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Report of Meeting

10th International Conference on Functional Equations and Inequalities, Będlewo, September 11-17, 2005

The Tenth International Conference on Functional Equations and Inequalities was held in Będlewo, from September 11 to September 17, 2005 at the Mathematical Research and Conference Center. It was organized by the Institute of Mathematics of the Pedagogical University of Cracow in cooperation with the Stefan Banach International Mathematical Center and with a financial support of the Mathematical Institute of the Polish Academy of Sciences and of the BPH Bank.

The Organizing Committee consisted of Professor Janusz Brzdęk (Chairman), Dr. Paweł Solarz, Miss Janina Wiercioch and Mr Władysław Wilk.

The Scientific Committee consisted of Professor Dobiesław Brydak (Chairman), Dr. Jacek Chmieliński (Scientific Secretary) and Professors Bogdan Choczewski, Roman Ger and Marek Cezary Zdun.

The 64 participants came from 10 countries: Austria (1, Innsbruck), France (1, Nantes), Germany (3, Clausthal-Zellerfeld, Karlsruhe, Munich), Hungary (1, Debrecen), Israel (2, Haifa, Tel-Aviv), People's Republic of China (1, Sichuan), Poland (50, Bielsko-Biała, Bydgoszcz, Gdańsk, Gliwice, Katowice, Kielce, Kraków, Rzeszów, Zielona Góra), Romania (2, Cluj-Napoca), Russia (2, Dolgoprudny, Nizhni Novgorod), USA (1, Louisville, KY). It was observed with satisfaction that a half of all participants were at most Ph.D. graduated Polish mathematicians.

Professor Brzdęk welcomed the participants in the name of the Organizing Committee. Opening address was given by Profesor Zdun, the Dean of the Faculty of Science of the Pedagogical University of Cracow. He spoke also on behalf of Professor Eugeniusz Wachnicki, Deputy Rector of the University. Then Professor Brydak officially opened the 10th ICFEI.

During 20 sessions 59 talk were delivered. The scientific program was preceded by a historical talk given by Professor Choczewski who presented statistical data and reminded some characteristic events from the previous nine meetings. He pointed out that there are 6 participants of the first meeting held at Sielpia in 1984 who were present at all the eight further ones. From among 15 colleagues who attends both the First and the Tenth ICFEI. Professor Peter Volkmann (Karlsruhe) is the only foreign guest. A slightly extended version of the talk is published in this volume under the title *International meetings organized by Polish schools of functional equations.*

The other talks at the conference focused on the following topics: *stability theory* (of d'Alembert's, Dhombres', Jensen's, translation and recurrence equations and of those of microperiodic functions, multiplicative symmetry, isometries, orthogonal-conditional and on metric groups, stemming from mean value Flett's type theorems, applications of generalized strong derivatives), *functional equations in several variables* (Cauchy, Dhombres, Gołąb–Schinzel, translation, addition formula, representing λ-affinity, orthogonality preserving property, relations among means including recurrences, characterizing the absolute value of n-additive functions), *functional equations in a single variable* (Dhombres, Schröder's in normed spaces, conjugacy, with iteration of unknown function (also on the unit circle), integral-functional; spectra of operators generated by de Rham equation), *convexity* (on Abelian groups, connections with Orlicz spaces, generalized Beckenbach – three parameters families of functions), *multifunctions* (iteration semigroups including concave ones, selections of multimeasures, $*$-convex, Drygas' equation, cosine families), *theory of iteration* (triangle mappings, near-iterability, roots of homeomorphisms of the unit circle), *inequalities* (systems (of iterative type), approximate integration), *varia and applications* (d'Alembert's and Schrödinger's partial differential equations, waveform relaxation method in difference, differential and differential-delay equations, Riemann–Hilbert problem and a related functional equation – composite materials, equations appearing in actuarial mathematics).

On Tuesday, September 13, there was a picnic in the park surrounding the Center, and the next afternoon was devoted to an excursion to Poznań including a visit to the Cathedral and to the Museum of Old Instruments. Also from Wednesday on we might enjoy each evening piano recitals performed by Professor Hans-Heinrich Kairies.

On Thursday, September 15, before the festive banquet, Professor Ger made a fine and hearty speech on the occasion of the jubilee of 10th ICFEI and of the 70th birthdays of Professors Dobiesław Brydak and Bogdan Choczewski, the organizers of the previous meetings. He also presented two special addresses of congratulations and wishes, signed by the 62 participants of the 10th ICFEI, in which one reads, among others: *The Conference has always been important for Polish mathematicians working in the field of functional equations. The international significance of this is shown by the enclosed list of the countries from which the participants of all the 10 Conferences have originated.* The addresses were accompanied by nice bouquets of flowers and by albums showing most precious monuments and miracles of nature, registered on the UNESCO lists of world heritage.

At the end of the last scientific session on Saturday, September 17, its chairman Professor Marian Kwapisz from the Casimir the Great University of Bydgoszcz, praised the members of the Organizing Committee for their efforts resulting in a very successful meeting.

Words of thanks said then by Professor Brzdęk to all the participants were followed by a closing address by Professor Choczewski. He first announced that Professor Brydak was elected the Honorary Chairman of the Scientific Committee of the next ICFEI. The announcement was warmly applauded by the audience. Next he continued with thanking:
— all the colleagues, and especially foreign guests, for their coming and acting nicely, friendly and effectively, so that the Conference would be worth to be kept in everyone's memory for a longer time,
— Professor Brzdęk and the whole Organizing Committee, which acted in a smooth and efficient way, being always at a careful disposal and helping many of participants in various ways,
— Professor Ger and Dr. Chmieliński who composed the program of the meeting and took care on its smooth realization,
— Professor Ger once again, who suggested the Center as the place for the 10th ICFEI and actively supported this idea by means of facilitating contacts with the managing institutions and persons. (Professor Ger was the member of a close council of Polish mathematicians especially chosen more than 10 years ago to create conceptually and organizationally the vision of the Center.)

The speaker expressed, also in the name of his friend Dobiesław (known to him since 53 years), very cordial personal thanks both to the organizers and to the participants of this surprising, touching and unforgettable anniversary celebration on Thursday evening. He also revealed that the 70th birthday of Professor Kwapisz had been honoured in June at the University of Gdańsk, and congratulated him on behalf of all the participants of the meeting.

Our Conference has been included into the programme of the Mathematical Research and Conference Center at Będlewo by the Banach Center Scientific Council. Professor Choczewski expressed the gratitude of the organizers to the Committee and also to Professor Bogdan Bojarski, the Honorary Director of the MRCC and Professor Łukasz Stettner, its Managing Director, for their decisions

Thanks were extended to Mrs. Anna Kreczmar-Puacz, the Manager of the Center, and for its staff as a whole, for a high standard of board and lodging, fully equipped with audiovisual facilities conference rooms and for the quality of service, which were highly appreciated by the participants.

The duties of the Chairman, both of the Scientific and Organizing Committees of the ICFEI would be now taken on by Professor Janusz Brzdęk. Professors Nicole Brillouet-Belluot (France), Hans-Heinrich Kairies (Germany) and Lászlo Losonczi (Hungary) kindly accepted the proposal to join the Scientific Committee as representatives of foreign participants.

Since the 45th International Symposium on Functional Equations is planned to be organized in 2007 in Bielsko-Biała, it has been decided that the 11th ICFEI will be held again at Będlewo, from September 18 to September 23, 2006.

Abstracts of the talks follow in alphabetical order of the authors. Contributions to several sessions devoted to problems and remarks (in chronological order of presentation) and the list of participants (with addresses) complete the report. All the scientific materials were collected and compiled by Dr. Chmieliński. The help, also in this respect, of Dr. Solarz, Miss Wiercioch and Mr Wilk is acknowledged with many thanks.

Bogdan Choczewski

Abstracts of Talks

Mirosław Adamek *An example connected with λ-affinity*

Let $\lambda\colon I^2 \longrightarrow (0,1)$ be a fixed function (I is a nonempty and open interval of $\mathbb{R}$). A function $f\colon I \longrightarrow \mathbb{R}$ is called *λ-affine* if

$$f(\lambda(x,y)x + (1-\lambda(x,y))y) = \lambda(x,y)f(x) + (1-\lambda(x,y))f(y), \qquad x,y \in I;$$

and f is *affine* if

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y), \qquad t \in [0,1]\,,\ x,y \in I.$$

In the talk we will show an example of a function λ for which each λ-affine function f is affine.

Roman Badora *On the stability of some functional equations*

Let G be a group and let $K = \{k_0 = \mathrm{Id}_G, k_1, \ldots, k_{N-1}\}$ be a subgroup of the automorphism group $\mathrm{Aut}(G)$ of G (the action of $k \in K$ on $x \in G$ is denoted by kx). We study the stability of the following functional equations

$$\frac{1}{N}\sum_{i=0}^{N-1} f(x + k_i y) = f(x), \qquad x,y \in G;$$

$$\frac{1}{N}\sum_{i=0}^{N-1} f(x + k_i y) = f(x)g(y), \qquad x,y \in G;$$

$$\frac{1}{N}\sum_{i=0}^{N-1} f(x + k_i y) = f(x) + g(y), \qquad x,y \in G$$

($f, g\colon G \longrightarrow \mathbb{C}$), which cover Jensen's functional equation, Cauchy's functional equation, d'Alembert's functional equation and the functional equation of the square of the norm.

Anna Bahyrycz *Conditional equation of exponential function*

We consider the conditional equation of exponential function:

$$f(x)\cdot f(y)\neq 0_m \implies f(x+y)=f(x)\cdot f(y),$$

where $n,m\in\mathbb{N}$,

$$f\colon \mathbb{R}(n):=[0,+\infty)^n\setminus\{0_n\}\longrightarrow \mathbb{R}(m),$$

$$x+y:=(x_1+y_1,\dots,x_k+y_k)\quad\text{and}\quad x\cdot y:=(x_1\cdot y_1,\dots,x_k\cdot y_k)$$

for $x=(x_1,\dots,x_k)$, $y=(y_1,\dots,y_k)\in\mathbb{R}(k)$.

We investigate systems of cones over $\mathbb{Q}$, which are one of the parameters determining the solutions of this equation.

Karol Baron *On a problem of József Bukszár*

Joint work with Witold Jarczyk.

Referring to [1, Problem 7 on p. 194] we show that any Lebesgue measurable function $f\colon\mathbb{R}\longrightarrow[0,\infty)$ satisfying

$$f(x)=\int_0^\infty f(x+y)f(y)\,dy \qquad \text{for } x\in\mathbb{R}$$

has the form

$$f(x)=2\lambda e^{-\lambda x}\qquad (x\in\mathbb{R})$$

with a $\lambda\in[0,\infty)$.

