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Some properties of convex and $*$-concave multifunctions

Abstract. We investigate some properties of $*$-concave and convex multifunctions on the real line with convex bounded closed values. In particularly we consider the Hadamard inequality and the Hardy–Littlewood–Pólya majorization theory in the case of multifunctions.

1. Basic definitions

Let X be a real Banach space. Denote by $clb(X)$ the set of all nonempty bounded closed convex subsets of X. For given $A, B \in clb(X)$ and $\lambda \geq 0$ we define $A + B = \{a + b : \ a \in A, \ b \in B\}$, $\lambda A = \{\lambda a : \ a \in A\}$,

$$A \overset{*}{+} B = cl(A + B) = cl(clA + clB).$$

The structure $(clb(X), \overset{*}{+})$ is an Abelian semigroup with the neutral element $\{0\}$. It is clear that

$$\lambda(A \overset{*}{+} B) = \lambda A \overset{*}{+} \lambda B, \quad (\lambda + \mu)A = \lambda A \overset{*}{+} \mu A, \quad \lambda(\mu A) = \lambda\mu A, \quad 1 \cdot A = A$$

for all $\lambda, \mu \geq 0$ and $A, B \in clb(X)$. Thus the triple $(clb(X), \overset{*}{+}, \cdot)$ is also an abstract convex cone (for definition see e.g. [11]). Since

$$A \overset{*}{+} C = B \overset{*}{+} C \implies A = B$$

(cf. [11]), the cancellation law is satisfied.

Let d be the Hausdorff metric in $clb(X)$ derived from the norm $||\cdot||$ in X, i.e. $d(A, B) = \max\{e(A, B), e(B, A)\}$, where $e(A, B) = \sup_{a\in A} \rho(a, B)$ and $\rho(a, B) = \inf_{b\in B} ||a - b||$ for $A, B \in clb(X)$. For given $A \in clb(X)$ we define $||A|| = \sup\{||a|| : \ a \in A\} = d(A, \{0\})$. The metric space $(clb(X), d)$ is complete (see e.g. [1, Theorem II-3, p. 40]). Moreover d is translation invariant since

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$$d(A \overset{*}{+} C, B \overset{*}{+} C) = d(A + C, B + C) = d(A, B)$$

and positively homogeneous

$$d(\lambda A, \lambda B) = \lambda d(A, B)$$

for all $A, B, C \in clb(X)$ (cf. [2, Lemma 2.2]).

A multifunction $F\colon [a, b] \longrightarrow clb(X)$ is said to be *$*$-concave ($*$-convex)* if

$$F(\lambda x + (1 - \lambda)y) \subset \lambda F(x) \overset{*}{+} (1 - \lambda)F(y),$$
$$\left(\lambda F(x) \overset{*}{+} (1 - \lambda)F(y) \subset F(\lambda x + (1 - \lambda)y) \right)$$

for all $x, y \in [a, b]$ and $\lambda \in (0, 1)$.

Remark 1

The concavity of multifunctions, defined as follows,

$$F(\lambda x + (1 - \lambda)y) \subset \lambda F(x) + (1 - \lambda)F(y), \qquad x, y \in [a, b],\ \lambda \in (0, 1)$$

implies the $*$-concavity, but not conversely. To see this we consider two sets $A, B \in clb(X)$ such that $A + B \neq cl(A + B)$ (an example could be found in [10, pp. 712-713]) and the multifunction $F\colon [0, 1] \longrightarrow clb(X)$ given by the formula $F(t) = tA \overset{*}{+} (1 - t)B$. It is easy to check that $F(\lambda t + (1 - \lambda)s) \subset \lambda F(t) \overset{*}{+} (1 - \lambda)F(s)$ for all $t, s \in [0, 1]$ and $\lambda \in (0, 1)$ but

$$F\left(\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1\right) = F\left(\frac{1}{2}\right) = \frac{1}{2}(A \overset{*}{+} B) \not\subset \frac{1}{2}(A + B) = \frac{1}{2}[F(0) + F(1)].$$

Remark 2

A multifunction $F\colon [a, b] \longrightarrow clb(X)$ is $*$-convex if and only if it is convex i.e.,

$$\lambda F(x) + (1 - \lambda)F(y) \subset F(\lambda x + (1 - \lambda)y), \qquad x, y \in [a, b],\ \lambda \in (0, 1).$$

We note that every convex multifunction with non-empty values has convex values. Indeed, $\lambda F(x) + (1 - \lambda)F(x) \subset F(x)$ for all $\lambda \geq 0$ and $x \in [a, b]$.

