AKADEMIA PEDAGOGICZNA im. KOMISJI EDUKACJI NARODOWEJ

45

Annales Academiae Paedagogicae Cracoviensis

Studia Mathematica VI

45

Annales Academiae Paedagogicae Cracoviensis

Studia Mathematica VI

Wydawnictwo Naukowe
Akademii Pedagogicznej
Kraków 2007

Contact: studmath@ap.krakow.pl
Available online at http://studmath.ap.krakow.pl

Contents

Editorial 5

Gheorghe Toader, *Complementary means and double sequences* 7

Anna Bahyrycz, *On sets associated to conditional equation of exponential function* 19

D.P. Patil, G. Tamone, *On type sequences and Arf rings* 35

Jadwiga Knop, Małgorzata Wróbel, *Some properties of i-connected sets* 51

Sergey Novikov, *Upper estimates of complexity of algorithms for multi-peg Tower of Hanoi problem* 57

Paweł Drygaś, Anna Król, *Some remarks on the domination between conjunctions and disjunctions* 67

Paweł Solarz, *Iterative roots of homeomorphisms possessing periodic points* 77

Jurij Povstenko, *Fractional calculus and diffusive stress* 95

Wioletta Syzdek, *Submaximal Riemann-Roch expected curve and symplectic packing* 101

Pavel Híc, Milan Pokorný, *Randomly* $C_n \cup C_m$ *graphs* 123

Vladimir Mityushev, *Boundary value problem and functional equations for overlapping disks* 131

Report of Meeting, *11th International Conference on Functional Equations and Inequalities, Będlewo, September 17-23, 2006* 137

Editorial

Annales Academiae Paedagogicae Cracoviensis Studia Mathematica (AAPC SM) has undergone radical changes during the last few months. These changes go in two directions.

The first, the most visible one is that we go online. As a matter of fact if you read this, it means that we are already online! The online version of the journal is not just an electronic version of the printed version. All papers accepted for publication will appear in the electronic form immediately after the typesetting procedures. I hope that in case of positive referees reports, between submitting the paper and its appearance online there should pass no more than six months. This way AAPC SM should become an attractive journal as far as the proceeding time is concern. Another advantage of publishing here is a practically unlimited dissemination of the results. At least for the opening period of five years we are about to maintain an open system with all articles available online to everybody at no charge. Depending on the number of accepted papers, at least once a year a printed version of the journal will appear.

The second direction of changes is less visible but quite vital to the perspective authors. So far each issue of AAPC SM was edited by a different team. From this issue on the journal has got fixed editors and clear submission procedures. I hope that with this change the journal will become on the one hand more homogeneous as far as the quality of published materials is concerned, and on the other hand it will attract authors and subjects so far not represented here. The journal is open to every branch of pure mathematics and to the entire mathematical community.

Jacek Chmieliński and Władysław Wilk who joined me in the editorial team prepared the online platform using Open Journal Systems. I would like to take the opportunity to thank them for the great job. At the same time I would like to thank Bogdan Nowecki, the Director of the Institute of Mathematics, as well as Tadeusz Budrewicz, the Rector of the Pedagogical University in Cracow responsible for scientific matters for their continuous support of that idea.

Well, here we are.

Tomasz Szemberg

Annales Academiae Paedagogicae Cracoviensis

Folia 45 Studia Mathematica VI

Gheorghe Toader

Complementary means and double sequences

Abstract. We look after the complementary means with respect to a weighted geometric mean of Stolarsky means in the family of Gini means and in the family of Stolarsky means.

1. Means

Usually, the means are given by the following

Definition 1
A *mean* (on the interval J) is a function $M\colon J^2 \longrightarrow J$, which has the property

$$\min(a,b) \le M(a,b) \le \max(a,b), \qquad \forall\, a,b \in J.$$

Each mean is *reflexive*, that is

$$M(a,a) = a, \qquad \forall\, a \in J,$$

which will be used also as a definition of $M(a,a)$ if it is necessary.

A mean can have additional properties.

Definition 2
The mean M is called:

a) *symmetric* if

$$M(a,b) = M(b,a), \qquad \forall\, a,b \in J;$$

b) *strict at the left* if

$$M(a,b) = a \Longrightarrow a = b,$$

strict at the right if

$$M(a,b) = b \Longrightarrow a = b,$$

and *strict* if is strict at the left and strict at the right.

AMS (2000) Subject Classification: 26E60.

