedicated to Professor Zenon Moszner on his 70th birthday, 197-206.

Witold Jarczyk *Improving regularity of some functions by Grosse-Erdmann's theorems*

Joint work with Karol Baron.

Making use of a theorem of K.-G. Grosse-Erdmann we prove a result providing the effect "measurability implies continuity". It can be applied to functions satisfying a pretty large class of equalities of the form

$$\varphi(s+t) = \Phi(s, \gamma(t)),$$

among others to functional equations. In particular, we get a slight generalization of a theorem of M.C. Zdun concerning the continuity of Lebesgue measurable solutions of the translation equation

$$F(s+t, x) = F(t, F(s, x)).$$

We also generalize some results of G. Guzik, among others that one dealing with the regularity of solutions of the cocycle equation

$$G(s+t, x) = G(s, x)G(t, F(s, x)).$$

Hans-Heinrich Kairies *Spectral properties of a sum type operator*

Joint work with Karol Baron.

The eigenspaces of the sum type operator $F\colon D_n \longrightarrow D_n$, given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x)$$

can be characterized as solution sets of a Schröder equation and (depending on the structure of the domain D_n) of other linear iterative equations, which are simultaneously satisfied.

Some particular cases are discussed in detail.

Zoltán Kaiser *Stability of the monomial functional equation in normed spaces over fields with valuation*

S.M. Ulam's problem was to give conditions for the existence of a linear mapping near an approximately linear mapping. The first solution for this problem was given by D.H. Hyers in real Banach spaces. Th.M. Rassias and Z. Gajda gave a generalized solution to Ulam's problem. A. Gilányi investigated the stability of the monomial functional equation in the same sense. This talk extends their results to a more general setting when we consider Banach spaces over fields of characteristic zero with valuation.

Zygfryd Kominek *On a problem of Rădulescu*

Vincentiu Rădulescu in American Mathematical Monthly [1] posed the following problem.

Let $g\colon (0,\infty) \longrightarrow (0,\infty)$ *be a continuous function satisfying the condition*

$$\lim_{x\to\infty} \frac{g(x)}{x^{1+\alpha}} = \infty, \tag{1}$$

where $\alpha > 0$ *is a given constant. Let* $f\colon \mathbb{R} \longrightarrow (0,\infty)$ *be twice differentiable function for which there exist* x_0 *and* $a > 0$ *such that*

$$f''(x) + f'(x) > ag(f(x)) \qquad \text{for every } x \ge x_0. \tag{2}$$

Prove that $\lim_{x\to\infty} f(x)$ *exists and is finite, and evaluate the limit.*

We prove that if a solution does exist then $\lim_{x\to\infty} f(x) = 0$.

[1] V. Rădulescu, *Problem 11024*, Amer. Math. Monthly, **110** no. 6 (June–July 2003), 543.

Dorota Krassowska *Measurable solutions of a pair of linear functional inequalities of iterative type*

Suppose that a Lebesgue measurable function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the system of inequalities

$$f(x+a) \le f(x) + \sum_{j=0}^{k} \alpha_j x^j; \quad f(x+b) \le f(x) + \sum_{j=0}^{k} \beta_j x^j, \qquad x \in \mathbb{R}, \tag{1}$$

where $k \in \mathbb{N}$, $a, b, \alpha_0, \alpha_1, \dots, \alpha_k, \beta_0, \beta_1, \dots, \beta_k \in \mathbb{R}$ are fixed.

Assuming that

$$a < 0 < b, \qquad \frac{b}{a} \notin \mathbb{Q}, \qquad \frac{\alpha_k}{a} - \frac{\beta_k}{b} = 0,$$

(where $\mathbb{Q}$ stands for the set of all rational numbers) we show that the function $g\colon \mathbb{R} \longrightarrow \mathbb{R}$,

$$g(x) := f(x) - \frac{\alpha_k}{a(k+1)} x^{k+1}, \qquad x \in \mathbb{R},$$

satisfies system (1) with the same a, b and some uniquely determined polynomials of the order not greater than $k-1$.

The Lebesgue measurable solutions of (1) in the case $\alpha_0 = \alpha_1 = \cdots = \alpha_k = \beta_0 = \beta_1 = \cdots = \beta_k = 0$ were considered by J. Brzdęk in [1].

Our result and the result of Brzdęk, under some algebraic conditions of involved constants, allow to characterize the Lebesgue measurable solutions of (1) as the functions which coincide with polynomials almost everywhere.

[1] J. Brzdęk, *On functions satisfying some inequalities*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 277-281.

Károly Lajkó *A special case of the generalized Hosszú equation on an interval*

The functional equation

$$F(xy) + G((1-x)y) = H(x) + K(y) \tag{1}$$

plays an important role in solving of the generalized Hosszú functional equation, introduced by I. Fenyő,

$$f(r_0 + (r_1 x + r_2)(r_3 y + r_4)) + g(s_0 + (s_1 x + s_2)(s_3 y + s_4)) = h(x) + k(y). \tag{2}$$

In this presentation we consider equation (1) for the unknown functions $F, G, H, K\colon]0,1[\longrightarrow \mathbb{R}$ on the restricted domain $D = \{(x,y) \mid x, y \in]0,1[\}$. We have found the general solution of (1) on D.

Piroska Lakatos *Zeros of Coxeter and reciprocal polynomials*

We give sufficient conditions (linear inequalities in the coefficients) for reciprocal polynomials to have all their zeros on the unit circle.

We apply these results for the construction of Salem and PV numbers as well as to get estimates for the spectral radius of Coxeter transformation.

Zbigniew Leśniak *On boundary and limit orbits of a flow on the plane*

We present some properties of an equivalence relation defined for a given flow of the plane which have no fixed points. In particular, we observe that each point belonging to the first prolongational limit set of the plane is contained in the union of boundaries of equivalence classes of the relation, which implies that each limit orbit is a boundary orbit. The main result says that in the strip between two orbits lying in different equivalence classes there exists a point such that its first prolongational limit set contains the intersection of boundaries of the two classes.

László Losonczi *Inequalities for the coefficients of some reciprocal polynomials*

We characterize reciprocal polynomials all of whose zeros are on the unit circle. Using this characterization theorem we obtain bounds for the coefficients of such reciprocal polynomials.

If all zeros of the complex reciprocal polynomial

$$p(z)=\sum_{k=0}^{m} A_k z^k \qquad (A_k\in\mathbb{C},\ A_0=1,\ A_k=A_{m-k} \text{ for } k=0,1,\dots,m)$$

of degree m $(m\in\mathbb{N})$ are on the unit circle then A_k are real and

$$|A_k|\le\binom{m}{k} \qquad (k=0,1,\dots,m).$$

Here equality holds for all reciprocal polynomials if $k=0,m$ and for the polynomials $p(z)=(z\pm1)^m$ equality holds for all $k=0,\dots,m$.

We also show how other necessary conditions can be obtained from the characterization theorem.

Grażyna Łydzińska *On iteration semigroups of set-valued functions*

We present some conditions under which a family of set-valued functions, naturally occuring in iteration theory, fulfils one of the following conditions

$$F(s+t,x)\subset F\big(t,F(s,x)\big), \tag{C}$$

$$F\big(t,F(s,x)\big)\subset F(s+t,x) \tag{E}$$

for every $x\in X,\ s,t\in(0,\infty)$ (where X is an arbitrary set). Moreover, we compare the above conditions and answer the question whether either (C) or (E) implies that F is an iteration semigroup:

$$F\big(t,F(s,x)\big)=F(s+t,x)$$

for every $x\in X,\ s,t\in(0,\infty)$.

Janusz Morawiec *On a property of continuous solutions of the dilation equation*

Assume N is a positive integer, $c_0,\dots,c_N$ are positive reals summing up to 2 and let $f\colon\mathbb{R}\longrightarrow\mathbb{R}$ be a continuous and compactly supported solution of the dilation equation

$$f(x)=\sum_{n=0}^{N} c_n f(2x-n).$$

We show that either $f=0$ or $f|_{(0,N)}>0$ or $f|_{(0,N)}<0$.

Jacek Mrowiec *A counterexample to the stability property for δ-Jensen-convex functions defined on a convex set $D \subset \mathbb{R}^n$*

We give an example of a δ-Jensen-convex function f defined on a convex subset of $\mathbb{R}^n$ such that there is no Jensen-convex function uniformly close to f.

A function f defined on a convex subset D of a linear space X is said to be *δ-Jensen-convex* if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \delta, \qquad x, y \in D$$

where $\delta \geq 0$ is a given constant. If $\delta = 0$, a function f is said to be Jensen-convex.

Anna Mureńko *On some conditional generalizations of the Gołąb–Schinzel equation*

Joint work with Janusz Brzdęk.

We show connections between solutions $f\colon (0,\infty) \longrightarrow \mathbb{R}$, $g\colon [0,\infty) \longrightarrow \mathbb{R}$ and $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ of the conditional equations

$$\begin{aligned} \text{if } x + f(x)y > 0, \ \text{ then } \ & f(x+f(x)y) = f(x)f(y), \\ \text{if } x + g(x)y \geq 0, \ \text{ then } \ & g(x+g(x)y) = g(x)g(y), \\ \text{if } x > 0,\ y > 0, \ \text{ then } \ & h(x+h(x)y) = h(x)h(y), \end{aligned}$$

respectively, and solutions $F\colon \mathbb{R} \longrightarrow \mathbb{R}$ of the Gołąb–Schinzel equation

$$F(x+F(x)y) = F(x)F(y).$$

In particular, we describe the solutions of the conditional equations that are continuous at a point, Lebesgue measurable or Baire measurable (i.e. have the Baire property).

Adam Najdecki *On a certain characterization of continuous functions*

Joint work with Jacek Tabor.

Let $f\colon X \longrightarrow Y$, where X is a Banach space and Y is a Hausdorff topological space. We show that if $f \circ \gamma$ is continuous for every curve $\gamma\colon [0,1] \longrightarrow X$ of class C^∞, then f is continuous.

Shin-ichi Nakagiri *Functional equations arising from the Cauchy-Riemann equations*

Joint work with Shigeru Haruki (Okayama University of Science).

We consider some functional equations arising from the Cauchy–Riemann equations, and certain related functional equations. First, we propose a new functional equation of the form

$$f(x+t,y) - f(x-t,y) = -i[f(x,y+t) - f(x,y-t)] \tag{1}$$

over a divisible Abelian group. This equation (1) is a discrete version of the Cauchy–Riemann equations, and we determine the general and regular solutions of (1). For the related functional equation of the form

$$f(x+t,y)-f(x,y)=-i[f(x,y+t)-f(x,y)] \tag{2}$$

it was shown in J. Aczél and S. Haruki [1], and S. Haruki [2] that (2) does not lead essentially beyond a linear function. However, for the functional equation of the form

$$f(x+t,y+t)-f(x-t,y-t)=-i[f(x-t,y+t)-f(x+t,y-t)] \tag{3}$$

such a result has not been obtained. We show that (3) is equivalent to (1), equations (1), (2) and (3) satisfy the Haruki functional equation, and that the general solutions of (3) are given by quadratic functions. Further we propose and solve partial differential-difference type and nonsymmetric type functional equations which are also arising from the Cauchy-Riemann equations.

[1] J. Aczél, S. Haruki, *Partial difference equations analogous to the Cauchy–Riemann equations*, Funkcial. Ekvac. **24** (1981), 95-102.

[2] S. Haruki, *Partial difference equations analogous to the Cauchy–Riemann equations II*, Funkcial. Ekvac. **29** (1986), 237-241.

Kazimierz Nikodem *Convexity triplets and t-convex functions*

Joint work with Zsolt Páles.

For a function $f\colon I \longrightarrow \mathbb{R}$ we denote by $C(f)$ the set of all triplets (x,y,z) at which the second order divided difference $f[x,y,z] \geq 0$. Some properties of the set $C(f)$ and their application to t-convex functions are presented. A characterization of t-convex functions in terms of a second order generalized derivative is also given.

Jolanta Olko *Metric on the space of multimeasures*

We define the metric on the space of set-valued measures, generated by the Fortet–Mourier norm on the space of signed measures. The properties of this metric space are studied.

Iwona Pawlikowska *A method used in solving functional equations stemming from MVTs*

Let X, Y be two linear spaces over a field $\mathbb{K} \subset \mathbb{R}$ and let K be a convex balanced set with $0 \in \operatorname{alg\,int} K$. Fix $N, M \in \mathbb{N} \cup \{0\}$ and a, $b \in \mathbb{K}$, $b \neq 0$. We denote by $I = \{(\alpha,\beta) \in \mathbb{K} \times \mathbb{K} : \ |\alpha| + |\beta| \leq 1\}$ and $I^+ = \{(\alpha,\beta) \in I : \ \beta \neq 0\}$. Assume that $I_0, \ldots, I_M$ are finite subsets of I^+. We prove the following lemma: *if functions* $\varphi_i\colon K \longrightarrow SA^i(X;Y)$, $i \in \{0,\ldots,N\}$ *and*

$\psi_{j,(\alpha,\beta)}\colon K \longrightarrow SA^j(X;Y)$, $(\alpha,\beta) \in I_j$, $j \in \{0,\dots,M\}$ *satisfy the equation*

$$\sum_{i=0}^{N} \varphi_i(x)((ax+by)^i) = \sum_{i=0}^{M} \sum_{(\alpha,\beta)\in I_i} \psi_{i,(\alpha,\beta)}(\alpha x+\beta y)((ax+by)^i)$$

for every $x, y \in K$ *then there exists a* $p \in \mathbb{N}$ *such that* φ_N *is a local polynomial function on* $\frac{1}{p}K$ *of order at most equal to*

$$\sum_{i=0}^{M} \operatorname{card}\left(\bigcup_{k=i}^{M} I_k\right) - 1.$$

This outcome in case of $N = 0$ is an extension of Corollary 2 from [1]. We also generalize some results of W.H. Wilson [4], L. Székelyhidi [3] and M. Sablik [2]. Then we use this lemma to solve functional equations derived from Mean Value Theorems.

[1] Z. Daróczy, Gy. Maksa, *Functional equations on convex sets*, Acta Math. Hungarica, **68**(3) (1995), 187-195.

[2] M. Sablik, *Taylor's theorem and functional equations*, Aequationes Math. **60** (2000), 258-267.

[3] L. Székelyhidi, *Convolution type functional equations on topological Abelian groups*, World Scientific Singapore – New Jersey – London – Hong Kong, 1991.

[4] W.H. Wilson, *On a Certain General Class of Functional Equations*, Amer. J. Math. **40** (1918), 263-282.