A multifunction $F\colon [a, b] \longrightarrow clb(X)$ is said to be *increasing* if $F(x) \subset F(y)$ whenever $x, y \in [a, b]$ and $x < y$.

A set $\Delta = \{y_0, y_1, \ldots, y_n\}$, where $a = y_0 < y_1 < \ldots < y_n = b$, is said to be a *partition* of $[a, b]$. For given partition Δ we set $\delta(\Delta) = \max\{y_i - y_{i-1}\colon\ i = 1, \ldots, n\}$. For the partition Δ and for a system $\tau = (\tau_1, \ldots, \tau_n)$ of intermediate points $\tau_i \in [y_{i-1}, y_i]$ we create the *Riemann sum*

$$S(\Delta, \tau) = (y_1 - y_0)F(\tau_1) \overset{*}{+} \cdots \overset{*}{+} (y_n - y_{n-1})F(\tau_n).$$

If for every sequence (Δ^ν) of partitions $\Delta^\nu = \left\{y_0^\nu, y_1^\nu, \ldots, y_{n_\nu}^\nu\right\}$ of $[a,b]$ such that $\lim_{\nu\to\infty}\delta(\Delta^\nu)=0$, and for every sequence (τ^ν) of systems of intermediate points, the sequence of the Riemann sums $(S(\Delta^\nu,\tau^\nu))$ tends to the same limit $I \in clb(X)$, then F is said to be *Riemann integrable* over $[a,b]$ and $\int_a^b F(y)\,dy := I$.

The Riemann integral for multifunction with compact convex values was investigated by A. Dinghas [3] and M. Hukuhara [4]. Some properties of Riemann integral of multifunctions with convex closed bounded values may be found in paper [8].

2. Hadamard inequality in case of multifunctions

We believe that the following theorem is known. Nevertheless we prove it for convenience of the reader.

THEOREM 1
Every $*$-concave multifunction $F\colon [a,b] \longrightarrow clb(X)$ is continuous on (a,b) with respect to the Hausdorff metric.

Proof. Since all values of F are bounded we may find a constant $M>0$ such that $||\lambda F(a) \overset{*}{+} (1-\lambda)F(b)|| \le M$ for $\lambda \in [0,1]$. Thus by $*$-concavity of F we have $||F(x)|| \le M$, $x \in [a,b]$.

Let us fix $x_0 \in (a,b)$ and let x be a point belonging to the interval (x_0,b). There exist $\lambda,\mu \in (0,1)$ such that $x = \lambda x_0 + (1-\lambda)b$ and $x_0 = \mu x + (1-\mu)a$. Hence $\lambda = \frac{x-b}{x_0-b} \to 1^-$ and $\mu = \frac{x_0-a}{x-a} \to 1^-$ as $x \to x_0^+$. Then by the $*$-concavity we obtain

$$\begin{aligned} e(F(x),F(x_0)) &\le e(\lambda F(x_0) \overset{*}{+} (1-\lambda)F(b), F(x_0)) \\ &\le d(\lambda F(x_0) \overset{*}{+} (1-\lambda)F(b), \lambda F(x_0) \overset{*}{+} (1-\lambda)F(x_0)) \\ &= (1-\lambda)d(F(b),F(x_0)) \end{aligned}$$

and

$$\begin{aligned} e(F(x_0),F(x)) &\le e(\mu F(x) \overset{*}{+} (1-\mu)F(a), F(x)) \\ &\le d(\mu F(x) \overset{*}{+} (1-\mu)F(a), \mu F(x) \overset{*}{+} (1-\mu)F(x)) \\ &= (1-\mu)d(F(a),F(x)) \\ &\le 2M(1-\mu), \end{aligned}$$

whence $d(F(x),F(x_0)) \to 0$ as $x \to x_0^+$. We have shown that F is right-hand side continuous at x_0. The similar argument can be used to get the left-hand side continuity of F at x_0.

