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Aggregations in classes of fuzzy relations

Abstract. We consider aggregations of fuzzy relations using aggregation functions of n variables. After recalling fundamental properties of fuzzy relations we examine aggregation functions which preserve reflexivity, symmetry, connectedness and transitivity of fuzzy relations.

1. Introduction

Aggregations of relations are important in the group choice theory (cf. [8]) and multiple-criteria decision making (cf. [14]). Formally, instead of crisp relations we aggregate their characteristic functions. However, the aggregation results appear to be fuzzy relations. Therefore, the most fruitful approach to such aggregations begins with fuzzy relations (cf. [12], [9] or [13]).

Since fuzzy relations have values in $[0, 1]$, for their transformations we use real functions $F\colon [0, 1]^n \longrightarrow [0, 1]$. This leads to new functional equations and functional inequalities connected with the particular properties of fuzzy relations. Usually, the properties are checked each time for concrete assumptions on the form of aggregation functions (cf. e.g. [15]). We shall consider obtained equations without additional assumptions about expected aggregation functions.

We consider the fundamental properties of fuzzy relations during aggregations of finite families of these relations. Firstly, we describe the problem of aggregation of fuzzy relations (Section 2). Next, we describe solutions of functional equations and inequalities connected with: reflexivity (Section 3), symmetry (Section 4), connectedness (Section 5) and transitivity (Section 6) of fuzzy relations. All sections are preceded by suitable definitions of commonly used properties of fuzzy relations.

2. Fuzzy relations

The notion of fuzzy relations is a generalization of that of the characteristic function of crisp relations.

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Definition 1 (Zadeh [17])
Let $X \neq \emptyset$. A fuzzy relation in X is an arbitrary function $R\colon X \times X \longrightarrow [0,1]$. The family of all fuzzy relations in X is denoted by $FR(X)$.

Fuzzy relations form a lattice $(FR(X), \vee, \wedge)$ with the induced partial order

$$R \leqslant S \iff \forall\, x, y \in X \quad R(x,y) \leqslant S(x,y)$$

and with the lattice operations (cf. [17])

$$(R \vee S)(x,y) = \max(R(x,y), S(x,y)),$$
$$(R \wedge S)(x,y) = \min(R(x,y), S(x,y)), \qquad x, y \in X.$$

For $R, S \in FR(X)$ we also use the sup-$\star$ composition of fuzzy relations (cf. [10])

$$(R \circ S)(x,z) = \sup_{y \in X} [R(x,y) \star S(y,z)], \qquad x, z \in X,$$

where $\star\colon [0,1]^2 \longrightarrow [0,1]$ is a binary operation. Case $\star = \min$ is referred to as the standard fuzzy relation composition.

Definition 2 (Fodor [9])
Let $n \geqslant 2$, $F\colon [0,1]^n \longrightarrow [0,1]$, $R_1, \ldots, R_n \in FR(X)$. We define the aggregated fuzzy relation R_F by the formula

$$R_F(x,y) = F(R_1(x,y), \ldots, R_n(x,y)), \qquad x, y \in X. \tag{1}$$

We shall examine properties of the relation (1) under suitable assumptions on fuzzy relations $R_1, \ldots, R_n$. We look for such aggregation functions F which preserve some properties of aggregated fuzzy relations $R_1, \ldots, R_n$. Examples of such properties and appropriate aggregation functions can be found in the papers: [5]-[7] and [14]-[16]. In particular, any projection function

$$P_k(t_1, \ldots, t_n) = t_k\,, \qquad t_1, \ldots, t_n \in [0,1],\ k = 1, \ldots, n \tag{2}$$

preserves arbitrary properties of fuzzy relations, because $R = R_k$ in (1).

3. Reflexivity

At first, we examine the reflexivity properties of the relation (1). Presented definitions of fuzzy relation classes are based on [4], Chapter 5.