Zsolt Páles *On higher-order convexity and Wright-convexity*

Joint work with Attila Gilányi.

Motivated by the notions of convexity, t-convexity, Wright-convexity and t-Wright convexity, higher-order analogoues of these concepts are introduced and several results known for the classical situation are extended to the higher-order setting.

Themistocles M. Rassias *On new properties of isometric mappings*

This talk is concerned with results on new properties of isometric mappings in the spirit of the Mazur–Ulam theorem and the Aleksandrov problem of conservative distances.

Some old and new problems will be discussed.

[1] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, Basel, Berlin, 1998.

[2] Th.M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.

Maciej Sablik *Applications of a method for characterizing polynomials*

A lemma proved in [3] turns out to be a useful tool in dealing with equations characterizing polynomial functions. We illustrate its application by examples taken from [1] and [2].

[1] T. Riedel, M. Sablik, *A different version of Flett's Mean Value Theorem and an associated functional equation.* Acta Math. Sinica **20** (2004), 1073-1078.

[2] T. Riedel, M. Sablik, A. Sklar, *Polynomials and divided differences.* Publ. Math. Debrecen **66**/3-4 (2005), 313-326.

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Fulvia Skof *About a functional equation related to d'Alembert and other classical equations*

The known d'Alembert equation $f(x+y)+f(x-y)=2f(x)f(y)$ forces each of its solutions f to be an even function. An interesting generalized form of this equation, admitting also non even solutions, is the following one,

$$f(x+y)+f(x-y)=f(x)[f(y)+f(-y)], \tag{1}$$

which is the matter of the present study, where f is assumed to be a complex valued function from a linear space (or from a commutative group, or a suitable more general algebraic structure).

Equation (1) turns out to have a rather wide class of solutions, which contains the classes of functions satisfying either the d'Alembert equation, or the exponential Cauchy equation or the Jensen equation (with $f(0)=1$); when f is split into its even and odd components, further interesting trigonometric equations, widely studied by other authors, are involved too.

On such ground, the general table of the solutions of (1) can be easily drawn without any regularity assumption on f.

Then, some remarks are added about the similar generalization of the Wilson equation, namely

$$f(x+y)+f(x-y)=f(x)[g(y)+g(-y)],$$

although equation (1) seems to be a more interesting one.

The variety of solutions of (1), satisfying more than one equation, allows us to observe how it can happen that the solutions of a given functional equation on some "bounded restricted domains" may be locally very different from the restriction of the solution of the same equation given over the whole space. Simple examples support the above remark.

Wilhelmina Smajdor *Local analytic solutions of some functional equations*

Joint work with Andrzej Smajdor.

We determine all analytic solutions of the functional equations

$$\left|f\left(re^{i\theta}\right)\right|^2 + |f(1)|^2 = |f(r)|^2 + \left|f\left(e^{i\theta}\right)\right|^2,$$
$$\left|f\left(re^{i\theta}\right)\right| = |f(r)|,$$
$$\left|f\left(re^{i\theta}\right)\right| = \left|f\left(e^{i\theta}\right)\right|,$$

in the domains

$$\{z \in \mathbb{C} : \ 1 - \varepsilon < |z| < 1 + \varepsilon\}$$

and

$$\{re^{i\theta} \in \mathbb{C} : \ 1 - \varepsilon < r < r + \varepsilon, \ -\delta < \theta < \delta\},$$

where $0 < \varepsilon \le 1$ and $0 < \delta \le \pi$.

Paweł Solarz *On some iterative roots with a rational rotation number*

Let S^1 be the unit circle with the positive orientation, i.e., $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Suppose that $F\colon S^1 \longrightarrow S^1$ is a homeomorphism such that $F^n = \mathrm{id}_{S^1}$, where $n \in \mathbb{N}$, $n \ge 2$, is the minimal such a number. If F preserves orientation, then for every integer $m \ge 2$ there exist infinitely many orientation-preserving roots of F, i.e., solutions of the following functional equation

$$G^m(z) = F(z), \qquad z \in S^1. \tag{1}$$

However, if $\gcd(m, n) > 1$, each of these roots depends on an arbitrary function. Otherwise, there also exists a solution such that $G = F^l$ for some $l \in \mathbb{N}$.

Orientation-reversing solutions of (1) either, if F preserves orientation, do not exist or, if F reverses orientation, are equal to F.

Peter Volkmann *A characterization of quasimonotonicity*

It is known that quasimonotonicity of a continuous function can be characterized by means of differential inequalities. Using this, Karol Baron and me give a characterization by means of functional inequalities.

Eugeniusz Wachnicki *Sur la monotonie des suites en moyenne*

Travail commun avec Zbigniew Powązka.

F. Leja a introduit différentes notions de monotonie des suites en moyenne et il a démontré que toute suite monotone en moyenne au sens considéré par lui est convergente. Nous présentons les cas plus généraux que F. Leja en considérant la moyenne quasi-arithmétique et quasi-arithmétique pondérée.

Janusz Walorski *On homeomorphic solutions of the Schröder equation in Banach spaces*

Let X be a Banach space, $f\colon X \longrightarrow X$ be a homeomorphism and $A\colon X \longrightarrow X$ be a continuous linear operator.

We establish conditions, different from that of Grobman–Hartman, under which there exists a homeomorphism $\varphi : X \longrightarrow X$ which solves the Schröder equation

$$\varphi(f(x)) = A\varphi(x)$$

and such that $\varphi - \mathrm{id}_X$ is bounded.

Szymon Wąsowicz *Separation by functions belonging to Haar spaces*

Joint work with Mircea Balaj (Oradea, Romania).

In 1996 M. Balaj gave the necessary and sufficient condition for two functions f, g mapping a real interval I into $\mathbb{R}$ to be separated by a polynomial (cf. [2, Theorem 2]). We generalize this result dealing with functions belonging to Haar spaces instead of polynomials. Recall that if D is a set containing at least n elements, then a linear subspace $\mathcal{H}_n(D)$ of $\mathbb{R}^D$ is called an *n-dimensional Haar space on D*, if for any n distinct elements $x_1, x_2, \ldots, x_n$ of D and any $y_1, y_2, \ldots, y_n \in \mathbb{R}$ there exists the unique function $h \in \mathcal{H}_n(D)$ such that $h(x_j) = y_j$, $j = 1, \ldots, n$.

Let $\Phi_1, \ldots, \Phi_{n+1}$ be multifunctions defined on D, which values are compact real intervals. We give conditions under which at least one multifunction Φ_{i_0} admits a selection belonging to $\mathcal{H}_n(D)$. Next we consider $n+1$ pairs of functions $f_i, g_i\colon I \longrightarrow \mathbb{R}$, $i = 1, \ldots, n+1$, and we give the conditions under which at least one pair of functions f_{i_0}, g_{i_0} can be separated by a function belonging to $\mathcal{H}_n(I)$. As a consequence we obtain an extended version of Theorem 2 of [2] and some result on the stability of Hyers–Ulam type for polynomials.

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[2] Sz. Wąsowicz, *Polynomial selections and separation by polynomials*, Studia Math. **120** (1996), 75-82.

Marek Cezary Zdun *On a limit formula for regular iteration groups in Banach space*

Let X be a real Banach space, $U \subset X$ be an open set, $f\colon U \longrightarrow U$ be a diffeomorphism and $0 \in U$ be a unique globally attractive fixed point of f. Assume that there exists a linear bounded operator $A\colon X \longrightarrow X$ such that $f'(0) = \exp A$. We give some conditions which imply the existence of the following limit

$$f_t(x) := \lim_{n\to\infty} f^{-n}((\exp tA)f^n(x)), \qquad x \in U,\ t \geq 0$$

and the property that the family of mappings $\{f_t,\ t \geq 0\}$ yields a C^1 iteration semigroup of f.

An application for C^1 iterative roots of f is given. In particular we consider the case $X = \mathbb{R}^n$.

Marek Żołdak *Nonhomogeneous iterative equation*

Using the Banach fixed point theorem we investigate the existence and uniqueness of Lipschizian solutions of the nonhomogeneous iterative equation

$$\sum_{i=1}^{\infty} a_i f^i(x) = F(x) \qquad \text{for } x \in I,$$

where I is compact, convex subset in $\mathbb{R}^N$ $(N \in \mathbb{N})$ with nonempty interior, $F\colon I \longrightarrow I$ – is a given Lipschitz function, $f\colon I \longrightarrow I$ – is an unknown function, a_i for $i = 1, 2, \ldots$ – are real constants.

Problems and Remarks

1. Problem.
In describing the 1-periodic solutions of a certain Schröder equation, the classes

$$K_\alpha := 2^{\mathbb{Z}}\alpha + \mathbb{D}$$

($\alpha \in \mathbb{R}$, $\mathbb{D}$ – the set of dyadic rationals) play an important role.

It is an open problem to find an explicit description of the set

$$S := \{\alpha \in \mathbb{R} : \ K_\alpha = K_{-\alpha}\}.$$

It is known that $\mathbb{D} \subset S$, $(\mathbb{R} \setminus \mathbb{Q}) \cap S = \emptyset$, $S \neq \mathbb{Q}$.

Hans-Heinrich Kairies

2. Problem.
Let $(G, +)$ be an abelian group. The operation $\circ : G^2 \to G$ is called *translative* (*quasicommutative*) if

$$(x + z) \circ (y + z) = x \circ y + z \qquad (\ x \circ (y \circ z) = y \circ (x \circ z)\ )$$

for all $x, y, z \in G$. Denote the set of all translative and quasicommutative operations $\circ\colon G^2 \longrightarrow G$ by $T(G)$.

Determine $T(\mathbb{Z})$ and $T(\mathbb{Q})$.

Zoltán Daróczy

3. Remark. (To *the problem of Z. Daróczy*)
Let us note that among the elements of $T(\mathbb{Z})$, apart from those being restrictions of translative and quasicommutative operations $\circ$ defined in $\mathbb{R} \times \mathbb{R}$ we also have

$$x \circ y = \begin{cases} y & \text{if } y - x \in 2\mathbb{Z}, \\ y + 2 & \text{if } y - x \in 2\mathbb{Z} + 1. \end{cases}$$

Maciej Sablik

4. Problem.
Define the Takagi function $T\colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$T(x) := \sum_{k=0}^{\infty} \frac{\operatorname{dist}(2^k, \mathbb{Z})}{2^k}.$$

Prove (or disprove) that it satisfies the following approximate Jensen-convexity inequality

$$T\left(\frac{x+y}{2}\right) \leq \frac{T(x) + T(y)}{2} + \frac{1}{2}|x - y| \qquad (x, y \in \mathbb{R}).$$

Zsolt Páles

5. Problems and Remark. *Functional-integral equations stemming from Steffensen's inequality* (presented by B. Choczewski).
The Steffensen inequality reads:
If $f\colon [a,b] \longrightarrow [0,+\infty)$ *is a decreasing function and* $g\colon [a,b] \longrightarrow [0,1]$*, and both are integrable, then*

$$\int_{b-c}^{b} f(t)\,dt \leq \int_{a}^{b} f(t)g(t)\,dt \leq \int_{a}^{a+c} f(t)\,dt, \qquad c = \int_{a}^{b} g(t)\,dt. \tag{S}$$

The question when the medial term in (S) is the arithmetic mean of side terms leads to the formula:

$$\int_{a}^{a+c} f(t)\,dt + \int_{b-c}^{b} f(t)\,dt = 2\int_{a}^{b} f(t)g(t)\,dt. \tag{E}$$

Study different functional equations which may stem from (E): the numbers a, b, c may be treated as constants, or variables, or some of them may be functions of some others.

The author studied the functional equation of type (E):

$$\int_{x}^{H(xy+x+y)} f(t)\,dt + \int_{H(xy-x-y)}^{y} f(t)\,dt = 2\int_{x}^{y} f(t)g(t)\,dt \tag{C}$$

with three unknown functions f, g and H (selfmappings of reals). Assuming that they satisfy (E) and are analytic the author derived from (C) the following relations:

$$\begin{cases} H'(x)f(H(x)) = \dfrac{\alpha x+\beta}{x+1}; & \alpha,\beta\in\mathbb{R},\ x\neq -1 \\ f(x)(1-2g(x)) = \dfrac{2(\beta-\alpha x^2)}{1-x^2}, & |x|\neq 1. \end{cases}$$

Ilie Corovei (Cluj-Napoca, Romania)

6. Problem.

A function $f\colon I \longrightarrow \mathbb{R}$ is called $\left(\frac{1}{3},\frac{2}{3}\right)$*-convex* if it satisfies

$$f\left(\frac{x+2y}{3}\right) \leq \frac{f(x)+2f(y)}{3}$$

for all $x, y \in I$ with $x < y$.

Clearly, the $\frac{1}{3}$-convexity implies $\left(\frac{1}{3},\frac{2}{3}\right)$-convexity.

Question 1. Does $\left(\frac{1}{3},\frac{2}{3}\right)$-convexity imply Jensen convexity?

Question 2. Is $\left(\frac{1}{3},\frac{2}{3}\right)$-convexity a localizable property?

Question 3. Can $\left(\frac{1}{3},\frac{2}{3}\right)$-convexity be characterized by the nonnegativity of the 2nd order derivative

$$\underline{\delta}^2_{\left(\frac{1}{3},\frac{2}{3}\right)} f(p) = \liminf_{\substack{(x,y)\to(p,p)\\ x\leq p\leq y}} 2\cdot \left[x, \frac{x+2y}{3}, y; f\right]$$

where $[x, y, z; f]$ is the 2nd order divided difference of f.

Zsolt Páles

7. Remark.

There are homeomorphisms $f\colon (0,\infty) \longrightarrow (0,\infty)$ with $f(x) < x$, $x > 0$, for which $d := \lim_{x\to 0^+} \frac{f(x)}{x} \in (0,1)$ and such that

(a) $\frac{f(x)}{x} - d = O\left(|\log x|^{1+\nu}\right)$, as $x \to 0$, where $\nu > 0$;

(b) $\frac{f(x)}{x} - d$ is of bounded variation in $x \in (0,\infty)$;

(c) $\varphi_f(x|1) := \lim_{n\to\infty} \frac{f^n(x)}{f^n(1)}$ is a non-bijective continuous function.

In particular there are examples of f satisfying the above conditions without any regular iterative root.

For given $d \in (0,1)$, $(a_n)_{n\in\mathbb{Z}} \subset (0,1)^{\mathbb{Z}}$, such a function can be built as follows:

$$f(x) = \begin{cases} d^{n+1} & \text{for } x = d^n,\ n\in\mathbb{N}, \\ d^{n+1}(1-(1-d)a_{n+1}) & \text{for } x = d^n(1-(1-d)a_n) \end{cases}$$

and then as a function piecewise linear between the indicated points. The only additional requirement is that $\lim_{n\to\infty} a_n = 0$.