[1] *Report of Meeting, 7th International Conference on Functional Equations and Inequalities, Złockie, September 12–18, 1999*, Ann. Acad. Paed. Cracoviensis Studia Math. **1** (2001), 163-201.

Lech Bartłomiejczyk *Solution of a problem of J. Smítal*

Let $J\subset(0,1)$ be an interval and $h\colon J\longrightarrow J$ be a strictly increasing function. The following solves the problem of Jaroslav Smítal concerning the existence of a very irregular solution $\varphi\colon(0,+\infty)\longrightarrow J$ of the equation

$$\varphi(x\varphi(x))=h(\varphi(x)) \tag{1}$$

posed during the 43rd ISFE (Batz-sur-Mer, May 15-21, 2005).

Assume $\mathcal{R}$ is a family of subsets of $(0,+\infty)\times J$ such that

$$\operatorname{card}\mathcal{R}\le\mathfrak{c}$$

and for every $R\in\mathcal{R}$ there is a $y\in J$ with

$$\operatorname{card}\{x\in(0,+\infty):\ (x,y)\in R\}=\mathfrak{c}.$$

Then there exists a solution $\varphi\colon (0,+\infty) \longrightarrow J$ of (1) such that its graph meets every set of $\mathcal{R}$.

The above conditions are fulfilled by, among others, the family of all the sets of the form $B \times \{y\}$ where $B \subset (0,+\infty)$ is Borel and uncountable and $y \in J$. If a subset G of $(0,+\infty) \times J$ meets every set of this specific family, then its complement $(0,+\infty) \times J \setminus G$ contains no set of the second category having the property of Baire and contains no set of positive Lebesgue measure.

To get the above presented result we use [1].

[1] L. Bartłomiejczyk, *Solutions with big graph of the equation of invariant curves*, Bull. Polish Acad. Sci. Math. **49** (2001), 309-317.

Bogdan Batko *On approximate solutions of Dhombres' functional equation*

Let $f\colon S \longrightarrow X$ map an abelian semigroup $(S,+)$ into a Banach space $(X, \|\ \|)$. We are going to deal with the stability of Dhombres' functional equation

$$f(x) + f(y) \neq 0 \Longrightarrow f(x+y) = f(x) + f(y) \qquad \text{for } x, y \in S. \tag{1}$$

We assume that f is an *approximate solution* of equation (1) with control functions $\Phi_1, \Phi_2\colon S \times S \longrightarrow \mathbb{R}^+$, i.e.,

$$||f(x)+f(y)|| > \Phi_1(x,y) \Longrightarrow ||f(x+y)-f(x)-f(y)|| \leq \Phi_2(x,y) \quad \text{for } x, y \in S$$

and ask for the existence of a solution $a\colon S \longrightarrow X$ of (1) with

$$||f(x) - a(x)|| \leq \Psi(x) \qquad \text{for } x \in S,$$

where $\Psi\colon S \longrightarrow \mathbb{R}^+$ is a function we can explicitly compute starting from Φ_1 and Φ_2 .

Zoltán Boros *Generalized strong derivatives*

Let I denote an open interval in the real line, and let us consider a function $f\colon I \longrightarrow \mathbb{R}$. For $x \in I$ and $h \in \mathbb{R}$, we define the lower and upper strong dyadic derivatives of f by

$$\underline{D}_h^{\beta} f(x) = \liminf_{\substack{y \to x \\ n \to \infty}} 2^n (f(y + 2^{-n}h) - f(y))$$

and

$$\overline{D}_h^{\beta} f(x) = \limsup_{\substack{y \to x \\ n \to \infty}} 2^n (f(y + 2^{-n}h) - f(y)),$$

respectively. We call f *strongly dyadically differentiable* if

$$\underline{D}_h^{\beta} f(x) = \overline{D}_h^{\beta} f(x) \in \mathbb{R}$$

holds for every $x \in I$ and $h \in \mathbb{R}$. We say that f has *increasing strong dyadic derivatives* if

$$-\infty < \overline{D}_h^\beta f(x) \leq \underline{D}_h^\beta f(y) < +\infty$$

holds for every $h > 0$ and $x, y \in I$ such that $x < y$. These properties are characterized by the following decomposition theorems:

Theorem 1
The function f is strongly dyadically differentiable if, and only if, there exist a continuously differentiable function $g\colon I \longrightarrow \mathbb{R}$ and an additive mapping $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = g(x) + \varphi(x)$ for every $x \in I$.

Theorem 2
The function f has increasing strong dyadic derivatives if, and only if, there exist a convex function $g\colon I \longrightarrow \mathbb{R}$ and an additive mapping $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = g(x) + \varphi(x)$ for every $x \in I$.

Applying these results, we characterize affine (respectively, Wright-convex) functions as approximately affine (respectively, approximately Wright-convex) functions in a specific sense. In what follows, let us consider a fixed real number $p > 1$.

Example 1
Suppose that, for every $x \in I$, f satisfies an inequality of the form

$$|f(y+u) - f(y) - \phi_x(u)| \leq \varepsilon(x)|u|^p \tag{1}$$

for every y taken from a neighbourhood of x and for every u taken from a neighbourhood of 0. It is derived from the inequality (1) that f is strongly dyadically differentiable. Applying our decomposition theorem, we obtain that $f = g + \varphi$, where g is continuously differentiable and φ is the restriction of an additive mapping to the interval I. Substitution into the inequality (1) yields that g' is constant and thus f is affine.

Example 2
Let $\varepsilon > 0$ and suppose that f satisfies the inequality

$$f(\lambda x + (1-\lambda)y) + f((1-\lambda)x + \lambda y) \leq f(x) + f(y) + \varepsilon(\lambda(1-\lambda)|x-y|)^p \tag{2}$$

for every $\lambda \in [0,1]$ and $x, y \in I$. It is derived from the inequality (2) that f has increasing strong dyadic derivatives, and thus $f = g + \varphi$, where g is convex and φ is the restriction of an additive mapping to the interval I. This yields that f satisfies the inequality (2) with $\varepsilon = 0$ as well.

Nicole Brillouët-Belluot *On a class of iterative-difference equations*

Joint work with Weinian Zhang (Sichuan University, P.R. China).

During the Thirty-eighth International Symposium on Functional Equations in Noszvaj, I posed the problem (Aequationes Math. **61** (2001), 304) of finding the continuous solutions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ of the functional equation

$$x + f(y + f(x)) = y + f(x + f(y)).$$

By letting $y = 0$, we see that this problem is related to the problem of finding the continuous solutions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ of the iterative-difference equation

$$f(f(x)) = f(x + a) - x$$

where a is a real number.

In this work, we consider the more general second order equation

$$f(f(x)) = \lambda_1 \; f(x+a) \; + \; \lambda_0 \; x, \qquad x \in J \tag{1}$$

where J is an interval of $\mathbb{R}$, λ_0, λ_1, a are real numbers with $a \neq 0$, $\lambda_1 \neq 0$.

We use three methods to approach equation (1). We find affine solutions of (1), we prove the existence of bounded continuous solutions of (1) on a compact interval and we construct piecewise continuous solutions of (1) on a finite interval.

Janusz Brzdęk *On approximately microperiodic mappings*

A function f mapping a group $(G, \cdot)$, endowed with a topology, into a nonempty set is said to be microperiodic provided the set

$$P_f := \{a \in G : \; f(a \cdot x) = f(x) \text{ for } x \in G\}$$

is dense in G. It is known that, under suitable assumptions, microperiodic functions that are continuous at a point must be constant and measurable microperiodic functions must be constant almost everywhere.

We generalize these statements in some directions. The main result is following.

Theorem

Let $(G, \cdot)$ be a semitopological group such that the mapping $G \ni x \to x^{-1} \in G$ is continuous, $\varepsilon \in [0, \infty)$, P be a dense subset of G, and $E \subset G$. Suppose that $g, h\colon E \longrightarrow \mathbb{R}$ satisfy

$$\begin{aligned} g(p \cdot x) - g(x) \le \varepsilon + h(p) \qquad & \textit{for } x \in E,\ p \in P \textit{ with } p \cdot x \in E, \\ h(p) + h(x) \le h(p \cdot x) \qquad & \textit{for } x \in E,\ p \in P \textit{ with } p \cdot x \in E. \end{aligned}$$

Then the following three assertions are valid.

(i) *If G is a locally compact topological group and g, h are Haar measurable on a set $D \in 2^E \setminus \mathcal{H}_0$, then there is $c \in \mathbb{R}$ such that $|g(x) - h(x) - c| \le \varepsilon$ $\mathcal{H}_0$-a.e. in E ($\mathcal{H}_0$ denotes the σ-ideal of subsets of G that are locally of Haar measure zero).*

(ii) *If g, h are Baire measurable on a set $D \in 2^E \setminus \mathcal{B}_0$, then there is $c \in \mathbb{R}$ such that $|g(x) - h(x) - c| \le \varepsilon$ $\mathcal{B}_0$-a.e. in E ($\mathcal{B}_0$ denotes the σ-ideal of subsets of G that are of first category of Baire).*

(iii) *If g, h are continuous at a point $x_0 \in \operatorname{int} E$, then there is $c \in \mathbb{R}$ such that $|g(x) - h(x) - c| \le \varepsilon$ for every $x \in E$.*

Remark
Taking $E = D = G$, $\varepsilon = 0$ and $h \equiv 0$ in the Theorem we obtain the classical results on microperiodic functions.

Jacek Chmieliński *Approximate functional relations stemming from orthogonality preserving property*

Let X and Y be two inner product spaces and $f\colon X \longrightarrow Y$. Then, the following two properties connected with orthogonality relation and its preservation can be considered.

Orthogonality preserving property:

$$\forall\, x, y \in X: \quad x \perp y \implies f(x) \perp f(y).$$

Right-angle preserving property:

$$\forall\, x, y, z \in X: \quad x - z \perp y - z \implies f(x) - f(z) \perp f(y) - f(z).$$

We deal with mappings satisfying the above properties *approximately.* In particular, some kind of stability of the considered properties is established.

Bogdan Choczewski *Nine International Conferences on Functional Equations and Inequalities*

The aim of this talk is to remind some facts from the history of our previous meetings (organization, participants, topics, events), starting from the first one which was held at Sielpia in 1984.