We can compose three means M, N and P on J to define another mean $P(M,N)$ by

$$P(M,N)(a,b) = P(M(a,b),N(a,b)), \qquad a,b \in J.$$

Most of the usual means are defined on $\mathbb{R}_+$. So are the *Stolarsky* (or *extended*) means given by

$$\mathcal{E}_{r,s}(a,b) = \left(\frac{s}{r}\cdot\frac{a^r-b^r}{a^s-b^s}\right)^{\frac{1}{r-s}}, \qquad r\cdot s\cdot(r-s)\neq 0,\ a\neq b$$

and the weighted Gini means defined by

$$\mathcal{B}_{r,s;\lambda}(a,b) = \left[\frac{\lambda\cdot a^r+(1-\lambda)\cdot b^r}{\lambda\cdot a^s+(1-\lambda)\cdot b^s}\right]^{\frac{1}{r-s}}, \qquad r\neq s\,,$$

with $\lambda\in[0,1]$ fixed. Weighted Lehmer means, $\mathcal{C}_{r;\lambda}=\mathcal{B}_{r,r-1;\lambda}$ and weighted power means $\mathcal{P}_{r;\lambda}=\mathcal{B}_{r,0;\lambda}$ $(r\neq 0)$ are also used. We can remark that $\mathcal{P}_{0;\lambda}=\mathcal{G}_\lambda=\mathcal{B}_{r,-r;\lambda}$ is the weighted geometric mean. Also

$$\mathcal{B}_{r,s;0}=\mathcal{C}_{r;0}=\mathcal{P}_{r;0}=\Pi_2 \quad\text{and}\quad \mathcal{B}_{r,s;1}=\mathcal{C}_{r;1}=\mathcal{P}_{r;1}=\Pi_1\,,$$

where we denote by Π_1 and Π_2 respectively the first and the second projection defined by

$$\Pi_1(a,b)=a, \quad \Pi_2(a,b)=b, \qquad \forall\, a,b\geq 0.$$

2. Gaussian double sequences

The well known arithmetic-geometric process of Gauss was generalized for arbitrary means as follows. Consider two means M and N defined on the interval J and two initial values $a,b\in J$.

Definition 3
The pair of sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ defined by

$$a_{n+1}=M(a_n,b_n) \quad\text{and}\quad b_{n+1}=N(a_n,b_n)\,, \qquad n\geq 0\,, \tag{1}$$

where $a_0=a$, $b_0=b$, is called a *Gaussian double sequence.*

Definition 4
The mean M is *compoundable in the sense of Gauss (or G -compoundable) with the mean* N if the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ defined by (1) are convergent to a common limit $M\otimes N(a,b)$ for each $a,b\in J$.

The function $M\otimes N$ defines a mean which is called *Gaussian compound mean* (or *G-compound mean*).

The study of convergence is quite complicated. A general result was proved in [6]. If the means M and N are continuous and strict at the left then M and N are G-compoudable. There is also a variant for means which are strict at the right. The result is not valid if we assume one mean to be strict at the left and the other strict at the right. For example the means Π_1 and Π_2 are not G-compoudable (in any order). But, as we proved in [11], we can G-compose a continuous strict mean with any mean. A similar result was given recently in [5].

Some G-compound means can be determined using a characterization based on the following result, proved in [2] (and rediscovered in [5]).

THEOREM 5 (Invariance Principle)
Suppose that $M \otimes N$ exists and is continuous. Then $M \otimes N$ is the unique mean P which is (M, N)-invariant, that is

$$P(M, N) = P. \tag{2}$$

In fact, this is the way in which Gauss proved that the arithmetic-geometric G-compound mean can be represented by

$$\mathcal{A} \otimes \mathcal{G}(a, b) = \frac{\pi}{2} \cdot \left[\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} \right]^{-1}.$$

As usual $\mathcal{A} = \mathcal{P}_{1;\frac{1}{2}}$ and $\mathcal{G} = \mathcal{G}_{\frac{1}{2}}$.

3. Complementary means

As we can see in the last example, the product of two simply means, like $\mathcal{A}$ and $\mathcal{G}$, can be very complicated. So, to obtain some results, we change the point of view from the Invariance Principle. The determination of an invariant mean is very difficult. To simplify the search, we start with the following definitions given in [7]: the means $2\mathcal{A} - M$ and $\frac{\mathcal{G}^2}{M}$ are called complementary of M and inverse of M, respectively. We proposed in [11] a more general definition (that was given again in [10]).

DEFINITION 6
The mean N is called *P-complementary to M* (or *complementary with respect to P of M*) if it satisfies (2).

If the P-complementary of M exists and is unique, we denote it by M^P. It is easy to verify that $M^{\mathcal{A}} = 2\mathcal{A} - M$ and $M^{\mathcal{G}} = \frac{\mathcal{G}^2}{M}$, thus the definitions given in [7] are indeed special cases. An existence theorem of complementary means for symmetric means, was proved in [10]. It is easy to verify the following results.

Proposition 7
For every continuous strict mean M we have

$$M^M = M, \quad \Pi_1^M = \Pi_2, \quad M^{\Pi_2} = \Pi_2$$

and if M is moreover symmetric then

$$\Pi_2^M = \Pi_1 \,.$$

We shall call these results as trivial cases *of complementariness.*

Many non trivial examples can be found in [12]. In fact, for the ten Greek means (or neo-Pytagorean means, as they are called in [2]), we determined the ninety complementaries of a mean with respect to another. They are done by direct computation. To make other determinations of complementaries, we use series expansions. We try to identify the complementary of a mean from a given family of means in an other family of means.