The details are in preparation for publication elsewhere.

Joachim Domsta

8. Remark and Problem.

S.-M. Jung [7] investigated the Hyers–Ulam stability of the orthogonality equation for a class of mappings defined on a closed ball in $\mathbb{R}^3$:

Theorem 1 ([7])
Let $D \subset \mathbb{R}^3$ be a closed ball of radius $d > 0$ and with center at the origin. If a mapping $T\colon D \longrightarrow D$ satisfies the conditions:

$$T(0) = 0 \qquad \text{and} \qquad |\langle Tx, Ty\rangle - \langle x, y\rangle| \le \varepsilon \tag{1}$$

for some ε such that $0 \le \varepsilon < \min\left\{\frac{1}{4}, \frac{d^2}{17}\right\}$ and for all $x, y \in D$, then there exists an isometry $I\colon D \longrightarrow D$ that satisfies, for any $x \in D$,

$$\|Tx - Ix\| \le \begin{cases} 16\sqrt{\varepsilon} & \text{for } d < \frac{\sqrt{17}}{2}, \\ (6d+3)\sqrt{\varepsilon} & \text{for } d \ge \frac{\sqrt{17}}{2}. \end{cases}$$

Jung's estimate was improved in [8] for the upper bound of the norm of the difference $Tx - Ix$ by proving the following:

Theorem 2 ([8])
Under assumptions of Theorem 1,

$$\|Tx - Ix\| \le \begin{cases} 13\sqrt{\varepsilon} & \text{for } d < \frac{\sqrt{17}}{2}, \\ (4.5d+3.5)\sqrt{\varepsilon} & \text{for } d \ge \frac{\sqrt{17}}{2} \end{cases} \tag{2}$$

for any $x \in D$.

In 1994, J. Chmieliński [1] was the first to prove the Hyers–Ulam stability of the orthogonality functional equation.

Theorem 3 ([1])
Let E be a real Hilbert space with dimension greater than 1. If a mapping $T\colon E \longrightarrow E$ satisfies the property

$$|\langle Tx, Ty\rangle - \langle x, y\rangle| \le \varepsilon, \qquad \text{for all } x, y \in E,$$

then there exists a unique isometry $I\colon E \longrightarrow E$ such that

$$\|Tx - Ix\| \le \sqrt{\varepsilon}, \qquad \text{for all } x \in E.$$

This estimate is the best possible. Chmieliński [2] proved the superstability of the orthogonality functional equation in the case $E = \mathbb{R}^n$ for $n \geq 2$. A similar result was also obtained by Jung [6]. In 1997, Chmieliński [3] proved the Hyers–Ulam stability of the orthogonality equation for complex inner product spaces. Furthermore, Chmieliński and Jung [4] investigated the Hyers–Ulam–Rassias stability of the orthogonality equation for mappings defined on restricted domains of Hilbert spaces.

The interested reader is referred to the book [5] for an extensive account on stability results for functional equations.

Problem: Determine the best coefficients of $\sqrt{\varepsilon}$ in (2).

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Report of Meeting

10th International Conference on Functional Equations and Inequalities, Będlewo, September 11-17, 2005

The Tenth International Conference on Functional Equations and Inequalities was held in Będlewo, from September 11 to September 17, 2005 at the Mathematical Research and Conference Center. It was organized by the Institute of Mathematics of the Pedagogical University of Cracow in cooperation with the Stefan Banach International Mathematical Center and with a financial support of the Mathematical Institute of the Polish Academy of Sciences and of the BPH Bank.

The Organizing Committee consisted of Professor Janusz Brzdęk (Chairman), Dr. Paweł Solarz, Miss Janina Wiercioch and Mr Władysław Wilk.

The Scientific Committee consisted of Professor Dobiesław Brydak (Chairman), Dr. Jacek Chmieliński (Scientific Secretary) and Professors Bogdan Choczewski, Roman Ger and Marek Cezary Zdun.

The 64 participants came from 10 countries: Austria (1, Innsbruck), France (1, Nantes), Germany (3, Clausthal-Zellerfeld, Karlsruhe, Munich), Hungary (1, Debrecen), Israel (2, Haifa, Tel-Aviv), People's Republic of China (1, Sichuan), Poland (50, Bielsko-Biała, Bydgoszcz, Gdańsk, Gliwice, Katowice, Kielce, Kraków, Rzeszów, Zielona Góra), Romania (2, Cluj-Napoca), Russia (2, Dolgoprudny, Nizhni Novgorod), USA (1, Louisville, KY). It was observed with satisfaction that a half of all participants were at most Ph.D. graduated Polish mathematicians.

Professor Brzdęk welcomed the participants in the name of the Organizing Committee. Opening address was given by Profesor Zdun, the Dean of the Faculty of Science of the Pedagogical University of Cracow. He spoke also on behalf of Professor Eugeniusz Wachnicki, Deputy Rector of the University. Then Professor Brydak officially opened the 10th ICFEI.

During 20 sessions 59 talk were delivered. The scientific program was preceded by a historical talk given by Professor Choczewski who presented statistical data and reminded some characteristic events from the previous nine meetings. He pointed out that there are 6 participants of the first meeting held at Sielpia in 1984 who were present at all the eight further ones. From among 15 colleagues who attends both the First and the Tenth ICFEI. Professor Peter Volkmann (Karlsruhe) is the only foreign guest. A slightly extended version of the talk is published in this volume under the title *International meetings organized by Polish schools of functional equations.*

The other talks at the conference focused on the following topics: *stability theory* (of d'Alembert's, Dhombres', Jensen's, translation and recurrence equations and of those of microperiodic functions, multiplicative symmetry, isometries, orthogonal-conditional and on metric groups, stemming from mean value Flett's type theorems, applications of generalized strong derivatives), *functional equations in several variables* (Cauchy, Dhombres, Gołąb–Schinzel, translation, addition formula, representing λ-affinity, orthogonality preserving property, relations among means including recurrences, characterizing the absolute value of n-additive functions), *functional equations in a single variable* (Dhombres, Schröder's in normed spaces, conjugacy, with iteration of unknown function (also on the unit circle), integral-functional; spectra of operators generated by de Rham equation), *convexity* (on Abelian groups, connections with Orlicz spaces, generalized Beckenbach – three parameters families of functions), *multifunctions* (iteration semigroups including concave ones, selections of multimeasures, $*$-convex, Drygas' equation, cosine families), *theory of iteration* (triangle mappings, near-iterability, roots of homeomorphisms of the unit circle), *inequalities* (systems (of iterative type), approximate integration), *varia and applications* (d'Alembert's and Schrödinger's partial differential equations, waveform relaxation method in difference, differential and differential-delay equations, Riemann–Hilbert problem and a related functional equation – composite materials, equations appearing in actuarial mathematics).

On Tuesday, September 13, there was a picnic in the park surrounding the Center, and the next afternoon was devoted to an excursion to Poznań including a visit to the Cathedral and to the Museum of Old Instruments. Also from Wednesday on we might enjoy each evening piano recitals performed by Professor Hans-Heinrich Kairies.

On Thursday, September 15, before the festive banquet, Professor Ger made a fine and hearty speech on the occasion of the jubilee of 10th ICFEI and of the 70th birthdays of Professors Dobiesław Brydak and Bogdan Choczewski, the organizers of the previous meetings. He also presented two special addresses of congratulations and wishes, signed by the 62 participants of the 10th ICFEI, in which one reads, among others: *The Conference has always been important for Polish mathematicians working in the field of functional equations. The international significance of this is shown by the enclosed list of the countries from which the participants of all the 10 Conferences have originated.* The addresses were accompanied by nice bouquets of flowers and by albums showing most precious monuments and miracles of nature, registered on the UNESCO lists of world heritage.

At the end of the last scientific session on Saturday, September 17, its chairman Professor Marian Kwapisz from the Casimir the Great University of Bydgoszcz, praised the members of the Organizing Committee for their efforts resulting in a very successful meeting.

Words of thanks said then by Professor Brzdęk to all the participants were followed by a closing address by Professor Choczewski. He first announced that Professor Brydak was elected the Honorary Chairman of the Scientific Committee of the next ICFEI. The announcement was warmly applauded by the audience. Next he continued with thanking:

— all the colleagues, and especially foreign guests, for their coming and acting nicely, friendly and effectively, so that the Conference would be worth to be kept in everyone's memory for a longer time,

— Professor Brzdęk and the whole Organizing Committee, which acted in a smooth and efficient way, being always at a careful disposal and helping many of participants in various ways,

— Professor Ger and Dr. Chmieliński who composed the program of the meeting and took care on its smooth realization,

— Professor Ger once again, who suggested the Center as the place for the 10th ICFEI and actively supported this idea by means of facilitating contacts with the managing institutions and persons. (Professor Ger was the member of a close council of Polish mathematicians especially chosen more than 10 years ago to create conceptually and organizationally the vision of the Center.)

The speaker expressed, also in the name of his friend Dobiesław (known to him since 53 years), very cordial personal thanks both to the organizers and to the participants of this surprising, touching and unforgettable anniversary celebration on Thursday evening. He also revealed that the 70th birthday of Professor Kwapisz had been honoured in June at the University of Gdańsk, and congratulated him on behalf of all the participants of the meeting.

Our Conference has been included into the programme of the Mathematical Research and Conference Center at Będlewo by the Banach Center Scientific Council. Professor Choczewski expressed the gratitude of the organizers to the Committee and also to Professor Bogdan Bojarski, the Honorary Director of the MRCC and Professor Łukasz Stettner, its Managing Director, for their decisions

Thanks were extended to Mrs. Anna Kreczmar-Puacz, the Manager of the Center, and for its staff as a whole, for a high standard of board and lodging, fully equipped with audiovisual facilities conference rooms and for the quality of service, which were highly appreciated by the participants.

The duties of the Chairman, both of the Scientific and Organizing Committees of the ICFEI would be now taken on by Professor Janusz Brzdęk. Professors Nicole Brillouet-Belluot (France), Hans-Heinrich Kairies (Germany) and Lászlo Losonczi (Hungary) kindly accepted the proposal to join the Scientific Committee as representatives of foreign participants.

Since the 45th International Symposium on Functional Equations is planned to be organized in 2007 in Bielsko-Biała, it has been decided that the 11th ICFEI will be held again at Będlewo, from September 18 to September 23, 2006.

Abstracts of the talks follow in alphabetical order of the authors. Contributions to several sessions devoted to problems and remarks (in chronological order of presentation) and the list of participants (with addresses) complete the report. All the scientific materials were collected and compiled by Dr. Chmieliński. The help, also in this respect, of Dr. Solarz, Miss Wiercioch and Mr Wilk is acknowledged with many thanks.

Bogdan Choczewski

Abstracts of Talks

Mirosław Adamek *An example connected with λ-affinity*

Let $\lambda\colon I^2 \longrightarrow (0,1)$ be a fixed function (I is a nonempty and open interval of $\mathbb{R}$). A function $f\colon I \longrightarrow \mathbb{R}$ is called *λ-affine* if

$$f(\lambda(x,y)x + (1-\lambda(x,y))y) = \lambda(x,y)f(x) + (1-\lambda(x,y))f(y), \qquad x,y \in I;$$

and f is *affine* if

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y), \qquad t \in [0,1]\,,\ x,y \in I.$$

In the talk we will show an example of a function λ for which each λ-affine function f is affine.

Roman Badora *On the stability of some functional equations*

Let G be a group and let $K = \{k_0 = \mathrm{Id}_G, k_1, \ldots, k_{N-1}\}$ be a subgroup of the automorphism group $\mathrm{Aut}(G)$ of G (the action of $k \in K$ on $x \in G$ is denoted by kx). We study the stability of the following functional equations

$$\frac{1}{N}\sum_{i=0}^{N-1} f(x + k_i y) = f(x), \qquad x,y \in G;$$

$$\frac{1}{N}\sum_{i=0}^{N-1} f(x + k_i y) = f(x)g(y), \qquad x,y \in G;$$

$$\frac{1}{N}\sum_{i=0}^{N-1} f(x + k_i y) = f(x) + g(y), \qquad x,y \in G$$

($f,g\colon G \longrightarrow \mathbb{C}$), which cover Jensen's functional equation, Cauchy's functional equation, d'Alembert's functional equation and the functional equation of the square of the norm.

Anna Bahyrycz *Conditional equation of exponential function*

We consider the conditional equation of exponential function:

$$f(x) \cdot f(y) \neq 0_m \implies f(x+y) = f(x) \cdot f(y),$$

where $n, m \in \mathbb{N}$,

$$f\colon \mathbb{R}(n) := [0, +\infty)^n \setminus \{0_n\} \longrightarrow \mathbb{R}(m),$$

$$x + y := (x_1 + y_1, \ldots, x_k + y_k) \quad \text{and} \quad x \cdot y := (x_1 \cdot y_1, \ldots, x_k \cdot y_k)$$

for $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k) \in \mathbb{R}(k)$.

We investigate systems of cones over $\mathbb{Q}$, which are one of the parameters determining the solutions of this equation.

Karol Baron *On a problem of József Bukszár*

Joint work with Witold Jarczyk.

Referring to [1, Problem 7 on p. 194] we show that any Lebesgue measurable function $f\colon \mathbb{R} \longrightarrow [0, \infty)$ satisfying

$$f(x) = \int_0^\infty f(x+y) f(y)\, dy \qquad \text{for } x \in \mathbb{R}$$

has the form

$$f(x) = 2\lambda e^{-\lambda x} \qquad (x \in \mathbb{R})$$

with a $\lambda \in [0, \infty)$.

[1] *Report of Meeting, 7th International Conference on Functional Equations and Inequalities, Złockie, September 12–18, 1999*, Ann. Acad. Paed. Cracoviensis Studia Math. **1** (2001), 163-201.

Lech Bartłomiejczyk *Solution of a problem of J. Smítal*

Let $J \subset (0, 1)$ be an interval and $h\colon J \longrightarrow J$ be a strictly increasing function. The following solves the problem of Jaroslav Smítal concerning the existence of a very irregular solution $\varphi\colon (0, +\infty) \longrightarrow J$ of the equation

$$\varphi(x\varphi(x)) = h(\varphi(x)) \tag{1}$$

posed during the 43rd ISFE (Batz-sur-Mer, May 15-21, 2005).

