Annales Academiae Paedagogicae Cracoviensis
Folia 33 Studia Mathematica V (2006)

Artur Figurski

A theorem on divergence of Fourier series

Abstract. The paper contains an extension of Calderon's work [2] on the optimality of the Dini test for Fourier series in a set of positive measure.

1. Introduction

Let us begin with recalling the classical Dini's result:

Proposition
If $\delta \in (0, \pi)$ *and* $f \in L^1_{2\pi}$ *satisfies at the point* x *the following condition*

$$\int_{-\delta}^{\delta} \frac{|f(x) - f(x-t)|}{|t|}\, dt < \infty, \tag{1}$$

then the sequence of partial sums of the Fourier expansion of f *at* x *is convergent to* $f(x)$.

The aim of this paper is to supplement the idea of Calderon [2] in which the author proves that condition (1) is optimal for x belonging to some set E of positive measure: $|E| > 0$. He also studies a weaker condition than (1), which is now recalled.

Condition (W)
The function $w\colon [0,1) \longrightarrow R$ *is continuous and increasing in* $[0,\delta), w(0) = 0$, *and*

$$\int_{0}^{\delta} \frac{w(t)}{t}\, dt = \infty. \tag{2}$$

We aim at proving the following

Theorem
Suppose that Condition (W) *is satisfied. Then there exists a function* $g \in L[0, 2\pi)$ *and a set of positive measure* F *such that*

AMS (2000) Subject Classification: 42A20.

$$\int_{-\delta}^{\delta} \frac{|g(x+t)-g(x)|}{|t|} w(|t|)\,dt < \infty \qquad \text{for } x \in F \tag{3}$$

and the sequence of partial sums of the Fourier expansion of g is divergent almost everywhere in F.

2. Auxiliary results

Before giving the proof of the Theorem we recall some basic results.

Marcinkiewicz's Theorem ([2], p. 382)
Let φ be a continuous, increasing function, defined on $[0,2\pi]$ such that $\varphi(0)=0$ and

$$[\varphi(t)]^{-1} = o\left(\ln\frac{1}{t}\right), t \to 0^+. \tag{4}$$

Then there exists a function $f \in L^1[0,2\pi]$ satisfying

$$\frac{1}{|h|}\int_0^h |f(x+t)-f(x)|\,dt = O(\varphi(|h|)), \quad |h| \to 0, \tag{5}$$

for almost every $x \in [0,2\pi]$, and the sequence of partial sums of the Fourier expansion of f is divergent almost everywhere.

Lemma 1
If

$$\int_0^{2\pi} dx \int_0^{\pi} \frac{[g(x+t)-g(x-t)]^2}{t}\,dt < \infty,$$

then the sequence of partial sums of the Fourier expansion of g is convergent almost everywhere.

Lemma 2 ([1], p. 383)
Let φ be given in the form

$$[\varphi(t)]^{-1} = \int_t^1 \frac{w(s)}{s}\,ds, \qquad 0<t<1,$$

where w obeys Condition (W). If $f \in L[0,2\pi]$ satisfies the asymptotic condition (5) almost everywhere in $[0,2\pi]$, then for each $\varepsilon > 0$ there is a perfect subset F of $[0,2\pi]$ and a constant C such that:

$$|F| > 2\pi - \varepsilon,$$

$$|f(x_1) - f(x_2)| \le C\varphi(|x_1 - x_2|), \qquad x_1, x_2 \in F,\ 0 < |x_1 - x_2| < \frac{1}{2},$$

$$\frac{1}{|h|}\int_0^h |f(x+t) - f(x)|\,dt < C\varphi(|h|) \qquad \textit{for } x \in F.$$

3. Proof of the Theorem

Consider the function

$$\bar{\varphi}(t) = \left[\int_t^1 \frac{\bar{w}(s)}{s}\,ds\right]^{-1},$$

where $0 < t < \frac{1}{2}$, $\bar{w}(s) = \max\{|\ln\ s|^{-\delta}, w(s)\}$, $0 < \delta < \frac{1}{4}$, $0 < s < \frac{1}{2}$, and w satisfies Condition (W). From the definition of $\bar{\varphi}$ we have condition (4) for $\bar{\varphi}$. Hence, the assumptions of Marcinkiewicz's Theorem are fulfilled for $\bar{\varphi}$. Then there exists a function f satisfying (4).