To illustrate this method, we study the complementariness with respect to the weighted geometric mean $\mathcal{G}_\lambda$. Denote the $\mathcal{G}_\lambda$-complementary of M by $M^{\mathcal{G}(\lambda)}$ and we call it *generalized inverse of M*. For example, in [3] one was looking after the generalized inverse of a weighted Gini mean in the same family. Here we determine the generalized inverse of a weighted Gini mean in the family of Stolarsky means and converse, the generalized inverse of a Stolarsky mean in the family of weighted Gini means.

We omit anywhere to write λ if it is equal to $\frac{1}{2}$. In this case the generalized inverse of M is simply the inverse of M. For instance, in [1] they are determined the inverses of Stolarsky means in the same family of means.

4. Series expansion of means

For the study of some problems related to means, in [9] the power series expansion is used. Usually, for a mean M the series of the normalized function $M(1, 1-x)$, $x \in (0,1)$ is considered.

For example, in [8] it is proved that the extended mean $\mathcal{E}_{r,s}$ has the following first terms of the power series expansion

$$\begin{aligned}
&\mathcal{E}_{r,s}(1,1-x)\\
&\quad = 1 - \frac{x}{2} + (r+s-3)\cdot\frac{x^2}{24} + (r+s-3)\cdot\frac{x^3}{48}\\
&\qquad - \left[2(r^3+r^2s+rs^2+s^3) - 5(r+s)^2 - 70(r+s) + 225\right]\cdot\frac{x^4}{5760}\\
&\qquad - \left[2(r^3+r^2s+rs^2+s^3) - 5(r+s)^2 - 30(r+s) + 105\right]\cdot\frac{x^5}{3840} + \cdots.
\end{aligned}$$

Also in [4] it is given the series expansion of the weighted Gini mean, for $r \neq 0$,

$$
\begin{aligned}
&\mathcal{B}_{q,q-r;\nu}(1,1-x)\\
&\quad = 1-(1-\nu)\cdot x+\nu\,(1-\nu)\,(2q-r-1)\cdot\frac{x^2}{2!}\\
&\qquad -\nu\,(1-\nu)\cdot\{\nu[6q^2-6q\,(r+1)+(r+1)\,(2r+1)]\\
&\qquad\qquad -3q\,(q-r)-(r-1)\,(r+1)\}\cdot\frac{x^3}{3!}\\
&\qquad -\nu\,(1-\nu)\cdot\{\nu^2[-24q^3+36q^2\,(r+1)-12q\,(r+1)\,(2r+1)\\
&\qquad\qquad +(r+1)\,(2r+1)\,(3r+1)]\\
&\qquad\qquad +\nu[24q^3-12q^2\,(3r+1)+12q\,(r+1)\,(2r-1)\\
&\qquad\qquad -3\,(r+1)\,(2r+1)\,(r-1)]\\
&\qquad\qquad -4q^3+6q^2\,(r-1)-2q\left(2r^2-3r-1\right)\\
&\qquad\qquad +(r-2)\,(r-1)\,(r+1)\}\cdot\frac{x^4}{4!}\\
&\qquad -\nu\,(1-\nu)\cdot\left\{\nu^3\left[120q^4-240q^3\,(r+1)+120q^2(r+1)(2r+1)\right.\right.\\
&\qquad\qquad -20q(r+1)(2r+1)(3r+1)\\
&\qquad\qquad \left.+(r+1)(2r+1)(3r+1)(4r+1)\right]\\
&\qquad\qquad +\nu^2\left[-180q^4+180q^3(2r+1)-90q^2(r+1)(4r-1)\right.\\
&\qquad\qquad +30q(r+1)(2r+1)(3r-2)\\
&\qquad\qquad \left.-6(r-1)(r+1)(2r+1)(3r+1)\right]\\
&\qquad\qquad +\nu\left[70q^4-20q^3(7r-2)+10q^2(14r^2-6r-9)\right.\\
&\qquad\qquad -10q(r+1)(7r^2-12r+3)\\
&\qquad\qquad \left.+(r-1)(2r+1)(7r-11)(r+1)\right]\\
&\qquad\qquad -5q^4+10q^3(r-2)-5q^2(2r^2-6r+3)\\
&\qquad\qquad +5q(r-2)(r^2-2r-1)\\
&\qquad\qquad \left.-(r+1)(r-1)(r-2)(r-3)\right\}\cdot\frac{x^5}{5!}+\cdots.
\end{aligned}
$$

In the special case $r=1$ we get the series expansion of the weighted Lehmer means, while for $q=r\neq 0$ we get the series expansion of the weighted power means.

5. Generalized inverses

In [3] the series expansion of the generalized inverse of $\mathcal{B}_{p,p-r;\mu}$ was given.