Jacek Chudziak *Continuous solutions of a composite addition formula*

Let X be a real vector space and J be a nontrivial real interval. We deal with the functional equation

$$g(x + M(g(x))y) = H(g(x), g(y)) \qquad \text{for } x, y \in X, \tag{1}$$

where $g: X \longrightarrow J$, $M: J \longrightarrow \mathbb{R}$ and $H: J^2 \longrightarrow J$ are unknown functions. The equation (1) is a generalization of equations of the form

$$g(x+y) = H(g(x), g(y)) \qquad \text{for } x, y \in X,$$

known as addition formulae. It is also a generalization of the Gołąb–Schinzel type functional equations

$$g(x + g(x)^k y) = tg(x)g(y) \qquad \text{for } x, y \in X,$$

where k is a positive integer and t is a real number.

We determine all solutions of (1) under the assumptions that $g: X \longrightarrow J$ is continuous on rays, $M: J \longrightarrow \mathbb{R}$ is continuous and $H: J^2 \longrightarrow J$ is associative.

Stefan Czerwik *A general Baker superstability criterium for the d'Alembert functional equation*

Joint work with Maciej Przybyla.

Let G be an abelian group. We define

$$U_1 := \{g: G \longrightarrow F\}, \quad U_2 := \{F: G^2 \longrightarrow F\},$$

$$g_a(x) := g(x+a), \qquad a, x \in G,$$

$$A(f)(x,y) := f(x+y) + f(x-y) - 2f(x)f(y), \qquad x, y \in G.$$

Lemma

Let G be an abelian group and F be a field. Let $f: G \longrightarrow F$ be a function. Then for all $x, u, v \in G$ we have

$$\begin{aligned} 2f(x)A(f)(u,v) = {} & A(f)(x+u,v) - A(f)(x,u+v) - A(f)(x,u-v) \\ & + A(f)(x-u,v) + 2f(v)A(f)(x,u). \end{aligned}$$

Theorem 1

Let $f: G \longrightarrow F$ be a function. Let U_1 be a linear space over F such that if $g \in U_1$, then for every $a \in G$, $g_a \in U_1$. If, moreover, for every $(u,v) \in G^2$ the function $f(\cdot)A(f)(u,v) \in U_1$, then

$$f \in U_1 \quad \textit{or} \quad A(f) = 0.$$

Theorem 2

Let $f: G \longrightarrow F$ be a function. Let U_1 be a linear space over F with the "translation property". If, moreover, $A(f)(\cdot,u) \in U_1$ for every fixed $u \in G$, then

$$f \in U_1 \quad \textit{or} \quad A(f) = 0.$$

Remark

The famous Baker result on superstability of the D'Alembert functional equation we get for

$$U_1 = B(G, C), \quad U_2 = B(G^2, C)$$

(the A-conjugate spaces of bounded functions on G and G^2 respectively). One may consider also the spaces X_q and X_q^2. For details see [1].

[1] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, New Jersey, London, 2002.

Joachim Domsta *On the regular and smooth conjugacy*

A self mapping f of $\mathbb{R}_+ := (0, \infty)$ is said to be *differentiable at the origin*, if $\lim_{x\to 0} \frac{f(x)}{x}$ exists in $\mathbb{R}_+$. Two differentiable at zero self-homeomorphisms f and g of $\mathbb{R}_+$ are said to be *regularly* [*differentiably*] *conjugate* if there exists an increasing self–homeomorphism Ψ on $\mathbb{R}_+$ regularly varying [differentiable] at the origin and such that $g(\Psi(x)) = \Psi(f(x))$, for $x \in \mathbb{R}_+$. Obviously, the differentiability implies regular variability. This has allowed to express the differentiable conjugacy through the regular conjugacy with the use of Szekeres principal function. From the main result a partial solution of problems related to the smooth conjugacy of diffeomorphisms is derived.

Lyudmila Efremova *Simplest skew products of interval maps with one-dimensional attractor*

Let us consider a simplest skew product of interval maps

$$F(x,\, y) = (f(x),\, g_x(y)) \qquad \text{for all } (x,\, y) \in I$$

($I = I_1 \times I_2$, where I_1, I_2 are the closed intervals). Then for any $m \geq 1$ we have

$$F^m(x, y) = (f^m(x), g_{x,m}(y)), \qquad \text{where } g_{x,m}(y) = g_{f^{m-1}(x)} \circ \ldots \circ g_{f(x)} \circ g_x(y).$$

The term "simplest" means that

(i) f has a sink x^0 with the period $n \geq 1$;

(ii) $g_{x^0,\, n}(y) = y$ for all $y \in I_2$;

(iii) the right-side basin of the immediate attraction of x^0 contains a subtrajectory $x_0 > x_n > \ldots > x_{nm} > \ldots > x^0$ of some point x_0 such that for all $y \in I_2$ and $m \geq 0$ the following holds

(iii.1) $g_{x_{n(2m+1)},\, n}(y) = y$;

(iii.2) $g_{x,\,n}(y) \le y$ for all $x \in (x_n,\, x_0] \cup \bigcup_{m\,\in\,2\mathbb{N}+1} (x_{n(2m+3)},\, x_{n(2m+1)})$, and the equality is valid only for $y = 0$;
$g_{x,\,n}(y) \ge y$ for all $x \in \bigcup_{m\,\in\,2\mathbb{N}} (x_{n(2m+3)},\, x_{n(2m+1)})$, and the equality is valid only for $y = 1$, where $y \in I_2$, $m \ge 0$.

1. Let us assign to a nonperiodic point $\overline{x}$ of the factor-map f and with a subtrajectory $\{f^{n_i}(\overline{x})\}_{i\ge 0}$ the set-valued function $\theta_{\overline{x},\,N_*}\colon \{f^{n_i}(\overline{x})\}_{i\ge 0} \longrightarrow (2^{I_2})_m$ so that the equality

$$\theta_{\overline{x},\,N_*}(x) = g_{\overline{x},\,n_i}(I_2)$$

holds for all $x = f^{n_i}(\overline{x})$, where $N_* = \{n_i\}_{i\ge 0}$, $(2^{I_2})_m$ is the space of all closed subsets of I_2, endowed with Hausdorff metric *dist*.

2. With the use of the concept of dynamical F-variation of function $\theta_{\overline{x},\,N_*}$ the explanation is given of the phenomenon of the existence of one-dimensional attractor $A = \bigcup_{i=0}^{n-1}\{f^i(x^0)\} \times g_{x^0,\,i}(I_2)$ such that $\omega_F((x,\,y)) = A$ (where $\omega_F((x,\,y))$ is ω-limit set of F-trajectory of a point $(x,\,y)$) for all $x \notin \mathrm{Orb}_f(x^0)$ from the basin of the attraction of the periodic orbit $\mathrm{Orb}_f(x^0)$ and all $y \in I_2$.

3. The influence of the smoothness of a simplest skew skew product of interval maps on the asymptotic behaviour of its trajectories is investigated.

The work is partially supported by RFBR, grant 04-01-00457.

[1] L.S. Efremova, *Nonhyperbolic periodic points and attracting sets of simplest skew products of interval maps*, J. Dyn. Control Syst. **10** (2004), 111-113.

Roman Ger *Almost Gołąb–Schinzel functions*

Recent studies of the classical Gołąb–Schinzel functional equation

$$f(x + yf(x)) = f(x)f(y) \tag{1}$$

and its generalizations show that there is an objective need of examining them on restricted domains (in general, depending upon a given solution). In a natural way, this leads also to similar questions regarding the associated Cauchy type equation

$$f(G(x,y)) = f(x)f(y) \tag{2}$$

with an (possibly “almost”) associative binary operation G. We deal with such kind of problems assuming the validity of (1) and/or (2) almost everywhere with respect to an abstract ideal of “small” sets.

Dorota Głazowska *Invariance of the geometric mean with respect to Lagrangean conditionally homogeneous mean-type mappings*

We determine all the Lagrangean conditionally homogeneous mean-type mappings for which the geometric mean is invariant. An important tool is a result on the generators of the Lagrangean conditionally homogeneous means.

Wojciech Jabłoński *Stability of the translation equation in rings of formal power series*

Joint work with L. Reich.

We will consider the stability problem for the translation equation in rings of formal power series

$$F(t_1+t_2,X)=F(t_1,F(t_2,X)) \qquad \text{for } t_1,t_2\in G,$$

where $(G,+)$ is an abelian group, $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, $F(t,X)=\sum_{i=1}^{\infty}c_i(t)X^i$ and $c_1\colon G\longrightarrow\mathbb{K}\setminus\{0\}$, $c_i\colon G\longrightarrow\mathbb{K}$ for $i\geq 2$.

We will show that, under some assumption on G, the translation equation in rings of formal power series is stable in Hyers–Ulam sense. What is more, stability of the translation equation in rings of formal power series is strictly connected with the problem of extensibility of the one-parameter group of truncated formal power series (cf. [1]).

[1] L. Reich, *Problem*, in: *Report of Meeting, The Twenty-eight International Symposium on Functional Equations, August 23 – September 1, 1990, Graz – Mariatrost, Austria*, Aequationes Math. **41** (1991), 248-310.

Justyna Jarczyk *Invariance in the class of weighted quasi-arithmetic means with continuous generators*

Let $I\subset\mathbb{R}$ be an open interval and $p,q,r\in(0,1)$. We find all continuous and strictly monotonic functions $\alpha,\beta,\gamma\colon I\longrightarrow\mathbb{R}$ satisfying the functional equation

$$\begin{aligned}\alpha^{-1}\big(p\alpha\big(\beta^{-1}(q\beta(x)+(1-q)\beta(y))\big)&\\ +(1-p)\alpha\big(\gamma^{-1}(r\gamma(x)+(1-r)\gamma(y))\big)\big)&\\ =\alpha^{-1}(p\alpha(x)+(1-p)\alpha(y))&\end{aligned} \tag{1}$$

generalizing the Matkowski–Sutô equation. In the proof we adopt a method elaborated by Z. Daróczy and Zs. Páles when solving the Matkowski–Sutô equation, some results of A. Járai on improving regularity of solutions and an extension theorem by Z. Daróczy and G. Hajdu. We also use a theorem giving the form of all twice continuously differentiable solutions of (1) proved jointly with J. Matkowski.

Witold Jarczyk *Convexity on Abelian groups*

Joint work with Miklós Laczkovich.

Let H be a subset of an Abelian group G. We say that $f\colon H \longrightarrow \mathbb{R}$ is convex if

$$2f(x) \leq f(x+h) + f(x-h)$$

holds whenever $x, h \in G$ and $x, x+h, x-h \in H$. It turns out that several classical theorems on convex functions on $\mathbb{R}^n$ or on (topological) linear spaces can also be proved in this general setting. In particular, we study the extendability of convex functions defined on subgroups of G as well as continuity properties of convex functions defined on open subsets of topological groups.