Remark 3
A $*$-concave multifunction on $[a,b]$ need not be continuous. To see this it is enough to take $F\colon[0,1]\longrightarrow clb(\mathbb{R})$ defined by

$$F(x)=\begin{cases}\{0\}, & x>0,\\ [0,1], & x=0.\end{cases}$$

The continuity of convex multifunctions can be obtained from Theorem 3.7 in [7]. We give here an independent, straightforward proof similar to that of Theorem 1.

Theorem 1′
Every convex multifunction $F\colon[a,b]\longrightarrow clb(X)$ is continuous on (a,b) with respect to the Hausdorff metric and bounded on $[a,b]$.

Proof. At first we will prove that F is bounded on $[a,b]$. We observe that for every $x\in[a,\frac{a+b}{2}]$ there exists $\lambda\in[\frac{1}{2},1]$ such that $\lambda x+(1-\lambda)b=\frac{a+b}{2}$. Let us fix $u\in F(b)$. The convexity of F yields

$$\lambda F(x)+(1-\lambda)u\subset F\left(\frac{a+b}{2}\right),$$

whence

$$F(x)\subset\frac{1}{\lambda}F\left(\frac{a+b}{2}\right)-\left(\frac{1}{\lambda}-1\right)u.$$

Thus F is bounded on $[a,\frac{a+b}{2}]$. In similar manner we show that F is bounded on $[\frac{a+b}{2},b]$. Consequently there is a constant M such that $||F(x)||\le M$ for $x\in[a,b]$.

Let us fix x_0 belonging to (a,b) and let $x_0<x<b$. We can find $\lambda,\mu\in(0,1)$ such that $x=\lambda x_0+(1-\lambda)b$, $x_0=\mu x+(1-\mu)a$. Clearly $\lambda F(x_0)+(1-\lambda)F(b)\subset F(x)$ and $\mu F(x)+(1-\mu)F(a)\subset F(x_0)$. We note that $\lambda,\mu\to1^-$ as $x\to x_0^+$. By the convexity of F and properties of e we obtain two inequalities

$$\begin{aligned}\lambda e(F(x_0),F(x))&=e(\lambda F(x_0),\lambda F(x))\\&=e(\lambda F(x_0)+(1-\lambda)F(b),\lambda F(x)+(1-\lambda)F(b))\\&\le e(F(x),\lambda F(x)+(1-\lambda)F(b))\\&\le d(F(x),\lambda F(x)+(1-\lambda)F(b))\\&=(1-\lambda)d(F(x),F(b))\\&\le 2M(1-\lambda),\end{aligned}$$

$$\begin{aligned}e(F(x),F(x_0))&=\sup_{v\in F(x)}\rho(v,F(x_0))\\&\le\sup_{v\in F(x)}\rho(v,\mu F(x)+(1-\mu)F(a))\end{aligned}$$

$$
\begin{aligned}
&= e(F(x), \mu F(x) + (1-\mu)F(a)) \\
&\le d(F(x), \mu F(x) + (1-\mu)F(a)) \\
&= (1-\mu) d(F(x), F(a)) \\
&\le 2M(1-\mu).
\end{aligned}
$$

Consequently

$$\lim_{x \to x_0^+} d(F(x), F(x_0)) = 0.$$

The left continuity of F at x_0 may be shown analogously.

A continuous multifunction $F\colon [a,b] \longrightarrow clb(X)$ is Riemann integrable on $[a,b]$ (cf. [8]). A $*$-concave multifunction on $[a,b]$ is commonly bounded on this interval. Therefore it is not difficult to see that a $*$-concave multifunction has to be Riemann integrable on each $[c,d] \subset [a,b]$ (cf. [8]).

In the case of convex functions on $[a,b]$ the following Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}$$

is well known (cf. [5, pp. 196-197]). We are going to deal with suitable inclusion for convex and $*$-concave multifunctions.

Theorem 2
If $F\colon [a,b] \longrightarrow clb(X)$ is $*$-concave multifunction, then

$$F\left(\frac{x+y}{2}\right) \subset \frac{1}{y-x}\int_x^y F(t)\,dt \subset \frac{F(x) \overset{*}{+} F(y)}{2} \tag{1}$$

for each x, y such that $x < y$ and $[x,y] \subset [a,b]$.