THEOREM 8
The first terms of the series expansion of the generalized inverse of $\mathcal{B}_{p,p-q;\mu}$ *are*

$$
\begin{aligned}
&\mathcal{B}^{\mathcal{G}(\lambda)}_{p,p-q;\mu}\left(1,1-x\right)\\
&= 1-(\alpha\mu-\alpha+1)\cdot x-\alpha\left(1-\mu\right)\left[\left(\alpha+2p-q\right)\mu-\left(\alpha-1\right)\right]\cdot\frac{x^2}{2!}\\
&\quad+\alpha\left(1-\mu\right)\Big\{\left[6p^2+6\left(\alpha-q\right)p+\left(\alpha-q\right)\left(\alpha-2q\right)\right]\,\mu^2\\
&\qquad\qquad-\left[3p^2\,-3p(q-2\alpha)+\left(2\alpha-q\right)\,\left(\alpha-q\right)\right]\mu\\
&\qquad\qquad+\,\left(\alpha-1\right)\left(\alpha+1\right)\Big\}\cdot\frac{x^3}{3!}\\
&\quad-\alpha\left(1-\mu\right)\Big\{\big[24p^3\,+36p^2\left(\alpha-q\right)\,+12\left(\alpha-q\right)\left(\alpha-2q\right)p\\
&\qquad\qquad+\left(\alpha-q\right)\left(\alpha-2q\right)\,\left(\alpha-3q\right)\big]\,\mu^3\\
&\qquad\qquad+\big[-24p^3\,+12p^2\left(3q-4\alpha-1\right)\\
&\qquad\qquad\quad-12\left(2\alpha-2q+1\right)\left(\alpha-q\right)p\\
&\qquad\qquad\quad-\left(\alpha-2q\right)\left(\alpha-q\right)\,\left(3\alpha+2-3q\right)\big]\,\mu^2\\
&\qquad\qquad+\big[4p^3\,+6\left(2\alpha-q+1\right)p^2\\
&\qquad\qquad\quad+2\left(6\alpha\left(2\alpha-2q+1\right)-3q+2q^2-1\right)p\\
&\qquad\qquad\quad+\left(\alpha-q\right)(3\alpha^2+4\alpha-3q\alpha-2q+q^2\,-1)\big]\,\mu\\
&\qquad\qquad-\,\left(\alpha-1\right)\left(\alpha+1\right)\left(\alpha+2\right)\Big\}\cdot\frac{x^4}{4!}\\
&\quad+\alpha\left(1-\mu\right)\Big\{\big[120p^4\,+240p^3\left(\alpha-q\right)+120\left(\alpha-q\right)\left(\alpha-2q\right)p^2\\
&\qquad\qquad+20\left(\alpha-q\right)\left(\alpha-2q\right)\left(\alpha-3q\right)p\\
&\qquad\qquad+\left(\alpha-q\right)\cdot\left(\alpha-2q\right)(\alpha-3q)\,\left(\alpha-4q\right)\big]\,\mu^4\\
&\qquad\qquad+\big[-180p^4\,+60\left(6q-7\alpha-2\right)p^3\\
&\qquad\qquad\quad-90p^2(3\alpha-4q+2)\left(\alpha-q\right)\\
&\qquad\qquad\quad-30\left(\alpha-q\right)\left(\alpha-2q\right)\left(2\alpha+2-3q\right)p\\
&\qquad\qquad\quad-\left(\alpha-q\right)\left(\alpha-2q\right)\left(\alpha-3q\right)\,\left(4\alpha+5-6q\right)\big]\,\mu^3\\
&\qquad\qquad+\big[70p^4\,+20\left(10\alpha-7q+6\right)p^3\\
&\qquad\qquad\quad+10(-30q\alpha+18\alpha^2+24\alpha+3+14q^2-18q)p^2\\
&\qquad\qquad\quad+10\left(\alpha-q\right)\left(6\alpha^2+12\alpha-12q\alpha+7q^2-12q+3\right)p\\
&\qquad\qquad\quad+\left(6\alpha^2\,-12q\alpha+15\alpha+5+7q^2\,-15q\right)\left(\alpha-2q\right)\,\left(\alpha-q\right)\big]\,\mu^2\\
&\qquad\qquad+\big[-5p^4\,+10(q-2-2\alpha)p^3\\
&\qquad\qquad\quad+\left(30q\alpha-30\alpha^2-60\alpha-15\,-10q^2+30q\right)p^2\\
&\qquad\qquad\quad-52\alpha+\left(2-q\right)\left(2\alpha^2\,-2q\alpha+4\alpha-2q+q^2\,-1\right)p\\
&\qquad\qquad\quad-\left(\alpha-q\right)\left(4\alpha^3\,-6q\alpha^2+15\alpha^2-15q\alpha+10\alpha+4q^2\alpha\right.\\
&\qquad\qquad\qquad\qquad\left.-5+5q^2-q^3\,-5q\right)\big]\,\mu
\end{aligned}
$$

$$+(\alpha-1)(\alpha+1)(\alpha+2)(\alpha+3)\} \cdot \frac{x^5}{5!}+\cdots,$$

where $\alpha = \frac{\lambda}{1-\lambda}$.