Definition 3
A fuzzy relation R is called

reflexive, if $\forall\, x \in X \quad R(x,x) = 1,$ (3)

irreflexive, if $\forall\, x \in X \quad R(x,x) = 0,$ (4)

weakly reflexive, if $\forall\, x \in X \quad R(x,x) > 0,$ (5)

weakly irreflexive, if $\forall\, x \in X \quad R(x,x) < 1.$ (6)

Theorem 1 (cf. [6], Theorem 1)
Let $R_1, \ldots, R_n \in FR(X)$ *be reflexive (resp. irreflexive). The relation* R_F *is reflexive (resp. irreflexive), if and only if the function* F *satisfies the condition* (7) *(resp.* (8)*), where*

$$F(1, \ldots, 1) = 1, \tag{7}$$
$$F(0, \ldots, 0) = 0. \tag{8}$$

Proof. Let $x \in X$. If $F(1, \ldots, 1) = 1$, then we get (3) for R_F whenever $R_1, \ldots, R_n$ are reflexive. Conversely, if $F(1, \ldots, 1) < 1$, then R_F does not fulfil (3). In the case of irreflexive fuzzy relations the proof is similar.

Example 1
Any idempotent function F,

$$F(t, \ldots, t) = t \qquad \text{for } t \in [0, 1] \tag{9}$$

fulfils the conditions (7) and (8).

Theorem 2
The fuzzy relation (1) *is weakly reflexive (resp. weakly irreflexive) for every weakly reflexive (resp. weakly irreflexive)* $R_1, \ldots, R_n \in FR(X)$*, if and only if the function* F *satisfies the condition* (10) *(resp.* (11)*), where*

$$t_1 > 0, \ldots, t_n > 0 \Longrightarrow F(t_1, \ldots, t_n) > 0, \qquad t_1, \ldots, t_n \in [0, 1], \tag{10}$$
$$t_1 < 1, \ldots, t_n < 1 \Longrightarrow F(t_1, \ldots, t_n) < 1, \qquad t_1, \ldots, t_n \in [0, 1]. \tag{11}$$

Proof. Let $x \in X$. If F fulfils (10), then we get (5) for R_F whenever $R_1, \ldots, R_n$ are weakly reflexive. Conversely, if $t_1 > 0, \ldots, t_n > 0$ in $[0, 1]$, then fuzzy relations $R_k \equiv t_k$, $k = 1, \ldots, n$ are weakly reflexive and from the condition (5) for R_F we obtain (10). In the case of weakly irreflexive fuzzy relations the proof is similar.

Example 2
Any increasing, idempotent function F fulfils the conditions (10) and (11) (cf. [9], Proposition 5.1).

Directly from the definition of increasing bijections we get

Lemma 1
If $\varphi \colon [0, 1] \longrightarrow [0, 1]$ *is an increasing bijection, then for every* $s \in [0, 1]$ *we have*

$$\varphi(s) = 0 \Longleftrightarrow s = 0, \qquad \varphi(s) = 1 \Longleftrightarrow s = 1, \tag{12}$$
$$\varphi(s) > 0 \Longleftrightarrow s > 0, \qquad \varphi(s) < 1 \Longleftrightarrow s < 1. \tag{13}$$

Using the above lemma for operations F_φ isomorphic with a given one,

$$F_\varphi(t_1,\dots,t_n)=\varphi^{-1}F(\varphi(t_1),\dots,\varphi(t_n)),\qquad t_1,\dots,t_n\in[0,1],\tag{14}$$

we can generate new transformations fulfilling conditions from Theorems 1 and 2.

Theorem 3
The conditions (7), (8), (10) *and* (11) *are invariant with respect to all increasing bijections, i.e., with any function F fulfilling one of these conditions, also the functions* (14) *fulfil the respective condition.*

Now, we examine the symmetry properties of the relation (1).