Proof. Let us fix $n \in \mathbb{N}$ and let $x_i = x + i\frac{y-x}{n}$ and $\tau_i = x + \frac{2i-1}{2n}(y-x)$ for $i \in \{1, \dots, n\}$. These points create the partition $\Delta_n = \{x, x_1, \dots, x_{n-1}, y\}$ of the interval $[x,y]$ and $\tau = (\tau_1, \dots, \tau_n)$ is a system of intermediate points. We note that

$$\tau_i = \frac{x_{i-1}+x_i}{2} = \frac{[2n-(2i-1)]x + (2i-1)y}{2n}.$$

Using the $*$-concavity of F we obtain

$$F(\tau_i) \subset \frac{2n-(2i-1)}{2n}F(x) \overset{*}{+} \frac{2i-1}{2n}F(y)$$

for $i \in \{1, \dots, n\}$. Summing up over i we get

$$
\begin{aligned}
&F(\tau_1) \overset{*}{+} \dots \overset{*}{+} F(\tau_n) \\
&\quad\subset \left(\frac{2n-1}{2n} + \frac{2n-3}{2n} + \dots + \frac{1}{2n}\right) F(x) \overset{*}{+} \left(\frac{1}{2n} + \frac{3}{2n} + \dots + \frac{2n-1}{2n}\right) F(y).
\end{aligned}
$$

Since $1+3+\cdots+(2n-1)=n^2$, we obtain

$$\frac{1}{y-x}\big[F(\tau_1)\overset{*}{+}\cdots\overset{*}{+}F(\tau_n)\big]\frac{y-x}{n}\subset\frac{F(x)\overset{*}{+}F(y)}{2}. \tag{2}$$

Now we let $n\to\infty$. Then $\delta(\Delta_n)\to 0$ and with respect to the definition of the integral, by (2) and by the closedness of the set $\frac{1}{2}(F(x)\overset{*}{+}F(y))$ we have

$$\frac{1}{y-x}\int_x^y F(t)\,dt\subset\frac{F(x)\overset{*}{+}F(y)}{2}.$$

To obtain the first inclusion of (1) we take an even positive integer n. Let $k=n/2$ and let us choose $i\in\{1,\dots,k\}$. We note that $\frac{1}{2}(\tau_i+\tau_j)=\frac{1}{2}(x+y)$ for $j=n+1-i$. Again by the $*$-concavity of F we infer

$$F\left(\frac{x+y}{2}\right)\subset\frac{1}{2}(F(\tau_i)\overset{*}{+}F(\tau_j)).$$

Summing up over $i\in\{1,\dots,k\}$ leads to

$$kF\left(\frac{x+y}{2}\right)\subset k\big[F(\tau_1)\overset{*}{+}\cdots\overset{*}{+}F(\tau_k)\overset{*}{+}F(\tau_{k+1})\overset{*}{+}\cdots\overset{*}{+}F(\tau_n)\big]$$

or

$$F\left(\frac{x+y}{2}\right)\subset\frac{1}{y-x}\big[F(\tau_1)\overset{*}{+}\cdots\overset{*}{+}F(\tau_n)\big]\cdot\frac{y-x}{n} \tag{3}$$

for all even n. The right-hand side of inclusion (3) tends to $\frac{1}{y-x}\int_x^y F(t)\,dt$ as $n\to\infty$. Hence

$$F\left(\frac{x+y}{2}\right)\subset\frac{1}{y-x}\int_x^y F(t)\,dt.$$

The proof of the next theorem runs similarly.

Theorem $2'$

If $F\colon[a,b]\longrightarrow clb(X)$ *is a convex multifunction, then the following inclusions hold*

$$\frac{F(x)\overset{*}{+}F(y)}{2}\subset\frac{1}{y-x}\int_x^y F(t)\,dt\subset F\left(\frac{x+y}{2}\right) \tag{4}$$

for all intervals $[x,y]\subset[a,b]$.

Inclusions (4) for the Aumann integral may be found in the paper of E. Sadowska [9, Theorem 1], where the integral Jensen inequality is applied (see the paper of J. Matkowski and K. Nikodem [6]). The assumptions of Theorem $2'$ differ somewhat from that of Theorem 1 in [9].

3. Hardy-Littlewood-Pólya majorization theorem for multifunctions

In this part of the note we are going to transfer the Hardy–Littlewood–Pólya majorization principle for convex functions (cf. [5, ch. 8, § 5]) to convex and $*$-concave multifunctions.