Hans-Heinrich Kairies *Continuous and residual spectra of operators connected with iterative functional equations*

The sum type operator F, given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x),$$

will be considered on the space of bounded real functions and on several subspaces. All the according restrictions are Banach space automorphisms. In their spectral theory some iterative functional equations arise in a natural way. We determine in all cases the resolvent set, the point spectrum, the continuous spectrum and the residual spectrum.

Dorota Krassowska *On a system of functional inequalities of iterative type*

Joint work with Janusz Matkowski.

We examine the functions $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying the system of simultaneous functional inequalities

$$f(\mathbf{a}_i + \mathbf{x}) \leq \alpha_i + f(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n,\ i = 1, 2, \ldots, n+1.$$

Assuming some algebraic conditions on given fixed reals $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ and the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n+1} \in \mathbb{R}^n$, we show that every continuous at least at one point solution of this system must be of the form

$$f(\mathbf{x}) = \mathbf{p}\mathbf{x} + f(\mathbf{0}), \qquad \mathbf{x} \in \mathbb{R}^n,$$

where $\mathbf{p} \in \mathbb{R}^n$ is uniquely determined.

An application for mappings having weakly constant sign is given.

Marian Kwapisz *Difference, differential, delay differential equations and convergence of WR methods*

The aim of the talk is to point out natural relations between difference equations and the waveform relaxation methods (in short WR methods) for solving large systems of ordinary differential equations, delay differential equations as well as neutral delay differential equations (NDDEs).

For difference equations the initial value problem is very easy. It can also be solved in finite steps by the use of successive iterations. We will look for this phenomenon for more complicated and more important equations. It will concern the convergence problem for the solutions of implicit difference equations considered in a function spaces.

We will present convergence results for WR methods applied to the general systems of NDDEs. We also will take care for the error evaluations in such a way that we will be able to detect the cases when the convergence will take place in a finite number of steps.

[1] Z. Bartoszewski, M. Kwapisz, *Delay dependent estimates for waveform relaxation methods for neutral differential-functional systems*, Comput. Math. Appl. **48** (2004), 1877-1892

Zbigniew Leśniak *On the d'Alembert equation and its generalizations*

We study the d'Alembert partial differential equation (and its generalizations) as a submanifold of the jet space $J^2(\mathbb{R}^2,\mathbb{R})$ (and $J^n(\mathbb{R}^2,\mathbb{R})$, respectively) with the contact system defined on it. We prove the existence and the uniqueness (for appropriate initial data) of solutions of these equations by using the Cartan–Kähler Theorem.

Grażyna Łydzińska *On semicontinuity of some set-valued iteration semigroups*

Let X be an arbitrary set, $A\colon X \longrightarrow 2^{\mathbb{R}}$, $q := \sup A(X)$ and $F\colon (0,\infty)\times X \longrightarrow 2^X$ be given by

$$F(t,x) := A^{-1}\left(A(x) + \min\{t, q - \inf A(x)\}\right), \tag{A}$$

where

$$A^{-1}(V) := \{x \in X :\ A(x)\cap V \neq \emptyset\}$$

for every $V \subset \mathbb{R}$.

The formula (A) is a set-valued counterpart of the well-known form of iteration semigroups of single-valued functions on an interval.

We present a few theorems about lower semicontinuity of the multifunctions $F(t,\cdot)$ and $F(\cdot,x)$ in the case when A is a single-valued function defined on a topological space.

Andrzej Mach *Translation equation on monoids*

Joint results with Zenon Moszner.

Large classes of solutions of the translation equation on a monoid $(G,\cdot)$ satisfying the identity condition and some results on stability of the translation equation are given.

[1] A. Mach, Z. Moszner, *Translation equation on monoids*, Ann. Polon. Math. **84** (2004), 137-146.

[2] A. Mach, Z. Moszner, *L'équation de translation sur le demi-groupe des éléments non négatifs d'un groupe ordonné et archimédien*, en preparation.

[3] A. Mach, Z. Moszner, *On stability of the translation equation in a class of functions*, Aequationes Math., submitted.

Janusz Matkowski *On a generalized Gołąb–Schinzel functional equation*

We consider the composite functional equation

$$f\left(p[f(y)x+y]+(1-p)[f(x)y+x]\right)=f(x)f(y)$$

where $p\in\mathbb{R}$ is arbitrarily fixed. For $p=0$ or $p=1$ it becomes the well-known Gołąb–Schinzel equation.

Vladimir Mityushev *Riemann–Hilbert problem for multiply connected domains and functional equations*

Consider mutually disjointed disks $D_k := \{z\in\mathbb{C}:\ |z-a_k|<r_k\}$ $(k=1,...,n)$ in the complex plane $\mathbb{C}$. Let $D := (\mathbb{C}\cup\{\infty\})\setminus\bigcup_{k=0}^{n}(D_k\cup\partial D_k)$. Given $\lambda_k(t)$, $f_k(t)$ as Hölder continuous functions on ∂D, $\lambda_k(t)\neq 0$. To find a function $\varphi(z)$ analytic in D continuous in $D\cup\partial D$ with the following boundary condition

$$\operatorname{Re}\overline{\lambda_k(t)}\varphi(t)=f_k(t),\qquad |t-a_k|=r_k,\ k=1,...,n.$$

This problem is called the *(Riemann–)Hilbert boundary value problem.* The scalar Riemann–Hilbert problem for any multiply connected domain has been reduced to functional equations in a class of analytic functions which has been solved in terms of the Poincaré series.

Janusz Morawiec *On L^1-solutions of a functional equation connected with the Grincevičjus series*

We consider the problem of the existence of non-trivial L^1-solutions of the equation

$$f(x)=\sum_{\varepsilon=\pm1}\sum_{n=-N}^{N}c_{n,\varepsilon}f(\varepsilon kx-n),$$

where $N\in\mathbb{N}$, $k\in\mathbb{N}\setminus\{1\}$ and $c_{n,\varepsilon}\geq 0$ for all $n\in\{-N,\dots,N\}$, $\varepsilon\in\{-1,1\}$.

Jacek Mrowiec *On the nonstability of the Jensen's equation*

Let $\mathbb{X}$ be a linear space and let $\mathbb{X}\supset D$ be a midconvex set, i.e.,

$$x,y\in D\Longrightarrow \frac{x+y}{2}\in D.$$

A function $g\colon D \longrightarrow \mathbb{R}$ is said to be *Jensen* if

$$g\Big(\frac{x+y}{2}\Big) = \frac{g(x)+g(y)}{2} \tag{1}$$

for every $x, y \in D$.

Let $\delta > 0$ be a given real number. We say that a function $f\colon D \longrightarrow \mathbb{R}$ is *approximately Jensen* (δ-Jensen) if

$$\left| f\Big(\frac{x+y}{2}\Big) - \frac{f(x)+f(y)}{2} \right| \le \delta$$

for every $x, y \in D$.

We say that the Jensen's functional equation (1) is *stable* on a given midconvex set D if for any δ-Jensen function $f\colon D \longrightarrow \mathbb{R}$ there exist a Jensen function $g\colon D \longrightarrow \mathbb{R}$ and a constant $C > 0$ such that

$$|f(x) - g(x)| \le C\delta, \qquad x \in D.$$

It is known, that if $D = \mathbb{X}$, $D = [-a, a]^N \subset \mathbb{R}^N$ (and in some other cases), then the equation (1) is stable.

However, the equation (1) is not stable in general, i.e., on any midconvex set $D \subset \mathbb{X}$. There exist a midconvex subset D of the real line and an approximately Jensen function $f\colon D \longrightarrow \mathbb{R}$ s.t.

$$\sup \big\{ |g(x) - f(x)| :\ x \in D \big\} = \infty$$

for every Jensen function $g\colon D \longrightarrow \mathbb{R}$.

Similar example works for affine functions in infinite dimensional spaces.

Anna Mureńko *On solutions of a generalization of the Gołąb–Schinzel functional equation*

We consider solutions $M, f\colon \mathbb{R} \longrightarrow \mathbb{R}$ and $\circ\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ of the functional equation

$$f(x + M(f(x))y) = f(x) \circ f(y),$$

under the following additional assumptions:

(a) f is Lebesgue measurable or Baire measurable;

(b) $M^{-1}(\{0\}) = \{0\}$;

(c) $\circ$ is commutative and associative.

Adam Najdecki *On the stability of some functional equations connected with the multiplicative symmetry*

Let $(X, \circ)$ be an abelian semigroup, $g\colon X \longrightarrow X$ and let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. We consider stability of the functional equation

$$f(x \circ g(y)) = f(x) \cdot f(y)$$

in the class of function $f\colon X \longrightarrow \mathbb{K}^n$, as well as of the equation

$$f(x \circ g(y)) = f(x) + f(y)$$

in the class of function mapping X into a Banach space over $\mathbb{K}$.

Kazimierz Nikodem *Three-parameter families and generalized convex functions*

Joint work with Attila Gilányi and Zsolt Páles.

We extend the notion of generalized convex functions introduced by E.F. Beckenbach to two-dimensional case in the following way: Let $\mathcal{F}$ be a family of continuous real functions defined on a convex set $D \subset \mathbb{R}^2$ such that for any three non-collinear points $x_1, x_2, x_3 \in D$ and any $t_1, t_2, t_3 \in \mathbb{R}$ there exists exactly one $\varphi = \varphi_{(x_1,t_1),(x_2,t_2),(x_3,t_3)} \in \mathcal{F}$ such that $\varphi(x_i) = t_i$ for $i = 1, 2, 3$. We say that a function $f\colon D \longrightarrow \mathbb{R}$ is *$\mathcal{F}$-convex* if for any non-collinear $x_1, x_2, x_3 \in D$

$$f(x) \le \varphi_{(x_1,f(x_1)),(x_2,f(x_2)),(x_3,f(x_3))}(x)$$

for every x in the triangle with vertices x_1, x_2, x_3. It is proved that every $\mathcal{F}$-convex function $f\colon D \longrightarrow \mathbb{R}$ is continuous on $\operatorname{int} D$. Some other properties of $\mathcal{F}$-convex and $\mathcal{F}$-midconvex functions are also given.