Definition 4
A fuzzy relation R is called

symmetric, if $\forall\, x,y\in X \quad R(y,x)=R(x,y)$, (15)

semi-symmetric, if $\forall\, x,y\in X \quad R(x,y)=0 \Longleftrightarrow R(y,x)=0$, (16)

asymmetric, if $\forall\, x,y\in X \quad \min(R(x,y),R(y,x))=0$, (17)

antisymmetric, if $\forall\, x,y\in X, x\neq y \quad \min(R(x,y),R(y,x))=0$, (18)

weakly symmetric, if $\forall\, x,y\in X \quad R(x,y)=1 \Longleftrightarrow R(y,x)=1$, (19)

weakly asymmetric, if $\forall\, x,y\in X \quad \min(R(x,y),R(y,x))<1$, (20)

weakly antisymmetric, if $\forall\, x,y\in X, x\neq y \quad \min(R(x,y),R(y,x))<1$. (21)

Symmetry appears to be the most stable property of fuzzy relations, because immediately we get

Theorem 4 (cf. [6], Theorem 2)
Let $R_1,\dots,R_n\in FR(X)$ be symmetric. For every function F the fuzzy relation R_F is also symmetric.

Definition 5
Let $p\in[0,1]$, $s=(s_1,\dots,s_n)\in[0,1]^n$, $t=(t_1,\dots,t_n)\in[0,1]^n$, $F(t)=F(t_1,\dots,t_n)$. We say that $s,t\in[0,1]^n$ are *p-equivalent* ($s\sim_p t$), if

$$\forall\, 1\leqslant k\leqslant n \quad s_k=p \Longleftrightarrow t_k=p.$$

Theorem 5
Let $\operatorname{card} X\geqslant 2$. *The relation R_F is semi-symmetric (resp. weakly symmetric) for every semi-symmetric (resp. weakly symmetric) $R_1,\dots,R_n\in FR(X)$, if and only if the function F satisfies the condition* (22) *(resp.* (23)*), where*

$$s\sim_0 t\Longrightarrow (F(s)=0\Leftrightarrow F(t)=0)\qquad \text{for } s,t\in[0,1]^n,\tag{22}$$

$$s\sim_1 t\Longrightarrow (F(s)=1\Leftrightarrow F(t)=1)\qquad \text{for } s,t\in[0,1]^n.\tag{23}$$

Proof. Let F fulfil (22), $x, y \in X$. If $R_1, \ldots, R_n \in FR(X)$ are semi-symmetric, then putting

$$s_k = R_k(x,y), \quad t_k = R_k(y,x), \qquad k = 1, 2, \ldots, n \tag{24}$$

we see that $s \sim_0 t$. Thus,

$$\begin{aligned} F(R_1(x,y), \ldots, R_n(x,y)) = 0 &\Leftrightarrow F(s) = 0 \\ &\Leftrightarrow F(t) = 0 \\ &\Leftrightarrow F(R_1(y,x), \ldots, R_n(y,x)) = 0, \end{aligned}$$

which proves (16) for R_F.

Conversely, let $x, y \in X$, $s, t \in [0,1]^n$, $s \sim_0 t$. Since $\operatorname{card} X \geqslant 2$, then there exist $a, b \in X$, $a \neq b$. The fuzzy relations

$$R_k(x,y) = \begin{cases} s_k, & \text{if } (x,y) = (a,b) \\ t_k, & \text{if } (x,y) = (b,a) \\ 1, & \text{otherwise} \end{cases}, \qquad k = 1, \ldots, n,$$

are semi-symmetric. Thus, the relation R_F is also semi-symmetric and we get

$$\begin{aligned} F(s) = 0 &\Leftrightarrow F(R_1(a,b), \ldots, R_n(a,b)) = 0 \\ &\Leftrightarrow F(R_1(b,a), \ldots, R_n(b,a)) = 0 \\ &\Leftrightarrow F(t) = 0, \end{aligned}$$

which proves (22). In the case of weakly symmetric fuzzy relations the proof is similar.

Example 3

There are many operations fulfilling the conditions (22) and (23). For example n-ary $F = \min$, $F = \max$ or the weighted mean:

$$F(t_1, \ldots, t_n) = \sum_{k=1}^{n} w_k t_k, \qquad t, w \in [0,1]^n, \ \textstyle\sum_{k=1}^{n} w_k = 1. \tag{25}$$

In virtue of Lemma 1, also quasilinear means (cf. [1], p. 287):

$$F(t_1, \ldots, t_n) = \varphi^{-1}\left(\sum_{k=1}^{n} w_k \varphi(t_k)\right), \tag{26}$$

fulfil (22), where $\varphi\colon [0,1] \longrightarrow [0,1]$ is an increasing bijection.