Using it, we can prove the following result.

Theorem 9
The relation

$$\mathcal{B}^{\mathcal{G}(\lambda)}_{p,p-q;\mu} = \mathcal{E}_{r,s}$$

holds if and only if we are in one of the following three cases:

$$\begin{aligned} &\text{(i)} \quad \mathcal{B}^{\mathcal{G}(\frac{1}{3})}_{p,p-q;0} = \mathcal{E}_{r,-r}\,; \\ &\text{(ii)} \quad \mathcal{B}^{\mathcal{G}}_{p,-p} = \mathcal{E}_{r,-r}\,; \\ &\text{(iii)} \quad \mathcal{B}^{\mathcal{G}}_{p,0} = \mathcal{E}_{-p,-2p}\,, \end{aligned}$$

or in equivalent cases, taking into account that $\mathcal{B}_{s,r;\nu} = \mathcal{B}_{r,s;\nu}$ *and* $\mathcal{E}_{r,s} = \mathcal{E}_{s,r}$.

Proof. Equating the coefficients of x , in $\mathcal{B}^{\mathcal{G}(\lambda)}_{p,p-q;\mu}(1,1-x)$ and in $\mathcal{E}_{r,s}(1,1-x)$, we have the condition

$$2\alpha\mu = 2\alpha - 1. \tag{3}$$

Then the coefficients of x^2 give the condition

$$(r+s)\,\alpha = 3\,(2\alpha-1)\,(q-2p)\,, \tag{4}$$

and the coefficients of x^3 are equal if, moreover,

$$\mu\,(2\mu-1)\left(3p^2-3pq+q^2\right) = 0. \tag{5}$$

This gives the cases:

1) $\mu = 0$; thus from (3), $\alpha = \frac{1}{2}$ and $\lambda = \frac{1}{3}$ and from (4), $s = -r$, so (i)

and

2) $\mu = \frac{1}{2}$ which implies, from (3) and (4), $\alpha = 1$ and

$$r+s = 3\,(q-2p)\,. \tag{6}$$

Equating also the coefficients of x^4, we obtain in this case:

$$(2p-q)\left(2q^2-3qr+r^2-13pq+6pr+13p^2\right) = 0.$$

So, the case 2) splits into the cases:

2.1) $2p = q$, giving from (4), $r = -s$ and leading to (ii)

and

2.2)

$$(q - r)(2q - r) = p(13q - 6r - 13p). \tag{7}$$

The coefficients of x^5 are equal in this case. Equating also the coefficients of x^6, we obtain

$$p(p - q)(2p - q)\left(11p^2 - 11pq + 3q^2\right) = 0. \tag{8}$$

Thus, we get a new splitting in the cases:

2.2.1) $p = 0$, $r = q$ (from (7)) and $s = 2q$ (from (6)), so we have (iii);

2.2.2) $p = 0$, $r = 2q$ (from (7)) and $s = q$ (from (6)), so we have again (iii);

2.2.3) $p = q$, $r = -q$ (from (7)) and $s = -2q$ (from (6)), so we have (iii);

2.2.4) $p = q$, $r = -2q$ (again from (7)) and $s = -q$ (from (6)), so we have also (iii);

2.2.5) $q = 2p$, $r = \pm p\sqrt{5}$ (from (7)) and $s = -r$ (from (6)), so we have a special case of (ii).

We have no other possibilities in (8) or (5). By direct computation, we verify that the four obtained cases are valid. In fact they reduce to the following results:

$$\text{(i)}\ \ \Pi_1^{\mathcal{G}(\frac{1}{3})} = \mathcal{G}; \qquad \text{(ii)}\ \ \mathcal{G}^{\mathcal{G}} = \mathcal{G}; \qquad \text{(iii)}\ \ \mathcal{P}_p^{\mathcal{G}} = \mathcal{P}_{-p}\,.$$

Corollary 10

The relation

$$\mathcal{B}_{p,p-q;\mu}^{\mathcal{G}(\lambda)} = \mathcal{E}_{r,s}$$

holds only in the following nontrivial cases:

$$\text{(i)}\qquad \mathcal{B}_{p,p-q;0}^{\mathcal{G}(\frac{1}{3})} = \mathcal{E}_{r,-r}\,;$$

$$\text{(ii)}\qquad \mathcal{B}_{p,0}^{\mathcal{G}} = \mathcal{E}_{-p,-2p}\,,$$

or in equivalent cases, taking into account that $\mathcal{B}_{s,r;\nu} = \mathcal{B}_{r,s;\nu}$ *and* $\mathcal{E}_{r,s} = \mathcal{E}_{s,r}$ *.*