Jolanta Olko *On the set of measure selections of a multimeasure*

Let $(T, \mathcal{A})$ be a measurable space, X be a Banach space. We say that a multifunction $M\colon \mathcal{A} \longrightarrow 2^X$ with nonempty, closed values is a multimeasure if for every $x^* \in X^*$ the function $A \mapsto \sup\{x^*(x) : x \in M(A)\}$ is an $\mathbb{R} \cup \{+\infty\}$-valued signed measure. A measure $m\colon \mathcal{A} \longrightarrow X$ is called the *measure selection* of M if $m(A) \in M(A)$ for every $A \in \mathcal{A}$.

Some properties of the set of measure selections of a multimeasure are discussed.

Boris Paneah *On the general theory of the Cauchy type functional equations*

The main object of this talk is so called Cauchy type functional equation on some interval I. This is the equation of the form

$$F\big((\delta_1 + \delta_2)(t)\big) - F\big(\delta_1(t)\big) - F\big(\delta_2(t)\big) = h(t), \qquad t \in I,$$

with δ_1 and δ_2 being continuous maps from I into itself and h and F being given and unknown functions, respectively. The two themes are of the main interest for us: 1) the solvability properties of this equation; 2) all possible connections of this equation with diverse fields in mathematics and, may be, outside. The solvability properties, as will be clarified, are completely determined by mutual properties of the maps δ_1 and δ_2 (so called, "configurations of maps"). The two different configurations will be discussed. In the case of a Z-configuration it is successful to solve the Cauchy type functional equation in an explicit form. In the case of a $\mathcal{P}$-configuration (considerably more difficult case) the key role in formulating results and in their obtaining belong to some completely new dynamical system determined by the semigroup of maps in I generated by the two maps δ_1 and δ_2 . All mentioned solvability properties are discussed in this talk.

The second part of the talk is devoted to applications of the above results in such diverse fields as functional equations, integral geometry and partial differential equations. There are many unsolved problems concerning the topic.

Iwona Pawlikowska *Theorems of Flett's type and stability*

T.M. Flett [2] proved a version of the Lagrange Mean Value Theorem saying that for every differentiable function f on $[a, b]$ with $f'(a) = f'(b)$ there exists an intermediate point η such that

$$f(\eta) - f(a) = f'(\eta)(\eta - a).$$

We will show some generalizations of these two theorems. Das, Riedel and Sahoo, [1], proved the Hyers–Ulam stability of points η for which Flett's Mean Value Theorem holds true. We will also discuss stability of points for which our generalizations of Flett's MVT hold true.

[1] M. Das, T. Riedel, P.K. Sahoo, *Hyers–Ulam Stability of Flett's points*, Applied Math. Letters **16** (2003), 269-271.

[2] T.M. Flett, *A mean value theorem*, Math. Gazette **42** (1958), 38-39.

Bożena Piątek *On ∗-concave and convex multifunctions*

Let Y be a real Banach space. We shall show that the inclusion

$$\frac{1}{t-s}\int_s^t F(x)\,dx \subset \frac{F(s) \overset{*}{+} F(t)}{2}$$

for all $a \le s < t \le b$ can be used to characterize $*$-concave multifunctions $F\colon [a, b] \longrightarrow \mathrm{clb}(Y)$.

Magdalena Piszczek *Second Hukuhara derivative and a cosine family of linear set-valued functions*

Let K be a closed convex cone with the nonempty interior in a real Banach space and let $\mathrm{cc}(K)$ denote the family of all nonempty convex compact subsets of K. If $\{F_t :\ t \ge 0\}$ is a regular cosine family of continuous linear set-valued functions $F_t\colon K \longrightarrow \mathrm{cc}(K)$, $x \in F_t(x)$ for $t \geq 0$, $x \in K$ and $F_t \circ F_s = F_s \circ F_t$ for $s, t \geq 0$, then

$$D^2 F_t(x) = F_t(H(x))$$

for $x \in K$ and $t \geq 0$, where $D^2 F_t(x)$ denotes the second Hukuhara derivative of $F_t(x)$ with respect to t and $H(x)$ is the second Hukuhara derivative of this multifunction at $t = 0$.

Dorian Popa *Hyers–Ulam stability of some linear recurences*

In this talk we give a Hyers–Ulam–Rassias stability result for the first order linear recurrence in Banach spaces. As a consequence we obtain a Hyers–Ulam stability result for the p-order linear recurrence with constant coefficients.

Barbara Przebieracz *Near-iterability*

Inspired by Problem (3.1.12) posed by E. Jen in [2] we present various approaches to the concept of near-iterability. We deal with selfmappings of a real compact interval, characterize and compare a few classes of near-iterable functions in a sense. That includes the class of *almost iterable functions*, that is continuous $f\colon I \longrightarrow I$, for which there exists an iterable $g\colon I \longrightarrow I$ such that

$$f^n - g^n \text{ converges to } 0, \text{ everywhere in } I, \tag{E}$$

and the convergence is uniform on every interval with endpoints being two consecutive fixed points of f (cf. [1]); of functions satisfying (E); and that of functions f for which there exists an iterable g such that

$$f^n - g^n \text{ converges to } 0 \text{ almost everywhere in } I \tag{AE}$$

or

$$f^n - g^n \text{ converges to } 0 \text{ in measure.} \tag{M}$$

The measure appearing in (AE) and (M) is any Borel measure vanishing at points and taking positive values on nondegenerate intervals. We provide some conditions under which conditions (M) and (AE) are equivalent.

[1] W. Jarczyk, *Almost iterable functions*, Aequationes Math., **42** (1991), 202-219.

[2] Gy. Targonski, *New directions and open problems in iteration theory*, Ber. Math.-Statist. Sekt. Forschungsgesellsch. Joanneum, No. **229**. Forschungszentrum, Graz, 1984.

Maciej Sablik *More functional equations stemming from actuarial mathematics*

We recall some functional equations that have been motivated by natural questions asked in actuarial mathematics. Also, we discuss some new problems leading to functional equations and arising when some (generally accepted in the actuarial calculus) hypotheses are admitted.

Vsevolod Sakbaev *On the averaging of the family of regularizing solutions of Schrödinger equation with degeneration*

In this work we consider the ill-posed boundary-value problem (BVP) and define the procedure of regularization of it by a sequence of well-posed BVP which approximates the considered problem. The sequence of solutions of regularizing BVP can diverge. We choose a measure on the set of all subsequences of this sequence and define the procedure of averaging of the set of particular limits of the sequence by this measure.

As an example of the ill-posed BVP we choose the Cauchy problem for degenerated Schrödinger equation with the mixed type Hamilton operator $\mathbf{L}$ in the Hilbert space $H = L_2(\mathbb{R})$ and the initial data $u_0 \in H$. According to regularization method we consider the directed family of regularizing Cauchy problems for Schrödinger equations with the uniformly elliptic Hamilton operators $\mathbf{L}_\varepsilon$, $\varepsilon \in (0,1)$, and study the convergence of the family of regularizing solutions $u_\varepsilon(t)$ as $\varepsilon \to 0$. In the paper [1] we prove the weak convergence of the whole family of regularizing solutions $u_\varepsilon(t)$ as $\varepsilon \to 0$ and obtain the necessary and sufficient conditions of its strong convergence.

The aim of our investigation is the convergence of the family of the linear continuous functionals $\{f_\varepsilon(t,u_0,\cdot)\}$ on the Banach space $B(H)$ of bounded selfadjoint operators in H which are defined by the formula:

$$f_\varepsilon(t,u_0,\mathbf{A}) = (u_\varepsilon(t), \mathbf{A}u_\varepsilon(t)), \qquad \mathbf{A} \in B(H).$$

We consider the convergence of the sequence $\{f_\varepsilon\}$ in the $*$-weak topology of $B(H)^*$.

THEOREM 1
If the sequence $u_{\varepsilon_n}(t)$ diverges in the norm of space H then there is a bounded operator $\mathbf{A} \in B(H)$ such that the sequence $f_{\varepsilon_n}(t,u_0,\mathbf{A})$ diverges.

Therefore the pointwise convergence on the space $B(H)$ of the functionals $\{f_\varepsilon(t,u_0,\cdot)\}$ as $\varepsilon \to 0$ for any $u_0 \in H$ is impossible. Note by the symbol $\mathrm{Ls}_{\varepsilon\to 0} f_\varepsilon(t,u_0,\mathbf{A})$ the set of all limit points of the family $f_\varepsilon(t,u_0,\mathbf{A})$ as $\varepsilon \to 0$. According to idea of regularization we define the multi-valued map

$$F(t,u_0,\cdot)\colon B(H) \longrightarrow 2^{\mathbb{R}},$$

where $2^{\mathbb{R}}$ is the metric space of the subsets of $\mathbb{R}$ with the Hausdorff distance function, which acts on the space $B(H)$ by the rule

$$F(t,u_0,\mathbf{A}) = \mathrm{Ls}_{\varepsilon\to 0} f_\varepsilon(t,u_0,\mathbf{A}).$$

Lemma 1
Many-valued map $F(\cdot)$ is the continuous map of Banach space $\mathbb{R} \times H \times B(H)$ into the metric space $2^{\mathbb{R}}$ such that for any $\mathbf{A} \in B(H)$ the set $F(t, u_0, \mathbf{A})$ is the segment.

To construct the rule of the averaging of multifunction F we choose some bounded-additive measure on the set $F(t, u_0, \mathbf{A})$. We denote by W the set of bounded-additive nonnegative normalized measures on the σ-algebra of all subsets of the set of regularizing parameters ε such that the measure of any set M is equal to zero if the point 0 is not a limit point of the set M.

Theorem 2
The set W is nonempty and convex.

For any $\mu \in W$ we define the procedure of averaging of multifunction F by the measure μ.

Theorem 3
For any $\mu \in W$ there is the unique function $f_\mu(\cdot)$ which is continuous function on the Banach space $\mathbb{R} \times H \times B(H)$ such that $f_\mu(t, u_0, \cdot)$ is the positive normalized continuous linear functional on $B(H)$.

To prove this result we construct the regular method of generalized summation such that any bounded sequence is summable.

This work is partially supported by RFBR, grant No 04-01-00457.

[1] V.Zh. Sakbaev, *On the functionals on solutions of Cauchy problem for Schrödinger equation with degeneration on semiaxe*, Comp. Math. and Math. Phys. **44** (2004), 1654-1673.

Ekaterina Shulman *On addition theorems of rational type*

We investigate the functions $f\colon \Lambda \longrightarrow \mathbb{C}$, $\Lambda \subset \mathbb{C}$, which admit an addition theorem of the form

$$f(t+s) = \frac{\sum_{i=1}^{n} y_i(t) u_i(s)}{\sum_{j=1}^{m} z_j(t) v_j(s)}.$$

Here all functions are supposed to be continuously differentiable on some interval. Our approach is based on the reduction to a system of differential equations. The concepts of *joint linear dependence* and *joint quadratic dependence* of two families of functions are introduced. It's proved that there are only two possibilities in the case of *jointly linearly independent* $\{u_i\}$ and $\{v_j\}$:

a) the function f is a ratio of quasi-polynomials,

b) the families $\{y_i\}$ and $\{z_j\}$ are jointly quadratically dependent.