THEOREM 6
Let card $X \geqslant 2$. *The relation* R_F *is asymmetric (resp. antisymmetric) for every asymmetric (resp. antisymmetric)* $R_1, \ldots, R_n \in FR(X)$, *if and only if the function* F *satisfies the condition* (27), *where*

$$\forall\, s,t \in [0,1]^n \quad (\forall\, 1 \leqslant k \leqslant n \quad \min(s_k, t_k) = 0) \Longrightarrow \min(F(s), F(t)) = 0. \quad (27)$$

Proof. Let F fulfil (27), $x, y \in X$. If $R_1, \ldots, R_n \in FR(X)$ are asymmetric, then using (24) we see that

$$\forall\, 1 \leqslant k \leqslant n \quad \min(s_k, t_k) = 0 \tag{28}$$

and the relation R_F is asymmetric by (27).

Conversely, let $s, t \in [0,1]^n$ fulfil (28). Since card $X \geqslant 2$, then there exist $a, b \in X$, $a \neq b$. The fuzzy relations

$$R_k(x,y) = \begin{cases} s_k\,, & \text{if } (x,y) = (a,b) \\ t_k\,, & \text{if } (x,y) = (b,a)\,, \\ 0, & \text{otherwise} \end{cases} \qquad k = 1, \ldots, n \tag{29}$$

are asymmetric. Thus, the relation R_F is also asymmetric and we get

$$\begin{aligned} \min(F(s), F(t)) &= \min(F(R_1(a,b), \ldots, R_n(a,b)), F(R_1(b,a), \ldots, R_n(b,a))) \\ &= 0, \end{aligned}$$

which proves (27). In the case of antisymmetric fuzzy relations the proof is similar.

EXAMPLE 4
As the first example of functions fulfilling (27) we can consider $F = \min$. A simple condition sufficient for (27) is connected with zero element $z = 0$ of operation F with respect to a certain coordinate:

$$\exists\, 1 \leqslant k \leqslant n \quad \forall\, i \neq k \quad \forall\, t_i \in [0,1] \quad F(t_1, \ldots, t_{k-1}, 0, t_{k+1}, \ldots, t_n) = 0.$$

In particular, the weighted geometric mean:

$$F(t_1, \ldots, t_n) = \prod_{k=1}^{n} t_k^{w_k}, \qquad t, w \in [0,1]^n,\ \textstyle\sum_{k=1}^n w_k = 1,$$

fulfils (27). As another example we consider the median (cf. [3], p. 21):

$$\operatorname{med}(t_1, \ldots, t_n) = \begin{cases} \dfrac{s_k + s_{k+1}}{2}, & \text{if } n = 2k \\ s_{k+1}\,, & \text{if } n = 2k+1 \end{cases}, \tag{30}$$

where $(s_1, \ldots, s_n)$ is an increasing permutation of $(t_1, \ldots, t_n)$, $(s_1 \leqslant \ldots \leqslant s_n)$.

If a function F fulfils the condition

$$\forall\, t \in [0,1]^n \quad \mathrm{card}\{k:\ t_k = 0\} > \frac{n}{2} \Longrightarrow F(t) = 0, \tag{31}$$

then we also get (27) (e.g. the median (30) fulfils (31)). However, the above condition is not necessary for (27), because it does not cover the projections (2).

Similarly as Theorem 6 we get

Theorem 7
Let $\mathrm{card}\, X \geqslant 2$*. The fuzzy relation* R_F *is weakly asymmetric (resp. weakly antisymmetric) for every weakly asymmetric (resp. weakly antisymmetric)* $R_1, \ldots, R_n \in FR(X)$*, if and only if the function* F *satisfies the condition* (32)*, where*

$$\forall\, s,t \in [0,1]^n \quad (\forall\, 1 \leqslant k \leqslant n \quad \min(s_k, t_k) < 1) \Longrightarrow \min(F(s), F(t)) < 1. \tag{32}$$

Proof. Let F fulfil (32), $x, y \in X$. If $R_1, \ldots, R_n \in FR(X)$ are weakly asymmetric, then using (24) we see that

$$\forall\, 1 \leqslant k \leqslant n \quad \min(s_k, t_k) < 1 \tag{33}$$

and the relation R_F is weakly asymmetric by (32).