Let $\varepsilon > 0$, F be a perfect subset of $[0, 2\pi]$ and C be a constant (see Lemma 2). Denote by $\bar{f}$ any continuous extension of f from F to $[0, 2\pi]$ such that

$$|\bar{f}(x_1) - \bar{f}(x_2)| \le C\bar{\varphi}(|x_1 - x_2|), \qquad x_1, x_2 \in [0, 2\pi],\ |x_1 - x_2| < \frac{1}{2}. \tag{6}$$

Let us define $g = f - \bar{f}$, hence $f = \bar{f} + g$.

Consider the double integral

$$J = \int_F \left(\int_0^{2\pi} |g(x) - g(y)|\frac{\bar{w}(|x-y|)}{|x-y|}\,dy\right)dx.$$

Since $g(u) = 0$ for any $u \in F$, then

$$J = \int_F \left(\int_G |g(y)|\frac{\bar{w}(|x-y|)}{|x-y|}\,dy\right)dx, \tag{7}$$

where $G = [0, 2\pi] \setminus F$.

We shall construct intervals I_k such that

$$G = \bigcup_{k=1}^{\infty} I_k\,,$$

where $\forall\, i, j \in N,\ i \ne j:\ I_i^0 \cap I_j^0 = \emptyset,\ I_i^0 = \operatorname{int} I_i$ and the distance $d(I_k, F)$ satisfies the inequalities

$$|I_k| \leq d(I_k, F) \leq 2|I_k| \qquad \text{for } k = 1, 2, \dots .$$

Let F_1, F_2 be subsets of F. For the simplicity, let us assume that the part of G lying between F_1 and F_2 is of length 1.

Let x be a point in G such that $d(x, F_1) = d(x, F_2)$. Let us define the intervals I_k as follows:

1) $I_1 = [x - \frac{1}{6}, x + \frac{1}{6}]$, $|I_1| = \frac{1}{3} = d(I_1, F) \leq \frac{2}{3} = 2|I_1|$;

2) I_1, I_2 are closed intervals, symmetric with respect to x and such that the right end of I_2 equals the left end of I_1 and moreover $|I_2| = d(F, I_2) \leq \frac{1}{6} = 2|I_2|$. I_3 has analogical properties.

n) At the n-th step we define I_{2n-2} , I_{2n-1} to be closed intervals, symmetric with respect to x and such that the right end of I_{2n-2} equals the left end of I_{2n-4} and moreover

$$|I_{2n-2}| = d(I_{2n-2}, F) = \left(\frac{1}{2}\right)^{n-2} \frac{1}{6} \leq 2|I_{2n-2}| = \left(\frac{1}{2}\right)^{n-2} \frac{2}{6}.$$

I_{2n-1} has analogical properties.

It is clear that $\sum_{k=1}^{\infty} |I_k| = 1$. Since I_k are closed, $\bigcup_{k=1}^{\infty} I_k$ fils totally the gap between F_1 and F_2. Because there is a countable number of gaps between the particular parts of F so we obtain a countable number of intervals I_k . Let us arrange all the intervals I_k in a sequence.

From the construction above it is seen that actually we have

$$\forall\, i, j \in N,\ i \neq j:\ I_i^0 \cap I_j^0 = \emptyset, \qquad \bigcup_{k=1}^{\infty} I_k = G.$$

We can assume that $\bar{f}(c) = f(c)$ for c being the midpoint of I_k . Such possible modification will not change the properties of f. Then

$$\begin{aligned}
\int\limits_{I_k} |g(y)|\, dy &= \int\limits_{I_k} |f(y) - \bar{f}(y)|\, dy \\
&\leq \int\limits_{I_k} |\bar{f}(c) - \bar{f}(y)|\, dy + \int\limits_{I_k} |f(y) - f(c)|\, dy + \int\limits_{I_k} |f(c) - \bar{f}(y)|\, dy \\
&= 2 \int\limits_{I_k} |\bar{f}(c) - \bar{f}(y)|\, dy + \int\limits_{I_k} |f(c) - f(y)|\, dy.
\end{aligned}$$