The second possibility is studied for $m = n = 2$. We apply our results to solving of the functional equation

$$f(t+s) = \frac{y_1(t)y_2(s) - y_2(t)y_1(s)}{z_1(t)z_2(s) - z_2(t)z_1(s)}.$$

Justyna Sikorska *On generalized stability of some orthogonal functional equations*

Starting with the papers of Th.M. Rassias [2] and Z. Gajda [1] new stability problems were introduced. For given Banach spaces $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$, $\varepsilon \geq 0$, they considered functions $f\colon X \longrightarrow Y$ satisfying the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p), \qquad x, y \in X.$$

We study this kind of stability for the orthogonally additive functional equation as well as some of its applications and generalizations.

[1] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci. **14** (1991), 431-434.

[2] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297-300.

Andrzej Smajdor *On concave iteration semigroups of linear set-valued functions*

The equality

$$G(x) + tG^2(x) = (I + tG)(G(x)) \tag{1}$$

is a necessary and sufficient condition under which a family $\{F^t : \ t \geq 0\}$ of linear continuous set-valued functions F^t, where

$$F^t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x), \tag{2}$$

is an iteration semigroup.

Moreover, a concave iteration semigroup of continuous linear set-valued functions with the infinitesimal generator G fulfilling (1) and such that $0 \in G(x)$ is of the form (2).

Wilhelmina Smajdor *On a set-valued version of a functional equation of Drygas*

We find the general solution of the functional equation

$$F(x+y)+F(x-y)=2F(x)+F(y)+F(-y). \tag{D}$$

Moreover, we show that every solution of (D) has a selection f satisfying the functional equation

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y).$$

Dariusz Sokołowski *Solutions with constant sign at infinity of a linear functional equation of infinite order*

Inspired by R.O. Davies and A.J. Ostaszewski [1] we investigate connections between the linear functional equation of the form

$$\varphi(x)=\int_S \varphi\left(x+M(s)\right)\sigma(ds) \tag{1}$$

and its characteristic equation

$$\int_S e^{\lambda M(s)}\sigma(ds)=1. \tag{2}$$

Here (S,Σ,σ) is a measure space with a finite measure σ and $M\colon S\longrightarrow\mathbb{R}$ is a Σ-measurable bounded function with $\sigma(M{\neq}0)>0$. By a *solution* of (1) we mean a Borel measurable real function φ defined on an interval of the form $(a,+\infty)$, Lebesgue integrable on every finite interval contained in $(a,+\infty)$ and such that for every $x>a+\sup\{|M(s)|:\ s\in S\}$ the integral $\int_S\varphi(x+M(s))\,\sigma(ds)$ exists and (1) holds.

According to [2, Theorem 2] if (1) has a solution with a constant sign (i.e., nonnegative and a.e. positive or nonpositive and a.e. negative), then (2) has a real root. It turns out that the existence of a solution of (1) with some additional properties guarantees the existence of a characteristic root with the specified sign. Namely we have the following two results.

Theorem 1
If (1) *has a solution with infinite limit at* $+\infty$, *then either* (2) *has a positive root or*

$$\sigma(S)=1 \qquad \textit{and} \qquad \int_S M(s)\,\sigma(ds)=0.$$

Theorem 2
If (1) *has a solution with a constant sign vanishing at* $+\infty$, *then* (2) *has a negative root.*

[1] R.O. Davies, A.J. Ostaszewski, *On a difference-delay equation*, J. Math. Anal. Appl. **247** (2000), 608-626.

[2] D. Sokołowski, *Solutions with constant sign at infinity of a linear functional equation of infinite order*, J. Math. Anal. Appl. **310** (2005), 144-160.

Paweł Solarz *Iterative roots for some homeomorphisms with infinitely many periodic points*

Let S^1 be the unit circle, $F\colon S^1 \longrightarrow S^1$ be an orientation-preserving homeomorphism and let $\operatorname{Per} F$ denote the set of all periodic points of F. Assume that the boundary of a set of cluster points of $\operatorname{Per} F$ is a finite set. We give the necessary and sufficient conditions for the existence of continuous and orientation-preserving solutions of the following equation:

$$G^m(z) = F(z), \qquad z \in S^1,$$

where $m \geq 2$ is an integer.

Joanna Szczawińska *On families of set-valued functions*

Let G be a linear continuous multifunction defined on a closed convex cone C in a Banach space X. J. Olko (see. [1]) has proved that for every $x \in C,\ t \geq 0$ a series $B^t(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} G^n(x)$ is convergent in the space of all nonempty compact convex subsets of X with the Hausdorff metric and $(B^t \circ B^s)(x) \subset B^{t+s}(x),\ x \in C,\ t, s \geq 0$.

We give a generalization of this result.

[1] J. Plewnia, *On a family of set-valued functions*, Publ. Math. Debrecen **46** (1995), 149-159.

Tomasz Szostok *Orlicz spaces and ω-convexity*

We deal with some modified version of convexity. Namely we fix an infinite interval $I \subset \mathbb{R}$ and a function $\omega\colon I \longrightarrow \mathbb{R}$. Then we consider a given function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$. Function f will be called *ω-convex* if and only if for all $x, y \in \mathbb{R}$, $x < y$ and for every $z \in (x, y)$ we have

$$f(z) \leq \omega(z + \alpha) + \beta,$$

where $\alpha, \beta \in \mathbb{R}$ are such that $\omega(x + \alpha) + \beta = f(x)$ and $\omega(y + \alpha) + \beta = f(y)$. It is clear that we have to make some assumption on ω in order to obtain the existence of such numbers. We present some applications of this notion to Orlicz spaces theory and our main result states that if an Orlicz function is ω-convex with $\omega(x) = x^p$ where $p > 1$, then f satisfies some inequalities used in Orlicz spaces theory.

Jacek Tabor *Shadowing and stability in metric groups I*

Joint work with Wojciech Jabłoński and Józef Tabor.

Let X be a complete metric space and let $\phi: X \longrightarrow X$ be a given mapping. We say that ϕ has the shadowing property if for every approximate orbit (x_k) there exists an exact orbit (y_k) which is close to (x_k).

One of the classical results from the stability of dynamical systems states that if ϕ is invertible and $\mathrm{Lip}(\phi^{-1}) < 1$ then ϕ has the shadowing property.

We generalize this result for the case when ϕ is locally invertible. As a corollary we obtain shadowing for the Julia set with a fixed parameter value. We also show that this shadowing result can be a useful tool in dealing with the stability of the Cauchy-type functional equations.

Józef Tabor *Shadowing and stability in metric groups II*

Joint work with Wojciech Jabłoński and Jacek Tabor.

We apply the concept of shadowing in dynamical systems to prove stability of functional equations. Using Hyers-like method we generalize some classical results to the case when the target space is a metric group. The condition of global 2-divisibility is replaced by a local one.

Gheorghe Toader *Complementary means and double sequences*

The well known arithmetic-geometric process of Gauss was generalized for arbitrary means as follows. Consider two means M and N defined on the interval J and two initial values $a, b \in J$.

By definition, the pair of sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ defined by

$$a_{n+1} = M(a_n, b_n) \quad \text{and} \quad b_{n+1} = N(a_n, b_n), \qquad n \geq 0\ ,$$

where $a_0 = a$, $b_0 = b$, is called a *(Gaussian) double sequence*. The mean M is *compoundable* with the mean N if the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are convergent to a common limit $M \otimes N(a, b)$ for each $a, b \in J$. The function $M \otimes N$ defines a mean which is called *G-compound mean.*

Some G-compound means can be determined using a characterization theorem proved in [1]: (Invariance Principle). *Suppose that $M \otimes N$ exists and is continuous. Then $M \otimes N$ is the unique mean P with the property $P(M, N) = P$.* The mean P is called (M, N)-*invariant*, and the mean N is called *P-complementary to M*.

Many non trivial examples of P-complementary means can be found in [2]. In fact, for the ten Greek means, we determined ninety complementaries of one mean with respect to another. They are done by direct computation. To make other determinations of complementaries, we use series expansions. We try to identify the complementary of one mean from a given family of means in an other family of means.

To illustrate this method, we study the complementariness with respect to the weighted geometric mean $\mathcal{G}_\lambda$, called also generalized inverse. We determine the generalized inverse of a weighted Gini mean in the family of extended means and converse, the generalized inverse of an extended mean in the family of weighted Gini means.

[1] J.M. Borwein, P.B. Borwein, *Pi and the AGM – a Study in Analytic Number Theory and Computational Complexity*, John Wiley & Sons, New York, 1986.

[2] Gh. Toader, Silvia Toader, *Greek means and the Arithmetic-Geometric Mean*, RGMIA Monographs, Victoria University, 2005. (ONLINE: http://rgmia.vu.edu.au/monographs)

Peter Volkmann *The absolute value of n-additive functions*

M being a *magma* (cf. Bourbaki), we call $f: M \longrightarrow \mathbb{R}$ *additive*, if $f(xy) = f(x) + f(y)$ $(x, y \in M)$. Suppose $F: M_1 \times \ldots \times M_n \longrightarrow \mathbb{R}$, $M_1, \ldots, M_n$ being magmas. Then $F(x_1, \ldots, x_n) = |f(x_1, \ldots, x_n)|$ with a function f being additive with respect to each variable if and only if all the functions $x_k \mapsto F(x_1, \ldots, x_n)$ $(x_k \in M_k)$ are absolute values of additive functions. Joint work with Attila Gilányi.

Janusz Walorski *On continuous and smooth solutions of the Schröder equation in normed spaces*

Let X and Y be normed spaces and $D \subset X$. We consider continuous and smooth solutions $\varphi: D \longrightarrow Y$ of the Schröder equation

$$\varphi(f(x)) = A\varphi(x),$$

where the function $f: D \longrightarrow D$ and the bounded linear operator $A: Y \longrightarrow Y$ are given.