Theorem 3
Let x_1, x_2, y_1, y_2 be real numbers such that $x_2 \le x_1$, $y_2 \le y_1$, $x_1 \le y_1$, $x_1 + x_2 = y_1 + y_2$. If $F\colon \mathbb{R} \longrightarrow clb(X)$ is $*$-concave, then

$$F(x_1) \overset{*}{+} F(x_2) \subset F(y_1) \overset{*}{+} F(y_2). \tag{5}$$

Proof. The assumptions of the theorem imply the inequality $y_2 \le x_2 \le x_1 \le y_1$. At first we assume that $y_1 \ne y_2$. Setting $\lambda = \frac{y_1 - x_2}{y_1 - y_2}$, $\mu = \frac{y_1 - x_1}{y_1 - y_2}$ by the $*$-concavity we have

$$\begin{aligned} F(x_2) &= F(\lambda y_2 + (1-\lambda) y_1) \subset \lambda F(y_2) \overset{*}{+} (1-\lambda) F(y_1), \\ F(x_1) &= F(\mu y_2 + (1-\mu) y_1) \subset \mu F(y_2) \overset{*}{+} (1-\mu) F(y_1). \end{aligned}$$

Multiplying the above inclusions by $y_1 - y_2$ and summing them up together we obtain

$$(y_1 - y_2)(F(x_1) \overset{*}{+} F(x_2)) \subset (x_2 - y_2 + x_1 - y_2) F(y_1) \overset{*}{+} (y_1 - x_2 + y_1 - x_1) F(y_2).$$

The equality $x_1 + x_2 = y_1 + y_2$ and the above inclusions lead to

$$F(x_1) \overset{*}{+} F(x_2) \subset F(y_1) \overset{*}{+} F(y_2).$$

If $y_1 = y_2$, then $y_1 = x_1 = x_2 = y_2$ and condition (5) holds true.

Theorem 3 for concave multifunctions can be found in [7, Theorem 2.14] in another formulation. The same concerns the next theorem. Its proof is similar to the previous one.

Theorem 3′
Let x_1, x_2, y_1, y_2 be real numbers such that $x_2 \le x_1$, $y_2 \le y_1$, $x_1 \le y_1$ and, $x_1 + x_2 = y_1 + y_2$. If $F\colon \mathbb{R} \longrightarrow clb(X)$ is convex, then

$$F(y_1) \overset{*}{+} F(y_2) \subset F(x_1) \overset{*}{+} F(x_2).$$

Corollary 1
Let a, b, c be non-negative numbers and let $a + b \le c$. If $F\colon [0, \infty) \longrightarrow clb(X)$ is a $*$-concave multifunction, then

$$F(a+b) \overset{*}{+} F(c) \subset F(a) \overset{*}{+} F(b+c).$$

Proof. To obtain the Corollary from Theorem 3 it is enough to set $x_1 = c$, $x_2 = a + b$, $y_1 = b + c$, $y_2 = a$ (see [5, pp. 194-195]).

COROLLARY 2
Let x_1, x_2, y_1, y_2 be real numbers satisfying the conditions: $x_2 \le x_1$, $y_2 \le y_1$, $x_1 \le y_1$ and $x_1 + x_2 \le y_1 + y_2$. If $F\colon \mathbb{R} \longrightarrow clb(X)$ is an increasing $*$-concave multifunction, then

$$F(x_1) \overset{*}{+} F(x_2) \subset F(y_1) \overset{*}{+} F(y_2)$$

holds true.

Proof. Taking $z_1 = y_1$ and $z_2 = x_1 + x_2 - y_1$ we can easily check that the numbers x_1, x_2, z_1, z_2 satisfy the assumption of Theorem 3. Hence

$$F(x_1) \overset{*}{+} F(x_2) \subset F(z_1) \overset{*}{+} F(z_2).$$

Moreover, F is increasing and $z_2 \le y_2$, so

$$F(x_1) \overset{*}{+} F(x_2) \subset F(y_1) \overset{*}{+} F(y_2).$$

THEOREM 4
Assume that x_i, y_i, $i \in \{1, \ldots, n\}$ are real numbers such that

$$x_n \le x_{n-1} \le \ldots \le x_1, \qquad y_n \le y_{n-1} \le \ldots \le y_1, \tag{6}$$