Conversely, let $s, t \in [0,1]^n$ fulfil (33). Since $\mathrm{card}\, X \geqslant 2$, then there exist $a, b \in X$, $a \neq b$. Fuzzy relations (29) are weakly asymmetric. Thus, the relation R_F is also weakly asymmetric and we get

$$\begin{aligned} \min(F(s), F(t)) &= \min(F(R_1(a,b), \ldots, R_n(a,b)), F(R_1(b,a), \ldots, R_n(b,a))) \\ &< 1, \end{aligned}$$

which proves (32). In the case of weakly antisymmetric fuzzy relations the proof is similar.

Example 5
As examples of n-ary operations fulfilling (32) we have $F = \min$ and the weighted mean (25).

In virtue of Lemma 1 we get

Theorem 8
The conditions (22)*,* (23)*,* (27) *and* (32) *are invariant with respect to increasing bijections.*

In particular, every quasilinear mean (26) fulfils (32).

4. Connectedness

Next we examine connectedness properties of the relation (1).

Definition 6
A fuzzy relation R is called

$$\begin{aligned} &\textit{connected}, \text{ if } \quad \forall\, x, y \in X, x \neq y \quad \max(R(x,y), R(y,x)) = 1, && (34)\\ &\textit{totally connected}, \text{ if } \quad \forall\, x, y \in X \quad \max(R(x,y), R(y,x)) = 1, && (35)\\ &\textit{weakly connected}, \text{ if } \quad \forall\, x, y \in X, x \neq y \quad \max(R(x,y), R(y,x)) > 0, && (36)\\ &\textit{weakly totally connected}, \text{ if } \quad \forall\, x, y \in X \quad \max(R(x,y), R(y,x)) > 0. && (37) \end{aligned}$$

The above definitions are very similar to those considered in Definition 4. This similarity can be described by the use of the complement R' of fuzzy relation R:

$$R'(x,y) = 1 - R(x,y), \qquad x, y \in [0,1].$$

Lemma 2
A fuzzy relation R is asymmetric (resp. antisymmetric, weakly asymmetric, weakly antisymmetric), if and only if its complement is totally connected (resp. connected, weakly totally connected, weakly connected).

In virtue of this lemma conditions for aggregated connected fuzzy relations can be obtained by negation of conditions considered above for aggregated asymmetric and antisymmetric fuzzy relations.

Similarly as Theorem 6 we get

Theorem 9
Let $\operatorname{card} X \geqslant 2$. *The relation R_F is connected (resp. totally connected) for every connected (resp. totally connected) $R_1, \ldots, R_n \in FR(X)$, if and only if the function F satisfies the condition* (38), *where*

$$\forall\, s, t \in [0,1]^n \ \ (\forall\, 1 \leqslant k \leqslant n \quad \max(s_k, t_k) = 1) \Longrightarrow \max(F(s), F(t)) = 1. \quad (38)$$

Example 6
As examples of functions fulfilling (38) we can consider $F = \max$, $F = \operatorname{med}$ or operations F with neutral element $z = 1$ with respect to a certain coordinate:

$$\exists\, 1 \leqslant k \leqslant n \ \ \forall\, i \neq k \ \ \forall\, t_i \in [0,1] \ \ F(t_1, \ldots, t_{k-1}, 1, t_{k+1}, \ldots, t_n) = 1.$$

Now a dual property for (31) have the form:

$$\forall\, t \in [0,1]^n \ \ \operatorname{card}\{k : \ t_k = 1\} > \frac{n}{2} \Longrightarrow F(t) = 1. \quad (39)$$