Assume $\mathcal{R}$ is a family of subsets of $(0, +\infty) \times J$ such that

$$\operatorname{card} \mathcal{R} \leq \mathfrak{c}$$

and for every $R \in \mathcal{R}$ there is a $y \in J$ with

$$\operatorname{card}\{x \in (0, +\infty) : \ (x, y) \in R\} = \mathfrak{c}.$$

Then there exists a solution $\varphi\colon (0,+\infty) \longrightarrow J$ of (1) such that its graph meets every set of $\mathcal{R}$.

The above conditions are fulfilled by, among others, the family of all the sets of the form $B \times \{y\}$ where $B \subset (0,+\infty)$ is Borel and uncountable and $y \in J$. If a subset G of $(0,+\infty) \times J$ meets every set of this specific family, then its complement $(0,+\infty) \times J \setminus G$ contains no set of the second category having the property of Baire and contains no set of positive Lebesgue measure.

To get the above presented result we use [1].

[1] L. Bartłomiejczyk, *Solutions with big graph of the equation of invariant curves*, Bull. Polish Acad. Sci. Math. **49** (2001), 309-317.

Bogdan Batko *On approximate solutions of Dhombres' functional equation*

Let $f\colon S \longrightarrow X$ map an abelian semigroup $(S,+)$ into a Banach space $(X,\|\ \|)$. We are going to deal with the stability of Dhombres' functional equation

$$f(x)+f(y) \neq 0 \Longrightarrow f(x+y) = f(x)+f(y) \qquad \text{for } x,y \in S. \tag{1}$$

We assume that f is an *approximate solution* of equation (1) with control functions $\Phi_1, \Phi_2\colon S \times S \longrightarrow \mathbb{R}^+$, i.e.,

$$||f(x)+f(y)|| > \Phi_1(x,y) \Longrightarrow ||f(x+y)-f(x)-f(y)|| \leq \Phi_2(x,y) \quad \text{for } x,y \in S$$

and ask for the existence of a solution $a\colon S \longrightarrow X$ of (1) with

$$||f(x)-a(x)|| \leq \Psi(x) \qquad \text{for } x \in S,$$

where $\Psi\colon S \longrightarrow \mathbb{R}^+$ is a function we can explicitly compute starting from Φ_1 and Φ_2.

Zoltán Boros *Generalized strong derivatives*

Let I denote an open interval in the real line, and let us consider a function $f\colon I \longrightarrow \mathbb{R}$. For $x \in I$ and $h \in \mathbb{R}$, we define the lower and upper strong dyadic derivatives of f by

$$\underline{D}_h^\beta f(x) = \liminf_{\substack{y \to x \\ n \to \infty}} 2^n(f(y+2^{-n}h)-f(y))$$

and

$$\overline{D}_h^\beta f(x) = \limsup_{\substack{y \to x \\ n \to \infty}} 2^n(f(y+2^{-n}h)-f(y)),$$

respectively. We call f *strongly dyadically differentiable* if

$$\underline{D}_h^\beta f(x) = \overline{D}_h^\beta f(x) \in \mathbb{R}$$

holds for every $x \in I$ and $h \in \mathbb{R}$. We say that f has *increasing strong dyadic derivatives* if

$$-\infty < \overline{D}_h^{\beta} f(x) \leq \underline{D}_h^{\beta} f(y) < +\infty$$

holds for every $h > 0$ and $x, y \in I$ such that $x < y$. These properties are characterized by the following decomposition theorems:

Theorem 1
The function f is strongly dyadically differentiable if, and only if, there exist a continuously differentiable function $g\colon I \longrightarrow \mathbb{R}$ and an additive mapping $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = g(x) + \varphi(x)$ for every $x \in I$.

Theorem 2
The function f has increasing strong dyadic derivatives if, and only if, there exist a convex function $g\colon I \longrightarrow \mathbb{R}$ and an additive mapping $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = g(x) + \varphi(x)$ for every $x \in I$.

Applying these results, we characterize affine (respectively, Wright-convex) functions as approximately affine (respectively, approximately Wright-convex) functions in a specific sense. In what follows, let us consider a fixed real number $p > 1$.

Example 1
Suppose that, for every $x \in I$, f satisfies an inequality of the form

$$|f(y+u) - f(y) - \phi_x(u)| \leq \varepsilon(x)|u|^p \tag{1}$$

for every y taken from a neighbourhood of x and for every u taken from a neighbourhood of 0. It is derived from the inequality (1) that f is strongly dyadically differentiable. Applying our decomposition theorem, we obtain that $f = g + \varphi$, where g is continuously differentiable and φ is the restriction of an additive mapping to the interval I. Substitution into the inequality (1) yields that g' is constant and thus f is affine.

Example 2
Let $\varepsilon > 0$ and suppose that f satisfies the inequality

$$f\left(\lambda x + (1-\lambda)y\right) + f\left((1-\lambda)x + \lambda y\right) \leq f(x) + f(y) + \varepsilon\left(\lambda(1-\lambda)|x-y|\right)^p \tag{2}$$

for every $\lambda \in [0,1]$ and $x, y \in I$. It is derived from the inequality (2) that f has increasing strong dyadic derivatives, and thus $f = g + \varphi$, where g is convex and φ is the restriction of an additive mapping to the interval I. This yields that f satisfies the inequality (2) with $\varepsilon = 0$ as well.

Nicole Brillouët-Belluot *On a class of iterative-difference equations*

Joint work with Weinian Zhang (Sichuan University, P.R. China).

During the Thirty-eighth International Symposium on Functional Equations in Noszvaj, I posed the problem (Aequationes Math. **61** (2001), 304) of finding the continuous solutions $f: \mathbb{R} \longrightarrow \mathbb{R}$ of the functional equation

$$x + f(y + f(x)) = y + f(x + f(y)).$$

By letting $y = 0$, we see that this problem is related to the problem of finding the continuous solutions $f: \mathbb{R} \longrightarrow \mathbb{R}$ of the iterative-difference equation

$$f(f(x)) = f(x + a) - x$$

where a is a real number.

In this work, we consider the more general second order equation

$$f(f(x)) = \lambda_1 \ f(x + a) \ + \ \lambda_0 \ x, \qquad x \in J \tag{1}$$

where J is an interval of $\mathbb{R}$, λ_0, λ_1, a are real numbers with $a \neq 0$, $\lambda_1 \neq 0$.

We use three methods to approach equation (1). We find affine solutions of (1), we prove the existence of bounded continuous solutions of (1) on a compact interval and we construct piecewise continuous solutions of (1) on a finite interval.

Janusz Brzdęk *On approximately microperiodic mappings*

A function f mapping a group $(G, \cdot)$, endowed with a topology, into a nonempty set is said to be microperiodic provided the set

$$P_f := \{a \in G : \ f(a \cdot x) = f(x) \text{ for } x \in G\}$$

is dense in G. It is known that, under suitable assumptions, microperiodic functions that are continuous at a point must be constant and measurable microperiodic functions must be constant almost everywhere.

We generalize these statements in some directions. The main result is following.

Theorem

Let $(G, \cdot)$ be a semitopological group such that the mapping $G \ni x \rightarrow x^{-1} \in G$ is continuous, $\varepsilon \in [0, \infty)$, P be a dense subset of G, and $E \subset G$. Suppose that $g, h: E \longrightarrow \mathbb{R}$ satisfy

$$g(p \cdot x) - g(x) \leq \varepsilon + h(p) \qquad \textit{for } x \in E, \ p \in P \textit{ with } p \cdot x \in E,$$
$$h(p) + h(x) \leq h(p \cdot x) \qquad \textit{for } x \in E, \ p \in P \textit{ with } p \cdot x \in E.$$

Then the following three assertions are valid.

(i) *If G is a locally compact topological group and g, h are Haar measurable on a set $D \in 2^E \setminus \mathcal{H}_0$, then there is $c \in \mathbb{R}$ such that $|g(x) - h(x) - c| \le \varepsilon$ $\mathcal{H}_0$-a.e. in E ($\mathcal{H}_0$ denotes the σ-ideal of subsets of G that are locally of Haar measure zero).*

(ii) *If g, h are Baire measurable on a set $D \in 2^E \setminus \mathcal{B}_0$, then there is $c \in \mathbb{R}$ such that $|g(x) - h(x) - c| \le \varepsilon$ $\mathcal{B}_0$-a.e. in E ($\mathcal{B}_0$ denotes the σ-ideal of subsets of G that are of first category of Baire).*

(iii) *If g, h are continuous at a point $x_0 \in \operatorname{int} E$, then there is $c \in \mathbb{R}$ such that $|g(x) - h(x) - c| \le \varepsilon$ for every $x \in E$.*

Remark
Taking $E = D = G$, $\varepsilon = 0$ and $h \equiv 0$ in the Theorem we obtain the classical results on microperiodic functions.

Jacek Chmieliński *Approximate functional relations stemming from orthogonality preserving property*

Let X and Y be two inner product spaces and $f\colon X \longrightarrow Y$. Then, the following two properties connected with orthogonality relation and its preservation can be considered.

Orthogonality preserving property:

$$\forall\, x, y \in X: \quad x \perp y \implies f(x) \perp f(y).$$

Right-angle preserving property:

$$\forall\, x, y, z \in X: \quad x - z \perp y - z \implies f(x) - f(z) \perp f(y) - f(z).$$

We deal with mappings satisfying the above properties *approximately*. In particular, some kind of stability of the considered properties is established.

Bogdan Choczewski *Nine International Conferences on Functional Equations and Inequalities*

The aim of this talk is to remind some facts from the history of our previous meetings (organization, participants, topics, events), starting from the first one which was held at Sielpia in 1984.

Jacek Chudziak *Continuous solutions of a composite addition formula*

Let X be a real vector space and J be a nontrivial real interval. We deal with the functional equation

$$g(x + M(g(x))y) = H(g(x), g(y)) \qquad \text{for } x, y \in X, \tag{1}$$

where $g\colon X \longrightarrow J$, $M\colon J \longrightarrow \mathbb{R}$ and $H\colon J^2 \longrightarrow J$ are unknown functions. The equation (1) is a generalization of equations of the form

$$g(x+y) = H(g(x), g(y)) \qquad \text{for } x, y \in X,$$

known as addition formulae. It is also a generalization of the Gołąb–Schinzel type functional equations

$$g(x+g(x)^k y) = tg(x)g(y) \qquad \text{for } x, y \in X,$$

where k is a positive integer and t is a real number.

We determine all solutions of (1) under the assumptions that $g\colon X \longrightarrow J$ is continuous on rays, $M\colon J \longrightarrow \mathbb{R}$ is continuous and $H\colon J^2 \longrightarrow J$ is associative.

Stefan Czerwik *A general Baker superstability criterium for the d'Alembert functional equation*

Joint work with Maciej Przybyla.

Let G be an abelian group. We define

$$U_1 := \{g\colon G \longrightarrow F\}, \quad U_2 := \{F\colon G^2 \longrightarrow F\},$$

$$g_a(x) := g(x+a), \qquad a, x \in G,$$

$$A(f)(x,y) := f(x+y) + f(x-y) - 2f(x)f(y), \qquad x, y \in G.$$

Lemma
Let G be an abelian group and F be a field. Let $f\colon G \longrightarrow F$ be a function. Then for all $x, u, v \in G$ we have

$$\begin{aligned} 2f(x)A(f)(u,v) = {} & A(f)(x+u,v) - A(f)(x,u+v) - A(f)(x,u-v) \\ & + A(f)(x-u,v) + 2f(v)A(f)(x,u). \end{aligned}$$

Theorem 1
Let $f\colon G \longrightarrow F$ be a function. Let U_1 be a linear space over F such that if $g \in U_1$, then for every $a \in G$, $g_a \in U_1$. If, moreover, for every $(u,v) \in G^2$ the function $f(\cdot)A(f)(u,v) \in U_1$, then

$$f \in U_1 \quad \textit{or} \quad A(f) = 0.$$

Theorem 2
Let $f\colon G \longrightarrow F$ be a function. Let U_1 be a linear space over F with the "translation property". If, moreover, $A(f)(\cdot,u) \in U_1$ for every fixed $u \in G$, then

$$f \in U_1 \quad \textit{or} \quad A(f) = 0.$$

Remark

The famous Baker result on superstability of the D'Alembert functional equation we get for

$$U_1 = B(G, C), \quad U_2 = B(G^2, C)$$

(the A-conjugate spaces of bounded functions on G and G^2 respectively). One may consider also the spaces X_q and X_q^2. For details see [1].

[1] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, New Jersey, London, 2002.

Joachim Domsta *On the regular and smooth conjugacy*

A self mapping f of $\mathbb{R}_+ := (0, \infty)$ is said to be *differentiable at the origin*, if $\lim_{x\to 0} \frac{f(x)}{x}$ exists in $\mathbb{R}_+$. Two differentiable at zero self-homeomorphisms f and g of $\mathbb{R}_+$ are said to be *regularly* [*differentiably*] *conjugate* if there exists an increasing self–homeomorphism Ψ on $\mathbb{R}_+$ regularly varying [differentiable] at the origin and such that $g(\Psi(x)) = \Psi(f(x))$, for $x \in \mathbb{R}_+$. Obviously, the differentiability implies regular variability. This has allowed to express the differentiable conjugacy through the regular conjugacy with the use of Szekeres principal function. From the main result a partial solution of problems related to the smooth conjugacy of diffeomorphisms is derived.

Lyudmila Efremova *Simplest skew products of interval maps with one-dimensional attractor*

Let us consider a simplest skew product of interval maps

$$F(x,\, y) = (f(x),\, g_x(y)) \qquad \text{for all } (x,\, y) \in I$$

($I = I_1 \times I_2$, where I_1, I_2 are the closed intervals). Then for any $m \geq 1$ we have

$$F^m(x, y) = (f^m(x), g_{x,m}(y)), \qquad \text{where } g_{x,m}(y) = g_{f^{m-1}(x)} \circ \ldots \circ g_{f(x)} \circ g_x(y).$$

The term "simplest" means that

(i) f has a sink x^0 with the period $n \geq 1$;

(ii) $g_{x^0,\, n}(y) = y$ for all $y \in I_2$;

(iii) the right-side basin of the immediate attraction of x^0 contains a subtrajectory $x_0 > x_n > \ldots > x_{nm} > \ldots > x^0$ of some point x_0 such that for all $y \in I_2$ and $m \geq 0$ the following holds

(iii.1) $g_{x_{n(2m+1)},\, n}(y) = y$;

(iii.2) $g_{x,\,n}(y) \le y$ for all $x \in (x_n,\, x_0] \cup \bigcup_{m \,\in\, 2\mathbb{N}+1} (x_{n(2m+3)},\, x_{n(2m+1)})$, and the equality is valid only for $y = 0$;
$g_{x,\,n}(y) \ge y$ for all $x \in \bigcup_{m \,\in\, 2\mathbb{N}} (x_{n(2m+3)},\, x_{n(2m+1)})$, and the equality is valid only for $y = 1$, where $y \in I_2$, $m \ge 0$.