$$\sum_{i=1}^{k} x_i \le \sum_{i=1}^{k} y_i, \quad k \in \{1, \ldots, n-1\}, \qquad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \tag{7}$$

and

$$x_{k+1} \le y_k, \qquad k \in \{2, \ldots, n-1\}. \tag{8}$$

If $F\colon \mathbb{R} \longrightarrow clb(X)$ is a $*$-concave multifunction, then

$$F(x_1) \overset{*}{+} \cdots \overset{*}{+} F(x_n) \subset F(y_1) \overset{*}{+} \cdots \overset{*}{+} F(y_n). \tag{9}$$

Proof. The theorem is valid for $n = 2$ thanks to Theorem 3.

Now we assume (9) true for an $n \in \mathbb{N}$, $n \ge 2$ and take arbitrary numbers x_i, y_i, $i \in \{1, \ldots, n, n+1\}$ satisfying

$$x_{n+1} \le x_n \le \ldots \le x_1, \qquad y_{n+1} \le y_n \le \ldots \le y_1, \tag{6_{n+1}}$$

$$\sum_{i=1}^{k} x_i \le \sum_{i=1}^{k} y_i, \quad k \in \{1, \ldots, n\}, \qquad \sum_{i=1}^{n+1} x_i = \sum_{i=1}^{n+1} y_i \tag{7_{n+1}}$$

and

$$x_{k+1} \le y_k, \qquad k \in \{2, \ldots, n\}. \tag{8_{n+1}}$$

By (7_{n+1}) we have

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n-1} y_i + (y_n + y_{n+1} - x_{n+1}).$$

According to the induction hypothesis

$$F(x_1) \overset{*}{+} \cdots \overset{*}{+} F(x_n) \subset F(y_1) \overset{*}{+} \cdots \overset{*}{+} F(y_{n-1}) \overset{*}{+} F(y_n + y_{n+1} - x_{n+1})$$

since $y_n + y_{n+1} - x_{n+1} \le y_{n-1}$ (see (6_{n+1}) and (7_{n+1})). If we show that

$$F(y_n + y_{n+1} - x_{n+1}) \overset{*}{+} F(x_{n+1}) \subset F(y_n) \overset{*}{+} F(y_{n+1}) \tag{10}$$

holds, the proof will be complete.

Consider two cases: (a) $x_{n+1} \le y_n + y_{n+1} - x_{n+1}$ and (b) $x_{n+1} > y_n + y_{n+1} - x_{n+1}$. In case (b) $(y_n + y_{n+1} - x_{n+1}) + x_{n+1} = y_n + y_{n+1}$, $y_{n+1} \le y_n$, $y_n + y_{n+1} - x_{n+1} < x_{n+1}$ and $x_{n+1} \le y_n$ according to (8_{n+1}). By Theorem 3 condition (10) holds. In case (a), $x_{n+1} + (y_n + y_{n+1} - x_{n+1}) = y_n + y_{n+1}$, $y_{n+1} \le y_n$, $x_{n+1} \le y_n + y_{n+1} - x_{n+1}$ and $y_n + y_{n+1} - x_{n+1} = y_n + (y_{n+1} - x_{n+1}) \le y_n$ because $y_{n+1} \le x_{n+1}$. By Theorem 3 condition (10) holds.

Theorem 4′
Assume that x_i, y_i, $i \in \{1, \dots, n\}$ are real numbers such that

$$x_n \le x_{n-1} \le \ldots \le x_1\,, \qquad y_n \le y_{n-1} \le \ldots \le y_1\,,$$

$$\sum_{i=1}^{k} x_i \le \sum_{i=1}^{k} y_i\,, \quad k \in \{1, \dots, n-1\}\,, \qquad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

and

$$x_{k+1} \le y_k\,, \qquad k \in \{2, \dots, n-1\}\,.$$

If $F\colon \mathbb{R} \longrightarrow clb(X)$ is a convex multifunction, then

$$F(y_1) \overset{*}{+} \cdots \overset{*}{+} F(y_n) \subset F(x_1) \overset{*}{+} \cdots \overset{*}{+} F(x_n).$$

Results of the same kind as Theorem 4 and 4′, formulated in some other language, were obtained by K. Nikodem (cf. [7, Theorem 2.14]).

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