As it is done in [3], we can give the series expansion of the generalized inverse of $\mathcal{E}_{r,s}$, using Euler's formula: if the function f has the Taylor series expansion $f(x) = \sum_{n=0}^{\infty} a_n x^n$, p is a real number, and $[f(x)]^p = \sum_{n=0}^{\infty} b_n x^n$, then we have the recurrence relation

$$\sum_{k=0}^{n} [k(p+1) - n]\, a_k b_{n-k} = 0, \qquad n \geq 0.$$

Theorem 11

The first terms of the series expansion of the generalized inverse of $\mathcal{E}_{r,s}$ *are*

$$\begin{aligned}
&\mathcal{E}_{r,s}^{\mathcal{G}(\lambda)}(1,1-x)\\
&\quad= 1+\frac{1}{2}(\alpha-2)\cdot x-\frac{\alpha}{24}[r+s-3(\alpha-2)]\cdot x^2\\
&\qquad-\frac{\alpha}{48}\left[\alpha(r+s)-\left(\alpha^2-4\right)\right]\cdot x^3\\
&\qquad+\frac{\alpha}{5760}\Big[2\left(r^3+s^3\right)+5\alpha(r+s)^2+2rs(r+s)\\
&\qquad\qquad+10(r+s)\left(2-6\alpha-3\alpha^2\right)+15\left(\alpha^2-4\right)(\alpha+4)\Big]\cdot x^4\\
&\qquad+\frac{\alpha(\alpha+2)}{11520}\Big[2\left(r^3+s^3\right)+5\alpha(r+s)^2+20rs(r+s)\\
&\qquad\qquad+10(r+s)\left(2-4\alpha-\alpha^2\right)\\
&\qquad\qquad+3(\alpha-2)(\alpha+4)(\alpha+6)\Big]\cdot x^5+\cdots,
\end{aligned}$$

where $\alpha=\frac{\lambda}{1-\lambda}$.

Using it we can prove some new results.

THEOREM 12
The relation

$$\mathcal{E}_{r,s}^{\mathcal{G}(\lambda)}=\mathcal{B}_{p,p-q;\mu}$$

holds if and only if we are in one of the following cases:

$$\begin{aligned}
&\text{(i)} && \mathcal{E}_{r,s}^{\mathcal{G}(0)}=\mathcal{B}_{p,p-q;0}\,;\\
&\text{(ii)} && \mathcal{E}_{r,-r}^{\mathcal{G}(\frac{2}{3})}=\mathcal{B}_{p,p-q;1}\,;\\
&\text{(iii)} && \mathcal{E}_{r,-r}^{\mathcal{G}}=\mathcal{B}_{p,-p}\,;\\
&\text{(iv)} && \mathcal{E}_{2s,s}^{\mathcal{G}}=\mathcal{B}_{0,-s}
\end{aligned}$$

or in the equivalent cases, taking into account the properties $\mathcal{B}_{s,r;\nu}=\mathcal{B}_{r,s;\nu}$ *and* $\mathcal{E}_{r,s}=\mathcal{E}_{s,r}$ *.*

Proof. Equating the coefficients of x in $\mathcal{E}_{r,s}^{\mathcal{G}(\lambda)}(1,1-x)$ and in $\mathcal{B}_{p,p-q;\mu}(1,1-x)$ we have the condition

$$\alpha=2\mu. \tag{9}$$

Then, the equality of the coefficients of x^2 gives the condition

$$\alpha\left[r+s+(1-\mu)(2p-q)\right]=0.$$

We have thus

1) $\alpha=0$, which gives $\mu=0$ and so the equality (i)

or

2)

$$r + s + (1 - \mu)(2p - q) = 0. \tag{10}$$

Replacing (9) and (10) into the coefficients of x^3, we get the condition

$$\alpha(\alpha - 1)(\alpha - 2)\left(3p^2 - 3pq + q^2\right) = 0$$

The last factor cannot be zero for $q \neq 0$.

So, we have only the following possibilities:

2.1) $\alpha = 2$, so, by (9) and (10), $\mu = 1$ and $r = -s$, thus (ii)

or

2.2) $\alpha = 1$, for which we have to consider (9), (10) and also the equality of the coefficients of x^4, giving

$$(q - 2p)\left(2q^2 - 3qs - 13pq + 13p^2 + s^2 + 6ps\right) = 0.$$

This also splits into:

2.2.1) $q = 2p$, so $r = -s$, giving (iii);

and

2.2.2)

$$2q^2 - 3qs - 13pq + 13p^2 + s^2 + 6ps = 0. \tag{11}$$

In this case, the coefficients of x^5 are equal, while the coefficients of x^6 give

$$p(p - q)(2p - q)\left(11p^2 - 11pq + 3q^2\right) = 0.$$

We have so the possibilities:

2.2.2.1) $p = 0$, for which (11) becomes

$$(q - s)(2q - s) = 0,$$

obtaining two variants of (iv);

2.2.2.2) $p = q$, for which (11) becomes

$$(q + s)(2q + s) = 0,$$

obtaining again two variants of (iv);

2.2.2.3) $p = 2q$, for which (6) becomes

$$28q^2 + 9qs + s^2 = 0,$$

which has no solution $q \neq 0$.