1. Let us assign to a nonperiodic point $\overline{x}$ of the factor-map f and with a subtrajectory $\{f^{n_i}(\overline{x})\}_{i\ge 0}$ the set-valued function $\theta_{\overline{x},\,N_*}\colon \{f^{n_i}(\overline{x})\}_{i\ge 0} \longrightarrow (2^{I_2})_m$ so that the equality

$$\theta_{\overline{x},\,N_*}(x) = g_{\overline{x},\,n_i}(I_2)$$

holds for all $x = f^{n_i}(\overline{x})$, where $N_* = \{n_i\}_{i\ge 0}$, $(2^{I_2})_m$ is the space of all closed subsets of I_2, endowed with Hausdorff metric *dist*.

2. With the use of the concept of dynamical F-variation of function $\theta_{\overline{x},\,N_*}$ the explanation is given of the phenomenon of the existence of one-dimensional attractor $A = \bigcup_{i=0}^{n-1}\{f^i(x^0)\} \times g_{x^0,\,i}(I_2)$ such that $\omega_F((x,\,y)) = A$ (where $\omega_F((x,\,y))$ is ω-limit set of F-trajectory of a point $(x,\,y))$ for all $x \notin \mathrm{Orb}_f(x^0)$ from the basin of the attraction of the periodic orbit $\mathrm{Orb}_f(x^0)$ and all $y \in I_2$.

3. The influence of the smoothness of a simplest skew skew product of interval maps on the asymptotic behaviour of its trajectories is investigated.

The work is partially supported by RFBR, grant 04-01-00457.

[1] L.S. Efremova, *Nonhyperbolic periodic points and attracting sets of simplest skew products of interval maps*, J. Dyn. Control Syst. **10** (2004), 111-113.

Roman Ger *Almost Gołąb–Schinzel functions*

Recent studies of the classical Gołąb–Schinzel functional equation

$$f(x + yf(x)) = f(x)f(y) \tag{1}$$

and its generalizations show that there is an objective need of examining them on restricted domains (in general, depending upon a given solution). In a natural way, this leads also to similar questions regarding the associated Cauchy type equation

$$f(G(x,y)) = f(x)f(y) \tag{2}$$

with an (possibly “almost”) associative binary operation G. We deal with such kind of problems assuming the validity of (1) and/or (2) almost everywhere with respect to an abstract ideal of “small” sets.

Dorota Głazowska *Invariance of the geometric mean with respect to Lagrangean conditionally homogeneous mean-type mappings*

We determine all the Lagrangean conditionally homogeneous mean-type mappings for which the geometric mean is invariant. An important tool is a result on the generators of the Lagrangean conditionally homogeneous means.

Wojciech Jabłoński *Stability of the translation equation in rings of formal power series*

Joint work with L. Reich.

We will consider the stability problem for the translation equation in rings of formal power series

$$F(t_1+t_2,X)=F(t_1,F(t_2,X)) \qquad \text{for } t_1,t_2\in G,$$

where $(G,+)$ is an abelian group, $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, $F(t,X)=\sum_{i=1}^{\infty}c_i(t)X^i$ and $c_1\colon G\longrightarrow\mathbb{K}\setminus\{0\}$, $c_i\colon G\longrightarrow\mathbb{K}$ for $i\geq 2$.

We will show that, under some assumption on G, the translation equation in rings of formal power series is stable in Hyers–Ulam sense. What is more, stability of the translation equation in rings of formal power series is strictly connected with the problem of extensibility of the one-parameter group of truncated formal power series (cf. [1]).

[1] L. Reich, *Problem*, in: *Report of Meeting, The Twenty-eight International Symposium on Functional Equations, August 23 – September 1, 1990, Graz – Mariatrost, Austria*, Aequationes Math. **41** (1991), 248-310.

Justyna Jarczyk *Invariance in the class of weighted quasi-arithmetic means with continuous generators*

Let $I\subset\mathbb{R}$ be an open interval and $p,q,r\in(0,1)$. We find all continuous and strictly monotonic functions $\alpha,\beta,\gamma\colon I\longrightarrow\mathbb{R}$ satisfying the functional equation

$$\begin{aligned}\alpha^{-1}\big(p\alpha\big(\beta^{-1}(q\beta(x)+(1-q)\beta(y))\big)&\\ +(1-p)\alpha\big(\gamma^{-1}(r\gamma(x)+(1-r)\gamma(y))\big)\big)&\\ =\alpha^{-1}(p\alpha(x)+(1-p)\alpha(y))&\end{aligned} \tag{1}$$

generalizing the Matkowski–Sutô equation. In the proof we adopt a method elaborated by Z. Daróczy and Zs. Páles when solving the Matkowski–Sutô equation, some results of A. Járai on improving regularity of solutions and an extension theorem by Z. Daróczy and G. Hajdu. We also use a theorem giving the form of all twice continuously differentiable solutions of (1) proved jointly with J. Matkowski.

Witold Jarczyk *Convexity on Abelian groups*

Joint work with Miklós Laczkovich.

Let H be a subset of an Abelian group G. We say that $f\colon H \longrightarrow \mathbb{R}$ is convex if

$$2f(x) \leq f(x+h) + f(x-h)$$

holds whenever $x, h \in G$ and $x, x+h, x-h \in H$. It turns out that several classical theorems on convex functions on $\mathbb{R}^n$ or on (topological) linear spaces can also be proved in this general setting. In particular, we study the extendability of convex functions defined on subgroups of G as well as continuity properties of convex functions defined on open subsets of topological groups.

Hans-Heinrich Kairies *Continuous and residual spectra of operators connected with iterative functional equations*

The sum type operator F, given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x),$$

will be considered on the space of bounded real functions and on several subspaces. All the according restrictions are Banach space automorphisms. In their spectral theory some iterative functional equations arise in a natural way. We determine in all cases the resolvent set, the point spectrum, the continuous spectrum and the residual spectrum.

Dorota Krassowska *On a system of functional inequalities of iterative type*

Joint work with Janusz Matkowski.

We examine the functions $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying the system of simultaneous functional inequalities

$$f(\mathbf{a}_i + \mathbf{x}) \leq \alpha_i + f(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n,\ i = 1, 2, \dots, n+1.$$

Assuming some algebraic conditions on given fixed reals $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ and the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n+1} \in \mathbb{R}^n$, we show that every continuous at least at one point solution of this system must be of the form

$$f(\mathbf{x}) = \mathbf{p}\mathbf{x} + f(\mathbf{0}), \qquad \mathbf{x} \in \mathbb{R}^n,$$

where $\mathbf{p} \in \mathbb{R}^n$ is uniquely determined.

An application for mappings having weakly constant sign is given.

Marian Kwapisz *Difference, differential, delay differential equations and convergence of WR methods*

The aim of the talk is to point out natural relations between difference equations and the waveform relaxation methods (in short WR methods) for solving large systems of ordinary differential equations, delay differential equations as well as neutral delay differential equations (NDDEs).

For difference equations the initial value problem is very easy. It can also be solved in finite steps by the use of successive iterations. We will look for this phenomenon for more complicated and more important equations. It will concern the convergence problem for the solutions of implicit difference equations considered in a function spaces.

We will present convergence results for WR methods applied to the general systems of NDDEs. We also will take care for the error evaluations in such a way that we will be able to detect the cases when the convergence will take place in a finite number of steps.

[1] Z. Bartoszewski, M. Kwapisz, *Delay dependent estimates for waveform relaxation methods for neutral differential-functional systems*, Comput. Math. Appl. **48** (2004), 1877-1892

Zbigniew Leśniak *On the d'Alembert equation and its generalizations*

We study the d'Alembert partial differential equation (and its generalizations) as a submanifold of the jet space $J^2(\mathbb{R}^2, \mathbb{R})$ (and $J^n(\mathbb{R}^2, \mathbb{R})$, respectively) with the contact system defined on it. We prove the existence and the uniqueness (for appropriate initial data) of solutions of these equations by using the Cartan–Kähler Theorem.

Grażyna Łydzińska *On semicontinuity of some set-valued iteration semigroups*

Let X be an arbitrary set, $A\colon X \longrightarrow 2^{\mathbb{R}}$, $q := \sup A(X)$ and $F\colon (0,\infty) \times X \longrightarrow 2^X$ be given by

$$F(t,x) := A^{-1}\left(A(x) + \min\{t, q - \inf A(x)\}\right), \tag{A}$$

where

$$A^{-1}(V) := \{x \in X \colon\ A(x) \cap V \neq \emptyset\}$$

for every $V \subset \mathbb{R}$.

The formula (A) is a set-valued counterpart of the well-known form of iteration semigroups of single-valued functions on an interval.

We present a few theorems about lower semicontinuity of the multifunctions $F(t,\cdot)$ and $F(\cdot,x)$ in the case when A is a single-valued function defined on a topological space.

Andrzej Mach *Translation equation on monoids*

Joint results with Zenon Moszner.

Large classes of solutions of the translation equation on a monoid $(G,\cdot)$ satisfying the identity condition and some results on stability of the translation equation are given.

[1] A. Mach, Z. Moszner, *Translation equation on monoids*, Ann. Polon. Math. **84** (2004), 137-146.

[2] A. Mach, Z. Moszner, *L'équation de translation sur le demi-groupe des éléments non négatifs d'un groupe ordonné et archimédien*, en preparation.

[3] A. Mach, Z. Moszner, *On stability of the translation equation in a class of functions*, Aequationes Math., submitted.

Janusz Matkowski *On a generalized Gołąb–Schinzel functional equation*

We consider the composite functional equation

$$f\left(p[f(y)x+y]+(1-p)[f(x)y+x]\right)=f(x)f(y)$$

where $p \in \mathbb{R}$ is arbitrarily fixed. For $p=0$ or $p=1$ it becomes the well-known Gołąb–Schinzel equation.

Vladimir Mityushev *Riemann–Hilbert problem for multiply connected domains and functional equations*

Consider mutually disjointed disks $D_k := \{z \in \mathbb{C} : |z-a_k| < r_k\}$ $(k = 1,...,n)$ in the complex plane $\mathbb{C}$. Let $D := (\mathbb{C} \cup \{\infty\}) \setminus \bigcup_{k=0}^{n} (D_k \cup \partial D_k)$. Given $\lambda_k(t)$, $f_k(t)$ as Hölder continuous functions on ∂D, $\lambda_k(t) \neq 0$. To find a function $\varphi(z)$ analytic in D continuous in $D \cup \partial D$ with the following boundary condition

$$\operatorname{Re} \overline{\lambda_k(t)} \varphi(t) = f_k(t), \qquad |t-a_k| = r_k,\ k=1,...,n.$$

This problem is called the *(Riemann–)Hilbert boundary value problem.* The scalar Riemann–Hilbert problem for any multiply connected domain has been reduced to functional equations in a class of analytic functions which has been solved in terms of the Poincaré series.

Janusz Morawiec *On L^1-solutions of a functional equation connected with the Grincevičjus series*

We consider the problem of the existence of non-trivial L^1-solutions of the equation

$$f(x) = \sum_{\varepsilon=\pm1} \sum_{n=-N}^{N} c_{n,\varepsilon} f(\varepsilon kx - n),$$

where $N \in \mathbb{N}$, $k \in \mathbb{N} \setminus \{1\}$ and $c_{n,\varepsilon} \geq 0$ for all $n \in \{-N,\dots,N\}$, $\varepsilon \in \{-1,1\}$.

Jacek Mrowiec *On the nonstability of the Jensen's equation*

Let $\mathbb{X}$ be a linear space and let $\mathbb{X} \supset D$ be a midconvex set, i.e.,

$$x,y \in D \Longrightarrow \frac{x+y}{2} \in D.$$

A function $g\colon D \longrightarrow \mathbb{R}$ is said to be *Jensen* if

$$g\Big(\frac{x+y}{2}\Big) = \frac{g(x)+g(y)}{2} \tag{1}$$

for every $x, y \in D$.

Let $\delta > 0$ be a given real number. We say that a function $f\colon D \longrightarrow \mathbb{R}$ is *approximately Jensen* (δ-Jensen) if

$$\left| f\Big(\frac{x+y}{2}\Big) - \frac{f(x)+f(y)}{2} \right| \le \delta$$

for every $x, y \in D$.

We say that the Jensen's functional equation (1) is *stable* on a given midconvex set D if for any δ-Jensen function $f\colon D \longrightarrow \mathbb{R}$ there exist a Jensen function $g\colon D \longrightarrow \mathbb{R}$ and a constant $C > 0$ such that

$$|f(x) - g(x)| \le C\delta, \qquad x \in D.$$

It is known, that if $D = \mathbb{X}$, $D = [-a, a]^N \subset \mathbb{R}^N$ (and in some other cases), then the equation (1) is stable.

However, the equation (1) is not stable in general, i.e., on any midconvex set $D \subset \mathbb{X}$. There exist a midconvex subset D of the real line and an approximately Jensen function $f\colon D \longrightarrow \mathbb{R}$ s.t.

$$\sup\Big\{\, |g(x) - f(x)| :\ x \in D \Big\} = \infty$$

for every Jensen function $g\colon D \longrightarrow \mathbb{R}$.

Similar example works for affine functions in infinite dimensional spaces.

Anna Mureńko *On solutions of a generalization of the Gołąb–Schinzel functional equation*

We consider solutions $M, f\colon \mathbb{R} \longrightarrow \mathbb{R}$ and $\circ\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ of the functional equation

$$f(x + M(f(x))y) = f(x) \circ f(y),$$

under the following additional assumptions:

(a) f is Lebesgue measurable or Baire measurable;

(b) $M^{-1}(\{0\}) = \{0\}$;

(c) $\circ$ is commutative and associative.

Adam Najdecki *On the stability of some functional equations connected with the multiplicative symmetry*

Let $(X, \circ)$ be an abelian semigroup, $g\colon X \longrightarrow X$ and let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. We consider stability of the functional equation

$$f(x \circ g(y)) = f(x) \cdot f(y)$$

in the class of function $f: X \longrightarrow \mathbb{K}^n$, as well as of the equation

$$f(x \circ g(y)) = f(x) + f(y)$$

in the class of function mapping X into a Banach space over $\mathbb{K}$.