Szymon Wąsowicz *Some inequalities connected with an approximate integration*

Some classical and new inequalities of an approximate integration are obtained with use of Hadamard type inequalities and delta-convex functions of higher orders. Error bounds of midpoint, trapezoidal and Simpson's rules are reproved. As an example of some new inequalities we give the following

Theorem
Let f be three times differentiable on $[a, b]$ and assume that

$$M_3(f) := \sup\big\{\big|f'''(x)\big| : \ x \in [a, b]\big\} < \infty.$$

Then

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{4}\left(f(a) + 3f\left(\frac{a+2b}{3}\right)\right)\right| \le \frac{M_3(f)(b-a)^4}{216}$$

and

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{4}\left(f(b)+3f\left(\frac{2a+b}{3}\right)\right)\right| \le \frac{M_3(f)(b-a)^4}{216}.$$

[1] Sz. Wąsowicz, *Some inequalities connected with an approximate integration*, J. Inequal. Pure Appl. Math. (JIPAM) **6(2)** (2005), Article 47.

Bing Xu *Analytic solutions of a nonlinear iterative equation*

Joint work with Weinian Zhang.

Analytic solutions of the functional equation

$$\sum_{j=0}^{k}\sum_{t=1}^{\infty} C_{t,j}(z)(\varphi(q_j z))^t = G(z)$$

are discussed in various distributions of the vector $(q_0,\dots,q_k)$. In the special case that $q_j = q^j$, $j = 0,\dots,k$, our main theorems imply corresponding results for a q-difference equation and weaken the conditions given by Si and Zhang [1]. Moreover, we discuss invertible analytic solutions of the q-difference equation, which enable us to apply our theorems to iterative equations weaken the conditions given by Si and Zhang [2].

[1] J. Si, W. Zhang, *Analytic solutions of a nonlinear iterative equation near neutral fixed points and poles*, J. Math. Anal. Appl. **284** (2003), 373-388.

[2] J. Si, W. Zhang, *Analytic solutions of a q-difference equation and applications to iterative equations*, J. Difference Eq. Appl. **10** (2004), 955-962.

Marek Cezary Zdun *A general class of iterative equation on the unit circle*

Joint work with Weinian Zhang.

We consider the problem of the existence and uniqueness of solutions of the iterative equation

$$\Phi(f(z), f^2(z), \dots, f^n(z)) = F(z), \qquad z \in T^1,$$

on the unite circle T^1 in a subclass of the class of homeomorphisms

$$H_1^0(T^1,T^1) = \{f \in C^0(T^1,T^1)\colon f(T^1) = T^1 \text{ homeomorphically and } f(1) = 1\},$$

where $C^0(T^1,T^1)$ consists of all continuous maps from T^1 into itself.

We discuss the influence of the domain of the continuous mapping Φ for the existence of these solutions.

Marek Żołdak *Stability of isometries in p-homogeneous F-spaces*

Joint work with Józef Tabor and Jacek Tabor.

The equation of isometry in Banach spaces is stable in the Ulam–Hyers sense. It happens that in complete Fréchet spaces this equation is not stable. We discuss the problem of stability in the class of Fréchet spaces with p-homogeneous norm, where $p \in (0, 1]$.

Problems and Remarks

1. Remark. *To K. Baron's talk*
Similar equations arise in the correlation function theory. For instance, E. Wegert has solved equation

$$\int_0^{+\infty} f(x)f(x+y)\,dx = g(y), \qquad y > 0, \tag{1}$$

with given $g(y)$. It is interesting to study also triple equations of the type

$$\int_0^{+\infty}\int_0^{+\infty} f(x,t)f(x+y,t)f(x,t+z)\,dxdt = g(y,z), \tag{2}$$

or equations with linear combinations of double and triple terms.

Vladimir Mityushev

2. Remark. *On stable probability distribution function*

Definition
Let $P \subset \mathbb{R}_+ = (0,\infty)$ be non-void. A probability distribution function $G\colon \mathbb{R} \longrightarrow [0,1]$ is said to be P-stable if

$$\forall\, p \in P\ \exists\, a_p > 0, b_p \in \mathbb{R}: \quad (G(a_p x + b_p))^p = G(x). \tag{$*$}$$

Theorem (Gnedenko 1943)
For a p.d.f. $G \neq F_{\delta_a}$ (for $a \in \mathbb{R}$), the following conditions are equivalent

(i) *for some p.d.f. F and some sequences $(a_n > 0)_{n\in\mathbb{N}}$, $(b_n \in \mathbb{R})_{n\in\mathbb{N}}$*

$$G(x) = \lim_{n\to\infty} F^n(a_n x + b_n);$$

(ii) *G is $\mathbb{N}$-stable;*

(iii) *G is $\mathbb{R}_+$-stable;*

(iv) *up to a linear map in the domain, $G = G_\alpha$, for some $\alpha \in \mathbb{R}$, where*

$$G_\alpha(x) := \begin{cases} \exp(-x^\alpha),\ x > 0, & \textit{if } \alpha < 0, \\ \exp(-\exp(-x)),\ x \in \mathbb{R}, & \textit{if } \alpha = 0, \\ \exp(-|x|^\alpha),\ x < 0, & \textit{if } \alpha > 0. \end{cases} \tag{$**$}$$

Remark
G is $\mathbb{R}_+$-stable iff G is P-stable with any $P \subset \mathbb{R}_+$ which generates a dense multiplicative subgroup of $\mathbb{R}_+$. In particular iff G is $\{2,3\}$-stable.

Remark
If G is $\{p\}$-stable with $p \in \mathbb{R}_+ \setminus \{1\}$, then the stability equation of $(*)$ has solution dependent on arbitrary function. The solution will be necessarily of class $(**)$ under additional requirement that G is suitably regularly varying.

As we know from a talk by Professor Maciej Sablik, the stable distribution functions have found application in the financial and/or actuarial mathematics, especially as models of the duration of life of some populations. Due to the natural discretization of data, the usual way of representing them is

$$F(m|n) := P\{T \geq m | T \geq n\} = \frac{P\{T \geq m\}}{P\{T \geq n\}}, \qquad m, n \in \mathbb{N},\ m \geq n.$$

The form of $F(m|n)$ shows that this is a "multiplicative distance function" (m.d.f.). In some "ideal" models the given F should be treated as a restriction of some other m.d.f. $\tilde{F}\colon \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow [0,1]$ s.t.

$$\tilde{F}(t|s) = \frac{P\{T \geq t\}}{P\{T \geq s\}} = \frac{\tilde{F}_0(t)}{\tilde{F}_0(s)}, \qquad t \geq s,\ t, s \in \mathbb{R}_+ .$$

Then $\tilde{F}_0$ is the p.d.f. of the duration of life.

As a matter of fact, the experimental data concern the population at the different actual age, which implies differences of data when compared to the "ideal" model. Therefore a question appears, how to restore $\tilde{F}_0$ from F. One of the ways presented by Professor Sablik is to assume that

$$\tilde{\tilde{F}}(m+u|n) := \varphi_u(F(m|n), F(m+1|n)), \qquad u \in [0,1],\ m, n \in \mathbb{N}$$

where φ_u is an average (increasing in u).

I would like to give an example that homogeneous averages are not giving the $\tilde{F}$ equal to a multiplicative distance function for the stable p.d.f.

Assume $P\{T \geq t\} = \exp(-t^2)$, $t > 0$. Then

$$F(m|n) = e^{-m^2+n^2}, \quad m \geq n, \qquad \tilde{F}(t|s) = e^{-t^2+s^2}, \quad t \geq s.$$

For $\varphi_u(x,y) = x^{1-u}y^u$,

$$\tilde{\tilde{F}}(m+u|n) = e^{-m^2+n^2}e^{-2mu-u} \neq \tilde{F}(m+u|n) = e^{-(m+u)^2+n^2}.$$

For $\varphi_u(x,y) = ((1-u)x^\alpha + uy^\alpha)^{1/\alpha}$,

$$\tilde{\tilde{F}}(m+u|n) = e^{-m^2+n^2}\left[1-u+ue^{-2m\alpha-\alpha}\right]^{1/\alpha} \neq \tilde{F}(m+u|n).$$

Remark

If $\tilde{\tilde{F}}$ is a m.d.f. given by φ_u of the form

$$\varphi_u(x,y) = \psi^{-1}((1-u)\psi(x) + u\psi(y))$$

then

$$\begin{aligned}\psi^{-1}&\left((1-u)\psi\left(\tilde{\tilde{F}}_0(m)/\tilde{\tilde{F}}_0(n)\right) + u\psi\left(\tilde{\tilde{F}}_0(m+1)/\tilde{\tilde{F}}_0(n)\right)\right)\\ &= \tilde{\tilde{F}}_0(m+u)/\tilde{\tilde{F}}_0(n)\end{aligned}$$

in other words

$$\varphi_u\left(\tilde{\tilde{F}}_0(m)/\tilde{\tilde{F}}_0(n);\tilde{\tilde{F}}_0(m+1)/\tilde{\tilde{F}}_0(n)\right) = \varphi_u\left(\tilde{\tilde{F}}_0(m),\tilde{\tilde{F}}_0(m+1)\right)/\tilde{\tilde{F}}_0(n),$$

which means homogeneity of φ_u with respect to multiplication by $1/\tilde{\tilde{F}}_0(n)$ for $n \in \{0,\dots,m\}$ on the pair $\left(\tilde{\tilde{F}}_0(m),\tilde{\tilde{F}}_0(m+1)\right)$, for $m = 0,1,2,\dots$.

Joachim Domsta

3. Remark. *On a paper by A. Matkowska*

The first of functional equations modelling the perfect capital market reads, cf. [2] and also [1, pp. 3-4], is the Cauchy equation with parameter t:

$$A(K_1+K_2,t) = A(K_1,t) + A(K_2,t), \qquad K_1 \geq 0,\ K_2 \geq 0. \tag{1}$$

Here $A(K,t)$ is the amount to which a capital K increases during a time interval of length t by interest compounding.

Equation (1) being rather unrealistic, a component, $B(K,t)$ say, is proposed in [2] to be added to its right-hand side. On taking also $K_1 = K_2 = \frac{1}{2}K$ we arrive at the iterative functional equation

$$A(K,t) = 2A\left(\frac{K}{2},t\right) + B(K,t). \tag{2}$$

The result from [2] on solutions of linear iterative functional equations in the class Lip^α of Hölder's functions yields the following (cf. [3])

Theorem

Let $B(\cdot,t) \in \mathrm{Lip}^\alpha$, $\alpha > 1$, and let $B(0,t) = 0$. Then equation (2) has exactly one solution $A(\cdot,t) \in \mathrm{Lip}^\alpha$ such that $A(0,t) = 0$. The solution is given by the formula

$$A(K,t) = \sum_{n=0}^{\infty} 2^n B\left(\frac{K}{2^n}\right), \qquad K \geq 0$$

provided that the series is convergent (in the norm of Lip^α).