The validity of the cases (i)-(iv) can be verified directly. In fact they can be rewritten in order as:

(i) $\mathcal{B}^{\Pi_2}_{p,p-q,\mu} = \Pi_2$; (ii) $\mathcal{G}^{\mathcal{G}(2/3)} = \Pi_1$; (iii) $\mathcal{G}^{\mathcal{G}} = \mathcal{G}$; (iv) $\mathcal{P}^{\mathcal{G}}_s = \mathcal{P}_{-s}$.

COROLLARY 13
The relation

$$\mathcal{E}^{\mathcal{G}(\lambda)}_{r,s} = \mathcal{B}_{p,q;\mu}$$

holds only in the following two nontrivial cases:

$$\text{(i)} \quad \mathcal{E}^{\mathcal{G}(\frac{2}{3})}_{r,-r} = \mathcal{B}_{p,q;1} \,;$$
$$\text{(ii)} \quad \mathcal{E}^{\mathcal{G}}_{2s,s} = \mathcal{B}_{0,-s} \,.$$

THEOREM 14
The relation

$$\mathcal{E}^{\mathcal{G}(\lambda)}_{r,s} = \mathcal{E}_{p,q}$$

holds if and only if we are in one of the following cases:

$$\text{(i)} \quad \mathcal{E}^{\mathcal{G}}_{r,-r} = \mathcal{E}_{p,-p} \,;$$
$$\text{(ii)} \quad \mathcal{E}^{\mathcal{G}}_{r,s} = \mathcal{E}_{-r,-s} \,.$$

or in an equivalent case, taking into account the property $\mathcal{E}_{r,s} = \mathcal{E}_{s,r}$.

Proof. Equating the coefficients of x, in $\mathcal{E}^{\mathcal{G}(\lambda)}_{r,s}(1,1-x)$ and in $\mathcal{E}_{p,q}(1,1-x)$ we have the condition $\alpha = 1$, thus $\lambda = \frac{1}{2}$. Then, the equality of the coefficients of x^2 gives the condition

$$r + s + p + q = 0.$$

In these conditions, the coefficients of x^3 are allways equal, but those of x^4 are equal only if

$$(r+q)\,(s+q)\,(r+s) = 0.$$

This implies:

1) $r = -q$, $s = -p$ giving a variant of (ii);
2) $s = -q$, $r = -p$ giving (ii);
3) $s = -r$, $q = -p$ which implies (i).

COROLLARY 15
The relation

$$\mathcal{E}^{\mathcal{G}(\lambda)}_{r,s} = \mathcal{E}_{p,q}$$

holds only in the following nontrivial case:

$$\mathcal{E}^{\mathcal{G}}_{r,s} = \mathcal{E}_{-r,-s} \,.$$

References

[1] J. Błasińska-Lesk, D. Głazowska, J. Matkowski, *An invariance of the geometric mean with respect to Stolarsky mean-type mappings*, Result. Math. **43** (2003), 42-55.

[2] J.M. Borwein, P.B. Borwein, *Pi and the AGM – a Study in Analytic Number Theory and Computational Complexity*, John Wiley & Sons, New York, 1986.

[3] I. Costin, *Generalized inverses of means*, Carpathian J. Math. **20** (2004), 2, 169-175.

[4] I. Costin, G. Toader, *A weighted Gini mean.*, in: Proceedings of the International Symposium: Specialization, Integration and Development, Section: Quantitative Economics, Babeş-Bolyai University Cluj-Napoca, Romania, 2003, 137-142.

[5] Z. Daróczy, Zs. Páles, *Gauss-composition of means and the solution of the Matkowski–Sutô problem*, Publ. Math. Debrecen **61** (2002), 157-218.

[6] D.M.E. Foster, G.M. Phillips, *Double mean processes*, Bull. Inst. Math. Appl. **22** (1986), no. 11-12, 170-173.

[7] C. Gini, *Le Medie*, Unione Tipografico Torinese, Milano, 1958.

[8] H.W. Gould, M.E. Mays, *Series expansions of means*, J. Math. Anal. Appl. **101** (1984), 2, 611-621.

[9] D.H. Lehmer, *On the compounding of certain means*, J. Math. Anal. Appl. **36** (1971), 183-200.

[10] J. Matkowski, *Invariant and complementary quasi-arithmetic means*, Aequationes Math. **57** (1999), 87-107.

[11] G. Toader, *Some remarks on means*, Anal. Numér. Théor. Approx. **20** (1991), 97-109.