Kazimierz Nikodem *Three-parameter families and generalized convex functions*

Joint work with Attila Gilányi and Zsolt Páles.

We extend the notion of generalized convex functions introduced by E.F. Beckenbach to two-dimensional case in the following way: Let $\mathcal{F}$ be a family of continuous real functions defined on a convex set $D \subset \mathbb{R}^2$ such that for any three non-collinear points $x_1, x_2, x_3 \in D$ and any $t_1, t_2, t_3 \in \mathbb{R}$ there exists exactly one $\varphi = \varphi_{(x_1,t_1),(x_2,t_2),(x_3,t_3)} \in \mathcal{F}$ such that $\varphi(x_i) = t_i$ for $i = 1, 2, 3$. We say that a function $f: D \longrightarrow \mathbb{R}$ is $\mathcal{F}$*-convex* if for any non-collinear $x_1, x_2, x_3 \in D$

$$f(x) \leq \varphi_{(x_1,f(x_1)),(x_2,f(x_2)),(x_3,f(x_3))}(x)$$

for every x in the triangle with vertices x_1, x_2, x_3. It is proved that every $\mathcal{F}$-convex function $f: D \longrightarrow \mathbb{R}$ is continuous on $\operatorname{int} D$. Some other properties of $\mathcal{F}$-convex and $\mathcal{F}$-midconvex functions are also given.

Jolanta Olko *On the set of measure selections of a multimeasure*

Let $(T, \mathcal{A})$ be a measurable space, X be a Banach space. We say that a multifunction $M: \mathcal{A} \longrightarrow 2^X$ with nonempty, closed values is a multimeasure if for every $x^* \in X^*$ the function $A \mapsto \sup\{x^*(x) : x \in M(A)\}$ is an $\mathbb{R} \cup \{+\infty\}$-valued signed measure. A measure $m: \mathcal{A} \longrightarrow X$ is called the *measure selection* of M if $m(A) \in M(A)$ for every $A \in \mathcal{A}$.

Some properties of the set of measure selections of a multimeasure are discussed.

Boris Paneah *On the general theory of the Cauchy type functional equations*

The main object of this talk is so called Cauchy type functional equation on some interval I. This is the equation of the form

$$F\big((\delta_1 + \delta_2)(t)\big) - F\big(\delta_1(t)\big) - F\big(\delta_2(t)\big) = h(t), \qquad t \in I,$$

with δ_1 and δ_2 being continuous maps from I into itself and h and F being given and unknown functions, respectively. The two themes are of the main interest for us: 1) the solvability properties of this equation; 2) all possible connections of this equation with diverse fields in mathematics and, may be, outside. The solvability properties, as will be clarified, are completely determined by mutual properties of the maps δ_1 and δ_2 (so called, "configurations of maps"). The two different configurations will be discussed. In the case of a Z-configuration it is successful to solve the Cauchy type functional equation in an explicit form. In the case of a $\mathcal{P}$-configuration (considerably more difficult case) the key role in formulating results and in their obtaining belong to some completely new dynamical system determined by the semigroup of maps in I generated by the two maps δ_1 and δ_2 . All mentioned solvability properties are discussed in this talk.

The second part of the talk is devoted to applications of the above results in such diverse fields as functional equations, integral geometry and partial differential equations. There are many unsolved problems concerning the topic.

Iwona Pawlikowska *Theorems of Flett's type and stability*

T.M. Flett [2] proved a version of the Lagrange Mean Value Theorem saying that for every differentiable function f on $[a,b]$ with $f'(a)=f'(b)$ there exists an intermediate point η such that

$$f(\eta)-f(a)=f'(\eta)(\eta-a).$$

We will show some generalizations of these two theorems. Das, Riedel and Sahoo, [1], proved the Hyers–Ulam stability of points η for which Flett's Mean Value Theorem holds true. We will also discuss stability of points for which our generalizations of Flett's MVT hold true.

[1] M. Das, T. Riedel, P.K. Sahoo, *Hyers–Ulam Stability of Flett's points*, Applied Math. Letters **16** (2003), 269-271.

[2] T.M. Flett, *A mean value theorem*, Math. Gazette **42** (1958), 38-39.

Bożena Piątek *On $*$-concave and convex multifunctions*

Let Y be a real Banach space. We shall show that the inclusion

$$\frac{1}{t-s}\int_s^t F(x)\,dx \subset \frac{F(s)\overset{*}{+}F(t)}{2}$$

for all $a \le s < t \le b$ can be used to characterize $*$-concave multifunctions $F\colon [a,b] \longrightarrow \mathrm{clb}(Y)$.

Magdalena Piszczek *Second Hukuhara derivative and a cosine family of linear set-valued functions*

Let K be a closed convex cone with the nonempty interior in a real Banach space and let $\mathrm{cc}(K)$ denote the family of all nonempty convex compact subsets of K. If $\{F_t:\ t\ge 0\}$ is a regular cosine family of continuous linear set-valued functions $F_t\colon K \longrightarrow \mathrm{cc}(K)$, $x \in F_t(x)$ for $t \geq 0$, $x \in K$ and $F_t \circ F_s = F_s \circ F_t$ for $s, t \geq 0$, then

$$D^2 F_t(x) = F_t(H(x))$$

for $x \in K$ and $t \geq 0$, where $D^2 F_t(x)$ denotes the second Hukuhara derivative of $F_t(x)$ with respect to t and $H(x)$ is the second Hukuhara derivative of this multifunction at $t = 0$.

Dorian Popa *Hyers–Ulam stability of some linear recurences*

In this talk we give a Hyers–Ulam–Rassias stability result for the first order linear recurrence in Banach spaces. As a consequence we obtain a Hyers–Ulam stability result for the p-order linear recurrence with constant coefficients.

Barbara Przebieracz *Near-iterability*

Inspired by Problem (3.1.12) posed by E. Jen in [2] we present various approaches to the concept of near-iterability. We deal with selfmappings of a real compact interval, characterize and compare a few classes of near-iterable functions in a sense. That includes the class of *almost iterable functions*, that is continuous $f\colon I \longrightarrow I$, for which there exists an iterable $g\colon I \longrightarrow I$ such that

$$f^n - g^n \text{ converges to } 0, \text{ everywhere in } I, \tag{E}$$

and the convergence is uniform on every interval with endpoints being two consecutive fixed points of f (cf. [1]); of functions satisfying (E); and that of functions f for which there exists an iterable g such that

$$f^n - g^n \text{ converges to 0 almost everywhere in } I \tag{AE}$$

or

$$f^n - g^n \text{ converges to 0 in measure.} \tag{M}$$

The measure appearing in (AE) and (M) is any Borel measure vanishing at points and taking positive values on nondegenerate intervals. We provide some conditions under which conditions (M) and (AE) are equivalent.

[1] W. Jarczyk, *Almost iterable functions*, Aequationes Math., **42** (1991), 202-219.

[2] Gy. Targonski, *New directions and open problems in iteration theory*, Ber. Math.-Statist. Sekt. Forschungsgesellsch. Joanneum, No. **229**. Forschungszentrum, Graz, 1984.

Maciej Sablik *More functional equations stemming from actuarial mathematics*

We recall some functional equations that have been motivated by natural questions asked in actuarial mathematics. Also, we discuss some new problems leading to functional equations and arising when some (generally accepted in the actuarial calculus) hypotheses are admitted.

Vsevolod Sakbaev *On the averaging of the family of regularizing solutions of Schrödinger equation with degeneration*

In this work we consider the ill-posed boundary-value problem (BVP) and define the procedure of regularization of it by a sequence of well-posed BVP which approximates the considered problem. The sequence of solutions of regularizing BVP can diverge. We choose a measure on the set of all subsequences of this sequence and define the procedure of averaging of the set of particular limits of the sequence by this measure.

As an example of the ill-posed BVP we choose the Cauchy problem for degenerated Schrödinger equation with the mixed type Hamilton operator $\mathbf{L}$ in the Hilbert space $H = L_2(\mathbb{R})$ and the initial data $u_0 \in H$. According to regularization method we consider the directed family of regularizing Cauchy problems for Schrödinger equations with the uniformly elliptic Hamilton operators $\mathbf{L}_\varepsilon$, $\varepsilon \in (0,1)$, and study the convergence of the family of regularizing solutions $u_\varepsilon(t)$ as $\varepsilon \to 0$. In the paper [1] we prove the weak convergence of the whole family of regularizing solutions $u_\varepsilon(t)$ as $\varepsilon \to 0$ and obtain the necessary and sufficient conditions of its strong convergence.

The aim of our investigation is the convergence of the family of the linear continuous functionals $\{f_\varepsilon(t,u_0,\cdot)\}$ on the Banach space $B(H)$ of bounded selfadjoint operators in H which are defined by the formula:

$$f_\varepsilon(t,u_0,\mathbf{A}) = (u_\varepsilon(t), \mathbf{A}u_\varepsilon(t)), \qquad \mathbf{A} \in B(H).$$

We consider the convergence of the sequence $\{f_\varepsilon\}$ in the $*$-weak topology of $B(H)^*$.

Theorem 1
If the sequence $u_{\varepsilon_n}(t)$ diverges in the norm of space H then there is a bounded operator $\mathbf{A} \in B(H)$ such that the sequence $f_{\varepsilon_n}(t,u_0,\mathbf{A})$ diverges.

Therefore the pointwise convergence on the space $B(H)$ of the functionals $\{f_\varepsilon(t,u_0,\cdot)\}$ as $\varepsilon \to 0$ for any $u_0 \in H$ is impossible. Note by the symbol $\mathrm{Ls}_{\varepsilon\to 0} f_\varepsilon(t,u_0,\mathbf{A})$ the set of all limit points of the family $f_\varepsilon(t,u_0,\mathbf{A})$ as $\varepsilon \to 0$. According to idea of regularization we define the multi-valued map

$$F(t,u_0,\cdot)\colon B(H) \longrightarrow 2^{\mathbb{R}},$$

where $2^{\mathbb{R}}$ is the metric space of the subsets of $\mathbb{R}$ with the Hausdorff distance function, which acts on the space $B(H)$ by the rule

$$F(t,u_0,\mathbf{A}) = \mathrm{Ls}_{\varepsilon\to 0} f_\varepsilon(t,u_0,\mathbf{A}).$$

LEMMA 1
Many-valued map $F(\cdot)$ is the continuous map of Banach space $\mathbb{R} \times H \times B(H)$ into the metric space $2^{\mathbb{R}}$ such that for any $\mathbf{A} \in B(H)$ the set $F(t, u_0, \mathbf{A})$ is the segment.

To construct the rule of the averaging of multifunction F we choose some bounded-additive measure on the set $F(t, u_0, \mathbf{A})$. We denote by W the set of bounded-additive nonnegative normalized measures on the σ-algebra of all subsets of the set of regularizing parameters ε such that the measure of any set M is equal to zero if the point 0 is not a limit point of the set M.

THEOREM 2
The set W is nonempty and convex.

For any $\mu \in W$ we define the procedure of averaging of multifunction F by the measure μ.

THEOREM 3
For any $\mu \in W$ there is the unique function $f_\mu(\cdot)$ which is continuous function on the Banach space $\mathbb{R} \times H \times B(H)$ such that $f_\mu(t, u_0, \cdot)$ is the positive normalized continuous linear functional on $B(H)$.

To prove this result we construct the regular method of generalized summation such that any bounded sequence is summable.

This work is partially supported by RFBR, grant No 04-01-00457.

[1] V.Zh. Sakbaev, *On the functionals on solutions of Cauchy problem for Schrödinger equation with degeneration on semiaxe*, Comp. Math. and Math. Phys. **44** (2004), 1654-1673.

Ekaterina Shulman *On addition theorems of rational type*

We investigate the functions $f\colon \Lambda \longrightarrow \mathbb{C}$, $\Lambda \subset \mathbb{C}$, which admit an addition theorem of the form

$$f(t+s) = \frac{\sum_{i=1}^{n} y_i(t)u_i(s)}{\sum_{j=1}^{m} z_j(t)v_j(s)}.$$

Here all functions are supposed to be continuously differentiable on some interval. Our approach is based on the reduction to a system of differential equations. The concepts of *joint linear dependence* and *joint quadratic dependence* of two families of functions are introduced. It's proved that there are only two possibilities in the case of *jointly linearly independent* $\{u_i\}$ and $\{v_j\}$:

a) the function f is a ratio of quasi-polynomials,

b) the families $\{y_i\}$ and $\{z_j\}$ are jointly quadratically dependent.

The second possibility is studied for $m = n = 2$. We apply our results to solving of the functional equation

$$f(t+s) = \frac{y_1(t)y_2(s) - y_2(t)y_1(s)}{z_1(t)z_2(s) - z_2(t)z_1(s)}.$$

Justyna Sikorska *On generalized stability of some orthogonal functional equations*

Starting with the papers of Th.M. Rassias [2] and Z. Gajda [1] new stability problems were introduced. For given Banach spaces $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$, $\varepsilon \geq 0$, they considered functions $f\colon X \longrightarrow Y$ satisfying the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p), \qquad x, y \in X.$$

We study this kind of stability for the orthogonally additive functional equation as well as some of its applications and generalizations.

[1] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci. **14** (1991), 431-434.

[2] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297-300.

Andrzej Smajdor *On concave iteration semigroups of linear set-valued functions*

The equality

$$G(x) + tG^2(x) = (I + tG)(G(x)) \tag{1}$$

is a necessary and sufficient condition under which a family $\{F^t : \ t \geq 0\}$ of linear continuous set-valued functions F^t, where

$$F^t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x), \tag{2}$$

is an iteration semigroup.

Moreover, a concave iteration semigroup of continuous linear set-valued functions with the infinitesimal generator G fulfilling (1) and such that $0 \in G(x)$ is of the form (2).

Wilhelmina Smajdor *On a set-valued version of a functional equation of Drygas*

We find the general solution of the functional equation

$$F(x+y)+F(x-y)=2F(x)+F(y)+F(-y). \tag{D}$$

Moreover, we show that every solution of (D) has a selection f satisfying the functional equation

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y).$$

Dariusz Sokołowski *Solutions with constant sign at infinity of a linear functional equation of infinite order*

Inspired by R.O. Davies and A.J. Ostaszewski [1] we investigate connections between the linear functional equation of the form

$$\varphi(x)=\int_S \varphi\left(x+M(s)\right)\sigma(ds) \tag{1}$$

and its characteristic equation

$$\int_S e^{\lambda M(s)}\sigma(ds)=1. \tag{2}$$

Here (S,Σ,σ) is a measure space with a finite measure σ and $M\colon S\longrightarrow\mathbb{R}$ is a Σ-measurable bounded function with $\sigma(M{\neq}0)>0$. By a *solution* of (1) we mean a Borel measurable real function φ defined on an interval of the form $(a,+\infty)$, Lebesgue integrable on every finite interval contained in $(a,+\infty)$ and such that for every $x>a+\sup\{|M(s)|:\ s\in S\}$ the integral $\int_S \varphi(x+M(s))\,\sigma(ds)$ exists and (1) holds.