[1] W. Eichhorn, *Functional Equations in Economics*, Addison–Wesley Comp., London – Amsterdam – Don Mills, ON – Sydney – Tokyo, 1978.

[2] A. Matkowska, *Hölder's solutions of a linear functional equation*, Zeszyty Naukowe AGH, Zagadnienia Techniczno-Ekonomiczne **48** (Informatyka) No. 3 (2003), 899-905.

[3] A. Matkowska, *Term investments and Hölder's solution of a functional equation*, in: Zarządzanie przedsiębiorstwem w warunkach integracji europejskiej, cz. II: Ekonomia, Informatyka i Metody Matematyczne, Akademia Górniczo-Hutnicza im. St. Staszica w Krakowie, Uczelniane Wyd. Nauk.-Dydakt., Kraków, 2004, 435-440.

Bogdan Choczewski

4. Remark.
The remark concerns the talk by N. Brillouët-Belluot, and more exactly the equation

$$x + f(y + f(x)) = y + f(x + f(y)) \tag{1}$$

that motivated her joint work with Weinian Zhang.

1. If we consider the question of solving (1) for function f mapping $\mathbb{C}$ into $\mathbb{C}$, then it is easy to observe that

$$z \mapsto az$$

where $a \in \{\frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}\}$ yields a continuous and additive solution to (1).

2. Let H be a Hamel base of $\mathbb{R}$ over $\mathbb{Q}$, and split it into two sets of the same cardinality, say H_1 and H_2. Let $\gamma\colon H_1 \longrightarrow H_2$ be an arbitrary bijection. Define $g\colon H \longrightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} \gamma(x), & \text{if } x \in H_1 \\ x - \gamma^{-1}(x), & \text{if } x \in H_2. \end{cases}$$

Then g is bijectively mapping H onto $g(H)$, and it can be easily checked that $g(H)$ is also a Hamel base of $\mathbb{R}$ over $\mathbb{Q}$. It follows that $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, the additive extension of g, is bijective. Moreover, it can be verified quite easily, f is a solution of (1).

3. The problem of existence of continuous solutions of (1), mapping $\mathbb{R}$ into $\mathbb{R}$, remains open.

Maciej Sablik

5. Remark and Problem. *A functional equation with singularity*
Let $g(z)$ be a known function analytic in the unit disc and continuous on its closure ($g \in C_A$). Let s be a given constant such that $0 < s < 1$. The functional equation

$$\varphi(z) = \varphi(sz) + g(z), \qquad |z| \leq 1 \tag{1}$$

is solvable in C_A iff $g(0) = 0$. This and other general results were obtained by M. Kuczma, J. Matkowski, W. Smajdor (see Kuczma's book [1] for details).

Let us consider the simple functional equation (1), but in another disc

$$\varphi(z) = \varphi(sz) + 1, \qquad |z-1| \le 1,\ z \ne 0, \tag{2}$$

in a class of functions analytic in $\{z \in \mathbb{C} :\ |z-1| < 1\}$, continuous on its closure except the origin, where i) φ is bounded; ii) φ has an integrable singularity.

Such functional equations have applications in mechanics of composites.

[1] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, PWN, Uniwersytet Śląski, Warszawa – Kraków – Katowice, 1985.

Vladimir Mityushev

6. Problem. *A functional equation arisen in the diffraction theory*
Given entire functions a, b, c in $\mathbb{C}$ and a constant h find entire solutions of the functional equation

$$a(z)\varphi(z) + b(z)\varphi(z+h) + c(z)\varphi(z-h) = 0.$$

Vladimir Mityushev

7. Remark. *On generalized subadditive functions bounded by some functions*
Results by B. Choczewski and Z. Powązka, in preparation.
The following problem was proposed on the International Mathematical Olympic Competition in the USA (1978), cf. [2, p. 79]:

> Prove that if, for every $x, y \in \mathbb{R}$, we have $\psi(x) \le x$ and $\psi(x+y) \le \psi(x) + \psi(y)$, then $\psi(x) = x$, $x \in \mathbb{R}$.

Motivated by this problem we returned to study of relations (cf. [2]) between solutions ψ of the inequality

$$\psi(x+y) \le F(\psi(x), \psi(y)), \tag{$F_\le$}$$

and those φ of the equation

$$\varphi(x+y) = F(\varphi(x), \varphi(y)). \tag{$F_=$}$$

The result pertinent to the problem reads

Theorem
Assume that $I \subset \mathbb{R}$ is an open interval and $F\colon I^2 \longrightarrow I$ is a continuous function such that there is an $e \in I$ for which $F(x, e) = x$, $x \in I$. If $\psi\colon \mathbb{R} \longrightarrow I$ is a continuous solution to $(F_\le)$, $\psi(0) = e$, and there is a continuous solution $\varphi\colon \mathbb{R} \longrightarrow I$ of $(F_=)$ such that $\psi(x) \le \varphi(x)$, $x \in \mathbb{R}$, then $\psi = \varphi$.

Comments

1. The general solution of $(F_{\leq})$, continuous in $\mathbb{R}$ has been determined in [1] under the assumption that F is a continuous group operation on I.

2. The condition $\psi(0) = e$ in the Theorem is essential.

3. The Theorem fails to be true when $\psi(x) \geq \varphi(x)$, $x \in \mathbb{R}$.

4. We have found bounds of continuous solutions to $(F_{\leq})$ which are solutions to $(F_{=})$ and as a corollary we have obtained the following condition for a continuous solution ψ of the inequality to satisfy the equation: *There exists a strictly increasing continuous solution f of $(F_{=})$ such that*

$$-\infty < \inf\{\tfrac{1}{t} f^{-1}(\psi(t)),\ t < 0\} = \sup\{\tfrac{1}{t} f^{-1}(\psi(x)),\ x > 0\} < +\infty.$$

[1] B. Choczewski, Z. Powązka, *Generalized subadditivity and convexity*, General Inequalities 2 (ed. by E.F. Beckenbach), Birkhäuser Verlag, Basel, 1980, 185-192.

[2] H. Pawłowski, *Zadania z olimpiad matematycznych z całego świata*, Oficyna Wyd. Tutor, Toruń, 1997.

Bogdan Choczewski

8. Remark. *On the problem of V. Mityushev*
Observe that the equation

$$\varphi(z) = \varphi(sz) + 1$$

has no bounded solution at all. For, if φ is such a solution, then

$$\varphi(z) = \varphi(s^n z) + n \qquad \text{for } n \in \mathbb{N}$$

and the sequence $\varphi(z) - \varphi(s^n z)$ is bounded; a contradiction. It shows that considering bounded solution of equations of the form

$$\varphi(z) = \varphi(sz) + g(z)$$

it is necessary to assume that (for every z from the domain) the sequence $\sum_{k=0}^{n} g(s^k z)$ is bounded.

Janusz Walorski

9. Problem.
Let $I \subset \mathbb{R}$ be an interval and $\lambda\colon I^2 \longrightarrow (0,1)$ be a continuous function. We say that a function $f\colon I \longrightarrow \mathbb{R}$ is λ-convex if

$$f\left(\lambda(x,y)x + (1-\lambda(x,y))y\right) \leq \lambda(x,y)f(x) + (1-\lambda(x,y))f(y), \qquad x,y \in I.$$

1. Does there exist non-constant λ for which λ-convexity imply Jensen convexity?

2. Does λ-convexity imply Jensen convexity for arbitrary continuous λ?

Kazimierz Nikodem

10. Remark.
In the joint paper with A. Matkowska [3] (written in Polish) the following functional equation with unknown f and g is considered

$$A(x,y) + C(x,y) = 2B(x,y), \qquad a \le x < y \le b, \tag{E}$$

where $A(x,y)$ stands for the integral of f over the interval $[x, x+\gamma(x,y)]$, $C(x,y)$ — for that over $[y-\gamma(x,y), y]$, and $B(x,y)$ and $\gamma(x,y)$ are the integrals over $[x,y]$ of $f \cdot g$ and g, respectively.

Steffensen's inequality [4] says that

$$A(x,y) \le B(x,y) \le C(x,y) \tag{S}$$

and it was originated by a problem from actuarial mathematics.

We have proved the following theorem, using (cf. [2]) the theory of continuous solutions of linear homogeneous functional equations.

Theorem
If $g\colon [a,b] \longrightarrow (\frac{1}{2}, 1)$ *or* $g\colon [a,b] \longrightarrow (0, \frac{1}{2})$ *and* $g \in \mathcal{C}^1[a,b]$, *then the pair* (f,g) *satisfies* (E) *if and only if* f *is a constant function.*

Similar result has been obtained in [1] under other conditions imposed on g, with the aid of a different method.

[1] B. Choczewski, I. Corovei, A. Matkowska, *On some functional equations related to Steffensen's inequality*, Ann. Acad. Paed. Cracoviensis 23, Studia Mathematica **4** (2004), 31-37.

[2] B. Choczewski, M. Kuczma, *On the 'indeterminate case' in the theory of a linear functional equation*, Fund. Math. **58** (1966), 163-175.

[3] B. Choczewski, A. Matkowska, *A functional equation connected with Steffensen's integral inequality applicable in actuarial mathematics* [Polish], Zeszyty Naukowe Akademii Górniczo-Hutniczej w Krakowie. Zagadnienia Techniczno-Ekonomiczne **50** (2005), to appear.

[4] J.F. Steffensen, *On certain inequalities between mean values and their applications to actuarial problems*, Skandinavisk Aktuarietidskrift (1918), 28-97.

Bogdan Choczewski

11. Problem. *Of Matkowski–Sutô type*
Let $\mathcal{M} = \{C_{f,g};\ f,g > 0\}$ where

$$C_{f,g}(x,y) = \frac{xf(x) + yf(y)}{f(x) + g(y)}, \qquad x, y > 0.$$

Solve the Matkowski–Sutô problem for $\mathcal{M}$, thus determine all the functions F, G, f, g, h, k, such that

$$C_{F,G} \circ (C_{f,g}, C_{h,k}) = C_{F,G}.$$

Remark
For $F(x) = G(x) = \frac{1}{\sqrt{x}}$, we get $C_{F,G} = \mathcal{G}$ (the geometric mean). We have

$$\mathcal{G}(C_{f,g}, C_{h,k}) = \mathcal{G}$$

if and only if there exists a constant $d > 0$, such that

$$h(x) = \frac{d}{xf(x)}, \qquad x > 0,$$

and

$$k(x) = \frac{d}{yg(y)}, \qquad y > 0.$$

Therefore, the problem has solutions.

Gheorghe Toader

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