Similarly as Theorem 7 we get

Theorem 10
Let $\operatorname{card} X \geqslant 2$*. The fuzzy relation* R_F *is weakly connected (resp. weakly totally connected) for every weakly connected (resp. weakly totally connected)* $R_1, \ldots, R_n \in FR(X)$*, if and only if the function* F *satisfies the condition* (40)*, where*

$$\forall\, s,t \in [0,1]^n \ \ (\forall\, 1 \leqslant k \leqslant n \quad \max(s_k,t_k) > 0) \Longrightarrow \max(F(s),F(t)) > 0. \quad (40)$$

Example 7
As examples of operations fulfilling (40) we have $F = \max$ and the weighted mean (25).

In virtue of Lemma 1 we get

Theorem 11
The conditions (38)*,* (40) *are invariant with respect to increasing bijections.*

In particular, every quasilinear mean (26) fulfils (40).

5. Transitivity

Finally, we examine transitivity properties of the relation (1).

Definition 7 (cf. [2])
Let $\star\colon [0,1]^2 \longrightarrow [0,1]$ be a binary operation. A fuzzy relation R is called

$$\star\text{-}transitive\text{, if} \quad \forall\, x,y,z \in X \quad R(x,y) \star R(y,z) \leqslant R(x,z), \quad (41)$$

$$transitive\text{, if} \quad \forall\, x,y,z \in X \quad \min(R(x,y),R(y,z)) \leqslant R(x,z). \quad (42)$$

Definition 8 (cf. [11])
Binary operation $\star$ in $[0,1]$ is said to be a *triangular norm*, if it is increasing, associative, commutative and with the neutral element $e = 1$.

In particular, the Łukasiewicz multivalued conjunction

$$T_L(u,v) = \max(u+v-1,0), \qquad u,v \in [0,1]$$

is a triangular norm. The case of transitivity was discussed in details in [16].

Theorem 12 (Saminger et al. [16], Theorem 3.1)
Let $\operatorname{card} X \geqslant 3$*,* $\star$ *be a triangular norm and function* $F\colon [0,1]^n \longrightarrow [0,1]$ *be increasing with respect to the induced order in* $[0,1]^n$*, i.e.,*

$$s_k \leqslant t_k,\ k = 1,\ldots,n \Longrightarrow F(s_1,\ldots,s_n) \leqslant F(t_1,\ldots,t_n).$$

The relation R_F *is* $\star$*-transitive for every* $\star$*-transitive* $R_1,\ldots,R_n \in FR(X)$*, if and only if the function* F *dominates the operation* $\star$*, i.e.,*

$$\forall\, s,t \in [0,1]^n \quad F(s_1 \star t_1, \ldots, s_n \star t_n) \geqslant F(s_1,\ldots,s_n) \star F(t_1,\ldots,t_n). \quad (43)$$

Example 8
The main example of domination for $\star = \min$ is $F = \min$ (cf. [16], Proposition 5.1). Thus $F = \min$ preserves min-transitivity of fuzzy relations.

Example 9
Saminger et al. [16] presented some examples of aggregating functions preserving T_L-transitivity. In particular any weighted mean (25) preserves T_L-transitivity of fuzzy relations.

Let us observe that condition (43) is not invariant with respect to increasing bijections.

Example 10
Let $n = 2$, $\operatorname{card} X = 3$, $\varphi(x) = x^2$, $x \in [0,1]$. From the above example we know that the arithmetic mean $F(u,v) = \frac{u+v}{2}$, $u, v \in [0,1]$ dominates T_L. However, the operation $F_\varphi(u,v) = \sqrt{\frac{u^2+v^2}{2}}$, $u, v \in [0,1]$ does not dominate T_L. For $u = 0.9$, $v = 0.1$, $w = 0.8$, $z = 0.2$ it can be verified that

$$\sqrt{\frac{\max(u+v-1,0)^2 + \max(w+z-1,0)^2}{2}} < \max\left(\sqrt{\frac{u^2+w^2}{2}} + \sqrt{\frac{v^2+z^2}{2}} - 1, 0\right),$$

contrary to (43).

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