According to [2, Theorem 2] if (1) has a solution with a constant sign (i.e., nonnegative and a.e. positive or nonpositive and a.e. negative), then (2) has a real root. It turns out that the existence of a solution of (1) with some additional properties guarantees the existence of a characteristic root with the specified sign. Namely we have the following two results.

Theorem 1
If (1) *has a solution with infinite limit at* $+\infty$, *then either* (2) *has a positive root or*

$$\sigma(S)=1 \qquad \textit{and} \qquad \int_S M(s)\,\sigma(ds)=0.$$

Theorem 2
If (1) *has a solution with a constant sign vanishing at* $+\infty$, *then* (2) *has a negative root.*

[1] R.O. Davies, A.J. Ostaszewski, *On a difference-delay equation*, J. Math. Anal. Appl. **247** (2000), 608-626.

[2] D. Sokołowski, *Solutions with constant sign at infinity of a linear functional equation of infinite order*, J. Math. Anal. Appl. **310** (2005), 144-160.

Paweł Solarz *Iterative roots for some homeomorphisms with infinitely many periodic points*

Let S^1 be the unit circle, $F\colon S^1 \longrightarrow S^1$ be an orientation-preserving homeomorphism and let $\operatorname{Per} F$ denote the set of all periodic points of F. Assume that the boundary of a set of cluster points of $\operatorname{Per} F$ is a finite set. We give the necessary and sufficient conditions for the existence of continuous and orientation-preserving solutions of the following equation:

$$G^m(z) = F(z), \qquad z \in S^1,$$

where $m \geq 2$ is an integer.

Joanna Szczawińska *On families of set-valued functions*

Let G be a linear continuous multifunction defined on a closed convex cone C in a Banach space X. J. Olko (see. [1]) has proved that for every $x \in C,\ t \geq 0$ a series $B^t(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} G^n(x)$ is convergent in the space of all nonempty compact convex subsets of X with the Hausdorff metric and $(B^t \circ B^s)(x) \subset B^{t+s}(x),\ x \in C,\ t, s \geq 0$.

We give a generalization of this result.

[1] J. Plewnia, *On a family of set-valued functions*, Publ. Math. Debrecen **46** (1995), 149-159.

Tomasz Szostok *Orlicz spaces and ω-convexity*

We deal with some modified version of convexity. Namely we fix an infinite interval $I \subset \mathbb{R}$ and a function $\omega\colon I \longrightarrow \mathbb{R}$. Then we consider a given function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$. Function f will be called *ω-convex* if and only if for all $x, y \in \mathbb{R}$, $x < y$ and for every $z \in (x, y)$ we have

$$f(z) \leq \omega(z + \alpha) + \beta,$$

where $\alpha, \beta \in \mathbb{R}$ are such that $\omega(x + \alpha) + \beta = f(x)$ and $\omega(y + \alpha) + \beta = f(y)$. It is clear that we have to make some assumption on ω in order to obtain the existence of such numbers. We present some applications of this notion to Orlicz spaces theory and our main result states that if an Orlicz function is ω-convex with $\omega(x) = x^p$ where $p > 1$, then f satisfies some inequalities used in Orlicz spaces theory.

Jacek Tabor *Shadowing and stability in metric groups I*

Joint work with Wojciech Jabłoński and Józef Tabor.

Let X be a complete metric space and let $\phi\colon X \longrightarrow X$ be a given mapping. We say that ϕ has the shadowing property if for every approximate orbit (x_k) there exists an exact orbit (y_k) which is close to (x_k).

One of the classical results from the stability of dynamical systems states that if ϕ is invertible and $\mathrm{Lip}(\phi^{-1}) < 1$ then ϕ has the shadowing property.

We generalize this result for the case when ϕ is locally invertible. As a corollary we obtain shadowing for the Julia set with a fixed parameter value. We also show that this shadowing result can be a useful tool in dealing with the stability of the Cauchy-type functional equations.

Józef Tabor *Shadowing and stability in metric groups II*

Joint work with Wojciech Jabłoński and Jacek Tabor.

We apply the concept of shadowing in dynamical systems to prove stability of functional equations. Using Hyers-like method we generalize some classical results to the case when the target space is a metric group. The condition of global 2-divisibility is replaced by a local one.

Gheorghe Toader *Complementary means and double sequences*

The well known arithmetic-geometric process of Gauss was generalized for arbitrary means as follows. Consider two means M and N defined on the interval J and two initial values $a, b \in J$.

By definition, the pair of sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ defined by

$$a_{n+1} = M(a_n, b_n) \quad \text{and} \quad b_{n+1} = N(a_n, b_n), \qquad n \geq 0\,,$$

where $a_0 = a$, $b_0 = b$, is called a (*Gaussian) double sequence*. The mean M is *compoundable* with the mean N if the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are convergent to a common limit $M \otimes N(a, b)$ for each $a, b \in J$. The function $M \otimes N$ defines a mean which is called *G-compound mean*.

Some G-compound means can be determined using a characterization theorem proved in [1]: (Invariance Principle). *Suppose that $M \otimes N$ exists and is continuous. Then $M \otimes N$ is the unique mean P with the property $P(M, N) = P$.* The mean P is called *(M, N)-invariant*, and the mean N is called *P-complementary to M*.

Many non trivial examples of P-complementary means can be found in [2]. In fact, for the ten Greek means, we determined ninety complementaries of one mean with respect to another. They are done by direct computation. To make other determinations of complementaries, we use series expansions. We try to identify the complementary of one mean from a given family of means in an other family of means.

To illustrate this method, we study the complementariness with respect to the weighted geometric mean $\mathcal{G}_\lambda$, called also generalized inverse. We determine the generalized inverse of a weighted Gini mean in the family of extended means and converse, the generalized inverse of an extended mean in the family of weighted Gini means.

[1] J.M. Borwein, P.B. Borwein, *Pi and the AGM – a Study in Analytic Number Theory and Computational Complexity*, John Wiley & Sons, New York, 1986.

[2] Gh. Toader, Silvia Toader, *Greek means and the Arithmetic-Geometric Mean*, RGMIA Monographs, Victoria University, 2005. (ONLINE: http://rgmia.vu.edu.au/monographs)

Peter Volkmann *The absolute value of n-additive functions*

M being a *magma* (cf. Bourbaki), we call $f\colon M \longrightarrow \mathbb{R}$ *additive*, if $f(xy) = f(x) + f(y)$ $(x, y \in M)$. Suppose $F\colon M_1 \times \ldots \times M_n \longrightarrow \mathbb{R}$, $M_1, \ldots, M_n$ being magmas. Then $F(x_1, \ldots, x_n) = |f(x_1, \ldots, x_n)|$ with a function f being additive with respect to each variable if and only if all the functions $x_k \mapsto F(x_1, \ldots, x_n)$ $(x_k \in M_k)$ are absolute values of additive functions. Joint work with Attila Gilányi.

Janusz Walorski *On continuous and smooth solutions of the Schröder equation in normed spaces*

Let X and Y be normed spaces and $D \subset X$. We consider continuous and smooth solutions $\varphi\colon D \longrightarrow Y$ of the Schröder equation

$$\varphi(f(x)) = A\varphi(x),$$

where the function $f\colon D \longrightarrow D$ and the bounded linear operator $A\colon Y \longrightarrow Y$ are given.

Szymon Wąsowicz *Some inequalities connected with an approximate integration*

Some classical and new inequalities of an approximate integration are obtained with use of Hadamard type inequalities and delta-convex functions of higher orders. Error bounds of midpoint, trapezoidal and Simpson's rules are reproved. As an example of some new inequalities we give the following

Theorem
Let f be three times differentiable on $[a, b]$ and assume that

$$M_3(f) := \sup\big\{\big|f'''(x)\big| :\ x \in [a, b]\big\} < \infty.$$

Then

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{4}\left(f(a) + 3f\left(\frac{a+2b}{3}\right)\right)\right| \le \frac{M_3(f)(b-a)^4}{216}$$

and

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{4}\left(f(b) + 3f\left(\frac{2a+b}{3}\right)\right)\right| \le \frac{M_3(f)(b-a)^4}{216}.$$

[1] Sz. Wąsowicz, *Some inequalities connected with an approximate integration*, J. Inequal. Pure Appl. Math. (JIPAM) **6(2)** (2005), Article 47.

Bing Xu *Analytic solutions of a nonlinear iterative equation*

Joint work with Weinian Zhang.

Analytic solutions of the functional equation

$$\sum_{j=0}^{k}\sum_{t=1}^{\infty} C_{t,j}(z)(\varphi(q_j z))^t = G(z)$$

are discussed in various distributions of the vector $(q_0, \ldots, q_k)$. In the special case that $q_j = q^j$, $j = 0, \ldots, k$, our main theorems imply corresponding results for a q-difference equation and weaken the conditions given by Si and Zhang [1]. Moreover, we discuss invertible analytic solutions of the q-difference equation, which enable us to apply our theorems to iterative equations weaken the conditions given by Si and Zhang [2].

[1] J. Si, W. Zhang, *Analytic solutions of a nonlinear iterative equation near neutral fixed points and poles*, J. Math. Anal. Appl. **284** (2003), 373-388.

[2] J. Si, W. Zhang, *Analytic solutions of a q-difference equation and applications to iterative equations*, J. Difference Eq. Appl. **10** (2004), 955-962.

Marek Cezary Zdun *A general class of iterative equation on the unit circle*

Joint work with Weinian Zhang.

We consider the problem of the existence and uniqueness of solutions of the iterative equation

$$\Phi(f(z), f^2(z), \ldots, f^n(z)) = F(z), \qquad z \in T^1,$$

on the unite circle T^1 in a subclass of the class of homeomorphisms

$$H_1^0(T^1, T^1) = \{f \in C^0(T^1, T^1)\colon f(T^1) = T^1 \text{ homeomorphically and } f(1) = 1\},$$

where $C^0(T^1, T^1)$ consists of all continuous maps from T^1 into itself.

We discuss the influence of the domain of the continuous mapping Φ for the existence of these solutions.

Marek Żołdak *Stability of isometries in p-homogeneous F-spaces*

Joint work with Józef Tabor and Jacek Tabor.

The equation of isometry in Banach spaces is stable in the Ulam–Hyers sense. It happens that in complete Fréchet spaces this equation is not stable. We discuss the problem of stability in the class of Fréchet spaces with p-homogeneous norm, where $p \in (0, 1]$.

Problems and Remarks

1. Remark. *To K. Baron's talk*
Similar equations arise in the correlation function theory. For instance, E. Wegert has solved equation

$$\int_0^{+\infty} f(x)f(x+y)\,dx = g(y), \qquad y > 0, \tag{1}$$

with given $g(y)$. It is interesting to study also triple equations of the type

$$\int_0^{+\infty}\int_0^{+\infty} f(x,t)f(x+y,t)f(x,t+z)\,dxdt = g(y,z), \tag{2}$$

or equations with linear combinations of double and triple terms.

Vladimir Mityushev

2. Remark. *On stable probability distribution function*

Definition
Let $P \subset \mathbb{R}_+ = (0, \infty)$ be non-void. A probability distribution function $G\colon \mathbb{R} \longrightarrow [0, 1]$ is said to be P-stable if

$$\forall\, p \in P\ \exists\, a_p > 0, b_p \in \mathbb{R}: \quad (G(a_p x + b_p))^p = G(x). \tag{$*$}$$

Theorem (Gnedenko 1943)
For a p.d.f. $G \neq F_{\delta_a}$ (for $a \in \mathbb{R}$), the following conditions are equivalent

(i) *for some p.d.f. F and some sequences $(a_n > 0)_{n\in\mathbb{N}}$, $(b_n \in \mathbb{R})_{n\in\mathbb{N}}$*

$$G(x) = \lim_{n\to\infty} F^n(a_n x + b_n);$$

(ii) *G is $\mathbb{N}$-stable;*

(iii) *G is $\mathbb{R}_+$-stable;*

(iv) *up to a linear map in the domain, $G = G_\alpha$, for some $\alpha \in \mathbb{R}$, where*

$$G_\alpha(x) := \begin{cases} \exp(-x^\alpha),\ x > 0, & \textit{if } \alpha < 0,\\ \exp(-\exp(-x)),\ x \in \mathbb{R}, & \textit{if } \alpha = 0,\\ \exp(-|x|^\alpha),\ x < 0, & \textit{if } \alpha > 0. \end{cases} \tag{$**$}$$

Remark
G is $\mathbb{R}_+$-stable iff G is P-stable with any $P \subset \mathbb{R}_+$ which generates a dense multiplicative subgroup of $\mathbb{R}_+$. In particular iff G is $\{2,3\}$-stable.

Remark
If G is $\{p\}$-stable with $p \in \mathbb{R}_+ \setminus \{1\}$, then the stability equation of $(*)$ has solution dependent on arbitrary function. The solution will be necessarily of class $(**)$ under additional requirement that G is suitably regularly varying.

As we know from a talk by Professor Maciej Sablik, the stable distribution functions have found application in the financial and/or actuarial mathematics, especially as models of the duration of life of some populations. Due to the natural discretization of data, the usual way of representing them is

$$F(m|n) := P\{T \geq m | T \geq n\} = \frac{P\{T \geq m\}}{P\{T \geq n\}}, \qquad m, n \in \mathbb{N},\ m \geq n.$$

The form of $F(m|n)$ shows that this is a "multiplicative distance function" (m.d.f.). In some "ideal" models the given F should be treated as a restriction of some other m.d.f. $\tilde{F} \colon \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow [0,1]$ s.t.

$$\tilde{F}(t|s) = \frac{P\{T \geq t\}}{P\{T \geq s\}} = \frac{\tilde{F}_0(t)}{\tilde{F}_0(s)}, \qquad t \geq s,\ t, s \in \mathbb{R}_+ .$$

Then $\tilde{F}_0$ is the p.d.f. of the duration of life.