Applying (6) we have

$$\int\limits_{I_k} |\bar{f}(c) - \bar{f}(y)|\, dy \leq \int\limits_{I_k} C\varphi(|c - y|)\, dy \leq C|I_k|\varphi(|I_k|),$$

and by using Marcinkiewicz's Theorem we can estimate

$$\int_{I_k} |f(c) - f(y)|\, dy = \int_0^{|I_k|} |f(c+t) - f(c)| dt = |I_k| \cdot O(\varphi(|I_k|)).$$

Finally,

$$\int_{I_k} |g(y)|\, dy \le 2C \cdot \varphi(|I_k|) \cdot |I_k| + |I_k| \cdot O(\varphi(|I_k|)) = K_0 \cdot \varphi(|I_k|) \cdot |I_k|, \tag{8}$$

for any k, with a suitable constant K_0.

Using (8) in (7), we obtain

$$\begin{aligned} J &= \sum_{k=1}^{\infty} \int_{I_k} |g(y)| \left(\int_F \frac{\bar{w}(|x-y|)}{|x-y|}\, dx \right) dy \\ &< K_1 \sum_{k=1}^{\infty} \left(\int_{I_k} |g(y)|\, dy \right) [\varphi(|I_k|)]^{-1} < K_2 \sum_{k=1}^{\infty} |I_k|, \end{aligned}$$

provided that

$$\int_F \frac{\bar{w}(|x-y|)}{|x-y|}\, dx \le K_3 \cdot [\varphi(|I_k|)]^{-1}, \tag{9}$$

where K_1, K_2, K_3 are some constants. To prove (9) take $y \in I_k$, $x \in F$, such that $|x-y| \ge |I_k|$. Then

$$\int_F \frac{\bar{w}(|x-y|)}{|x-y|}\, dx \le |I_k|^{-1} \int_F \bar{w}(|x-y|)\, dx.$$

Let $|x-y| = s$. Since $\bar{w}(s) = 0$ for $s \in (1, 2\pi]$, we obtain

$$\begin{aligned} |I_k|^{-1} \int_F \bar{w}(|x-y|)\, dx &\le \int_{|I_k|}^{2\pi} \frac{\bar{w}(s)}{|I_k|}\, ds \le \int_{|I_k|}^{2\pi} \frac{\bar{w}(s)}{s}\, ds \\ &= \int_{|I_k|}^{1} \frac{\bar{w}(s)}{s}\, ds = [\varphi(|I_k|)]^{-1}. \end{aligned}$$

From Fubini's theorem it follows

$$\int_{-\delta}^{\delta} |g(x) - g(x+t)| \frac{\bar{w}(|t|)}{|t|}\, dt < \infty \tag{10}$$

for almost each x of F. The condition (10) is also satisfied for function w because $w(t) \leq \bar{w}(t)$. The Fourier series of f is divergent almost everywhere in F (see Marcinkiewicz's Theorem and the above construction of f). The Fourier series of $\bar{f}$ is convergent almost everywhere because

$$\int_0^{2\pi}\int_0^{2\pi} |\bar{f}(x) - \bar{f}(y)|^2 \frac{1}{|x-y|}\, dxdy < \infty$$

(see Lemma 1). If $g = f - \bar{f}$, so the Fourier series of g is divergent almost everywhere. This proves the theorem.

Remark

The apparent inconsistency of convergence of the Fourier series $\bar{f}$ almost everywhere and divergence of the Fourier series f almost everywhere in F ($|F| > 0$) results from the fact that set F contains no intervals.

References

[1] N.A. Bary, *Treatise on Trigonometric Series*, Vol. I, Pergamon Press, Inc., NY, 1964.

[2] C.P. Calderon, *On the Dini test and divergence of Fourier series*, Proceedings of the American Mathematical Society **82**, no. 3, (1981), 382-384.

Faculty of Applied Mathematics
AGH Univeristy of Science and Technology
Al. Mickiewicza 30
PL-30-059 Kraków
Poland
e-mail: figurski@wms.mat.agh.edu.pl