As a matter of fact, the experimental data concern the population at the different actual age, which implies differences of data when compared to the "ideal" model. Therefore a question appears, how to restore $\tilde{F}_0$ from F. One of the ways presented by Professor Sablik is to assume that

$$\tilde{\tilde{F}}(m+u|n) := \varphi_u(F(m|n), F(m+1|n)), \qquad u \in [0,1],\ m, n \in \mathbb{N}$$

where φ_u is an average (increasing in u).

I would like to give an example that homogeneous averages are not giving the $\tilde{F}$ equal to a multiplicative distance function for the stable p.d.f.

Assume $P\{T \geq t\} = \exp(-t^2)$, $t > 0$. Then

$$F(m|n) = e^{-m^2+n^2}, \quad m \geq n, \qquad \tilde{F}(t|s) = e^{-t^2+s^2}, \quad t \geq s.$$

For $\varphi_u(x,y) = x^{1-u}y^u$,

$$\tilde{\tilde{F}}(m+u|n) = e^{-m^2+n^2} e^{-2mu-u} \neq \tilde{F}(m+u|n) = e^{-(m+u)^2+n^2}.$$

For $\varphi_u(x,y) = ((1-u)x^\alpha + uy^\alpha)^{1/\alpha}$,

$$\tilde{\tilde{F}}(m+u|n) = e^{-m^2+n^2} \left[1 - u + ue^{-2m\alpha-\alpha}\right]^{1/\alpha} \neq \tilde{F}(m+u|n).$$

Remark

If $\tilde{\tilde{F}}$ is a m.d.f. given by φ_u of the form

$$\varphi_u(x, y) = \psi^{-1}((1-u)\psi(x) + u\psi(y))$$

then

$$\begin{aligned}\psi^{-1}&\left((1-u)\psi\left(\tilde{\tilde{F}}_0(m)/\tilde{\tilde{F}}_0(n)\right) + u\psi\left(\tilde{\tilde{F}}_0(m+1)/\tilde{\tilde{F}}_0(n)\right)\right)\\ &= \tilde{\tilde{F}}_0(m+u)/\tilde{\tilde{F}}_0(n)\end{aligned}$$

in other words

$$\varphi_u\left(\tilde{\tilde{F}}_0(m)/\tilde{\tilde{F}}_0(n); \tilde{\tilde{F}}_0(m+1)/\tilde{\tilde{F}}_0(n)\right) = \varphi_u\left(\tilde{\tilde{F}}_0(m), \tilde{\tilde{F}}_0(m+1)\right)/\tilde{\tilde{F}}_0(n),$$

which means homogeneity of φ_u with respect to multiplication by $1/\tilde{\tilde{F}}_0(n)$ for $n \in \{0, \ldots, m\}$ on the pair $\left(\tilde{\tilde{F}}_0(m), \tilde{\tilde{F}}_0(m+1)\right)$, for $m = 0, 1, 2, \ldots$.

Joachim Domsta

3. Remark. *On a paper by A. Matkowska*

The first of functional equations modelling the perfect capital market reads, cf. [2] and also [1, pp. 3-4], is the Cauchy equation with parameter t:

$$A(K_1 + K_2, t) = A(K_1, t) + A(K_2, t), \qquad K_1 \geq 0,\ K_2 \geq 0. \tag{1}$$

Here $A(K, t)$ is the amount to which a capital K increases during a time interval of length t by interest compounding.

Equation (1) being rather unrealistic, a component, $B(K, t)$ say, is proposed in [2] to be added to its right-hand side. On taking also $K_1 = K_2 = \frac{1}{2}K$ we arrive at the iterative functional equation

$$A(K, t) = 2A\left(\frac{K}{2}, t\right) + B(K, t). \tag{2}$$

The result from [2] on solutions of linear iterative functional equations in the class Lip^α of Hölder's functions yields the following (cf. [3])

Theorem

Let $B(\cdot, t) \in \mathrm{Lip}^\alpha$, $\alpha > 1$, and let $B(0, t) = 0$. Then equation (2) has exactly one solution $A(\cdot, t) \in \mathrm{Lip}^\alpha$ such that $A(0, t) = 0$. The solution is given by the formula

$$A(K, t) = \sum_{n=0}^{\infty} 2^n B\left(\frac{K}{2^n}\right), \qquad K \geq 0$$

provided that the series is convergent (in the norm of Lip^α).

[1] W. Eichhorn, *Functional Equations in Economics*, Addison–Wesley Comp., London – Amsterdam – Don Mills, ON – Sydney – Tokyo, 1978.

[2] A. Matkowska, *Hölder's solutions of a linear functional equation*, Zeszyty Naukowe AGH, Zagadnienia Techniczno-Ekonomiczne **48** (Informatyka) No. 3 (2003), 899 - 905.

[3] A. Matkowska, *Term investments and Hölder's solution of a functional equation*, in: Zarządzanie przedsiębiorstwem w warunkach integracji europejskiej, cz. II: Ekonomia, Informatyka i Metody Matematyczne, Akademia Górniczo-Hutnicza im. St. Staszica w Krakowie, Uczelniane Wyd. Nauk.-Dydakt., Kraków, 2004, 435 - 440.

Bogdan Choczewski

4. Remark.
The remark concerns the talk by N. Brillouët-Belluot, and more exactly the equation

$$x + f(y + f(x)) = y + f(x + f(y)) \tag{1}$$

that motivated her joint work with Weinian Zhang.

1. If we consider the question of solving (1) for function f mapping $\mathbb{C}$ into $\mathbb{C}$, then it is easy to observe that

$$z \mapsto az$$

where $a \in \{\frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}\}$ yields a continuous and additive solution to (1).

2. Let H be a Hamel base of $\mathbb{R}$ over $\mathbb{Q}$, and split it into two sets of the same cardinality, say H_1 and H_2. Let $\gamma\colon H_1 \longrightarrow H_2$ be an arbitrary bijection. Define $g\colon H \longrightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} \gamma(x), & \text{if } x \in H_1 \\ x - \gamma^{-1}(x), & \text{if } x \in H_2. \end{cases}$$

Then g is bijectively mapping H onto $g(H)$, and it can be easily checked that $g(H)$ is also a Hamel base of $\mathbb{R}$ over $\mathbb{Q}$. It follows that $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, the additive extension of g, is bijective. Moreover, it can be verified quite easily, f is a solution of (1).

3. The problem of existence of continuous solutions of (1), mapping $\mathbb{R}$ into $\mathbb{R}$, remains open.

Maciej Sablik

5. Remark and Problem. *A functional equation with singularity*
Let $g(z)$ be a known function analytic in the unit disc and continuous on its closure ($g \in C_A$). Let s be a given constant such that $0 < s < 1$. The functional equation

$$\varphi(z) = \varphi(sz) + g(z), \qquad |z| \le 1 \tag{1}$$

is solvable in C_A iff $g(0) = 0$. This and other general results were obtained by M. Kuczma, J. Matkowski, W. Smajdor (see Kuczma's book [1] for details).

Let us consider the simple functional equation (1), but in another disc

$$\varphi(z) = \varphi(sz) + 1, \qquad |z-1| \le 1,\ z \neq 0, \tag{2}$$

in a class of functions analytic in $\{z \in \mathbb{C} :\ |z-1| < 1\}$, continuous on its closure except the origin, where i) φ is bounded; ii) φ has an integrable singularity.

Such functional equations have applications in mechanics of composites.

[1] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, PWN, Uniwersytet Śląski, Warszawa – Kraków – Katowice, 1985.

Vladimir Mityushev

6. Problem. *A functional equation arisen in the diffraction theory*
Given entire functions a, b, c in $\mathbb{C}$ and a constant h find entire solutions of the functional equation

$$a(z)\varphi(z) + b(z)\varphi(z+h) + c(z)\varphi(z-h) = 0.$$

Vladimir Mityushev

7. Remark. *On generalized subadditive functions bounded by some functions*
Results by B. Choczewski and Z. Powązka, in preparation.
The following problem was proposed on the International Mathematical Olympic Competition in the USA (1978), cf. [2, p. 79]:

> Prove that if, for every $x, y \in \mathbb{R}$, we have $\psi(x) \le x$ and $\psi(x+y) \le \psi(x) + \psi(y)$, then $\psi(x) = x$, $x \in \mathbb{R}$.

Motivated by this problem we returned to study of relations (cf. [2]) between solutions ψ of the inequality

$$\psi(x+y) \le F(\psi(x), \psi(y)), \tag{$F_\le$}$$

and those φ of the equation

$$\varphi(x+y) = F(\varphi(x), \varphi(y)). \tag{$F_=$}$$

The result pertinent to the problem reads

Theorem
Assume that $I \subset \mathbb{R}$ is an open interval and $F\colon I^2 \longrightarrow I$ is a continuous function such that there is an $e \in I$ for which $F(x,e) = x$, $x \in I$. If $\psi\colon \mathbb{R} \longrightarrow I$ is a continuous solution to $(F_\le)$, $\psi(0) = e$, and there is a continuous solution $\varphi\colon \mathbb{R} \longrightarrow I$ of $(F_=)$ such that $\psi(x) \le \varphi(x)$, $x \in \mathbb{R}$, then $\psi = \varphi$.

Comments

1. The general solution of $(F_{\leq})$, continuous in $\mathbb{R}$ has been determined in [1] under the assumption that F is a continuous group operation on I.

2. The condition $\psi(0) = e$ in the Theorem is essential.

3. The Theorem fails to be true when $\psi(x) \geq \varphi(x)$, $x \in \mathbb{R}$.

4. We have found bounds of continuous solutions to $(F_{\leq})$ which are solutions to $(F_{=})$ and as a corollary we have obtained the following condition for a continuous solution ψ of the inequality to satisfy the equation: *There exists a strictly increasing continuous solution f of $(F_{=})$ such that*

$$-\infty < \inf\{\tfrac{1}{t} f^{-1}(\psi(t)),\ t < 0\} = \sup\{\tfrac{1}{t} f^{-1}(\psi(x)),\ x > 0\} < +\infty.$$

[1] B. Choczewski, Z. Powązka, *Generalized subadditivity and convexity*, General Inequalities 2 (ed. by E.F. Beckenbach), Birkhäuser Verlag, Basel, 1980, 185-192.

[2] H. Pawłowski, *Zadania z olimpiad matematycznych z całego świata*, Oficyna Wyd. Tutor, Toruń, 1997.

Bogdan Choczewski

8. Remark. *On the problem of V. Mityushev*

Observe that the equation

$$\varphi(z) = \varphi(sz) + 1$$

has no bounded solution at all. For, if φ is such a solution, then

$$\varphi(z) = \varphi(s^n z) + n \qquad \text{for } n \in \mathbb{N}$$

and the sequence $\varphi(z) - \varphi(s^n z)$ is bounded; a contradiction. It shows that considering bounded solution of equations of the form

$$\varphi(z) = \varphi(sz) + g(z)$$

it is necessary to assume that (for every z from the domain) the sequence $\sum_{k=0}^{n} g(s^k z)$ is bounded.

Janusz Walorski

9. Problem.

Let $I \subset \mathbb{R}$ be an interval and $\lambda\colon I^2 \longrightarrow (0,1)$ be a continuous function. We say that a function $f\colon I \longrightarrow \mathbb{R}$ is λ-convex if

$$f\left(\lambda(x,y)x + (1-\lambda(x,y))y\right) \leq \lambda(x,y)f(x) + (1-\lambda(x,y))f(y), \qquad x, y \in I.$$

1. Does there exist non-constant λ for which λ-convexity imply Jensen convexity?

2. Does λ-convexity imply Jensen convexity for arbitrary continuous λ?

Kazimierz Nikodem

10. Remark.
In the joint paper with A. Matkowska [3] (written in Polish) the following functional equation with unknown f and g is considered

$$A(x,y)+C(x,y)=2B(x,y), \qquad a\le x<y\le b, \tag{E}$$

where $A(x,y)$ stands for the integral of f over the interval $[x, x+\gamma(x,y)]$, $C(x,y)$ — for that over $[y-\gamma(x,y), y]$, and $B(x,y)$ and $\gamma(x,y)$ are the integrals over $[x,y]$ of $f\cdot g$ and g, respectively.

Steffensen's inequality [4] says that

$$A(x,y)\le B(x,y)\le C(x,y) \tag{S}$$

and it was originated by a problem from actuarial mathematics.

We have proved the following theorem, using (cf. [2]) the theory of continuous solutions of linear homogeneous functional equations.

THEOREM
If $g\colon [a,b]\longrightarrow (\frac{1}{2},1)$ *or* $g\colon [a,b]\longrightarrow (0,\frac{1}{2})$ *and* $g\in\mathcal{C}^1[a,b]$, *then the pair* (f,g) *satisfies* (E) *if and only if* f *is a constant function.*

Similar result has been obtained in [1] under other conditions imposed on g, with the aid of a different method.

[1] B. Choczewski, I. Corovei, A. Matkowska, *On some functional equations related to Steffensen's inequality*, Ann. Acad. Paed. Cracoviensis 23, Studia Mathematica **4** (2004), 31-37.

[2] B. Choczewski, M. Kuczma, *On the 'indeterminate case' in the theory of a linear functional equation*, Fund. Math. **58** (1966), 163-175.

[3] B. Choczewski, A. Matkowska, *A functional equation connected with Steffensen's integral inequality applicable in actuarial mathematics* [Polish], Zeszyty Naukowe Akademii Górniczo-Hutniczej w Krakowie. Zagadnienia Techniczno-Ekonomiczne **50** (2005), to appear.

[4] J.F. Steffensen, *On certain inequalities between mean values and their applications to actuarial problems*, Skandinavisk Aktuarietidskrift (1918), 28-97.

Bogdan Choczewski

11. Problem. *Of Matkowski–Sutô type*
Let $\mathcal{M}=\{C_{f,g};\ f,g>0\}$ where

$$C_{f,g}(x,y)=\frac{xf(x)+yf(y)}{f(x)+g(y)}, \qquad x,y>0.$$

Solve the Matkowski–Sutô problem for $\mathcal{M}$, thus determine all the functions F, G, f, g, h, k, such that

$$C_{F,G}\circ(C_{f,g},C_{h,k})=C_{F,G}.$$

Remark
For $F(x) = G(x) = \frac{1}{\sqrt{x}}$, we get $C_{F,G} = \mathcal{G}$ (the geometric mean). We have

$$\mathcal{G}(C_{f,g}, C_{h,k}) = \mathcal{G}$$

if and only if there exists a constant $d > 0$, such that

$$h(x) = \frac{d}{xf(x)}, \qquad x > 0,$$

and

$$k(x) = \frac{d}{yg(y)}, \qquad y > 0.$$

Therefore, the problem has solutions.

Gheorghe Toader

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