[12] G. Toader, S. Toader, *Greek means and the Arithmetic-Geometric Mean*, RGMIA Monographs, Victoria University, 2005.
(ONLINE: http://rgmia.vu.edu.au/monographs).

Department of Mathematics
Technical University of Cluj
Romania
e-mail: Gheorghe.Toader@math.utcluj.ro

Received: 12 October 2005; final version: 12 May 2006;
available online: 15 June 2006.

Annales Academiae Paedagogicae Cracoviensis

Folia 45 Studia Mathematica VI

Anna Bahyrycz

On sets associated to conditional equation of exponential function

Abstract. In the present paper we give a description and properties of the system of cones over $\mathbb{Q}$ which are one of parameters determining the solutions of the conditional equation of exponential function.

1. Introduction

F.S. Roberts, generalizing the mathematical description of choices introduced by himself in [10, 11], considers functions $f\colon \mathbb{R}(n) \longrightarrow \mathbb{R}(n)$, where $\mathbb{R}(n) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n :\ x_j \geq 0 \text{ for } j = 1, \ldots, n\}$, $0_n := (0, \ldots, 0) \in \mathbb{R}^n$, which satisfy, among others, the following conditional functional equation:

$$f(x) \cdot f(y) \neq 0_n \implies f(x+y) = f(x) \cdot f(y),$$

for $x, y \in \mathbb{R}(n)$. Here $x + y$ and $x \cdot y$ are defined in the following way:

$$x + y := (x_1 + y_1, \ldots, x_n + y_n), \qquad x \cdot y := (x_1 \cdot y_1, \ldots, x_n \cdot y_n).$$

Mathematical theory of this approach was developed by Z. S. Rosenbaum [12], Z. Moszner [3,7,8,9,], G.L. Forti and L. Paganoni [4,5] and A. Bahyrycz [1,2,3].

As a generalization, one may consider functions $f\colon \mathbb{R}(n) \longrightarrow \mathbb{R}(m)$ (where n, m are arbitrary natural numbers, independent of each other) satisfying the condition

$$\forall\, x, y \in \mathbb{R}(n) :\ f(x) \cdot f(y) \neq 0_m \implies f(x+y) = f(x) \cdot f(y). \tag{1}$$

It may be shown that in such a case the description of all the solutions $f = (f_1, \ldots, f_m)$ of equation (1) takes the form:

$$f_\nu(x) = \begin{cases} \exp a_\nu(x) & \text{for } x \in Z_\nu\,, \\ 0 & \text{for } x \in \mathbb{R}(n) \setminus Z_\nu\,, \end{cases} \tag{2}$$

AMS (2000) Subject Classification: 39B22, 39B72.

where $a_\nu: \mathbb{R}^n \longrightarrow \mathbb{R}$ are additive functions for $\nu = 1, \ldots, m$, whereas the sets Z_ν satisfy the conditions

$$Z_1 \cup \ldots \cup Z_m = \mathbb{R}(n), \tag{3}$$

$$ij \neq 0_m \implies Z_1^{i_1} \cap \ldots \cap Z_m^{i_m} + Z_1^{j_1} \cap \ldots \cap Z_m^{j_m} \subset Z_1^{i_1 j_1} \cap \ldots \cap Z_m^{i_m j_m}, \tag{4}$$

where $i = (i_1, \ldots, i_m)$, $j = (j_1, \ldots, j_m) \in 0(m) := \{0,1\}^m \setminus \{0_m\}$,
$E_1 + E_2 := \{x + y : x \in E_1 \text{ and } y \in E_2\}$ for $E_1, E_2 \subset \mathbb{R}^n$,
$E^1 := E$, $E^0 := \mathbb{R}(n) \setminus E$ for $E \subset \mathbb{R}(n)$.
The proof of this fact is analogous to the proof for the case of $n = m$ in [7].

Because of the form of the solutions, equation (1), will be called the conditional equation of exponential function.

Let us notice that the parameters determining the solutions of equation (1) are systems of sets $Z_1, ..., Z_m$ satisfying conditions (3) and (4), as well as additive functions $a_\nu: \mathbb{R}^n \longrightarrow \mathbb{R}$. For this reason, it is interesting to find conditions equivalent to condition (4) under the assumption of condition (3).

2. Auxiliary lemma

Let us recall the following definition:

Definition 1
A set C is called a cone over an ordered field $\mathbb{K}$ ($\mathbb{K} = \mathbb{Q}$ or $\mathbb{K} = \mathbb{R}$), if $x + y \in C$ for all $x, y \in C$ and $\alpha x \in C$ for all $x \in C$ and $\alpha \in \mathbb{K} \cap (0, +\infty)$.

Let us observe that if the sets $Z_1, \ldots, Z_m$ satisfy condition (3), then for every $a \in \mathbb{R}(n)$ and every $k \in \{1, \ldots, m\}$ there exists a unique $i_k \in \{0,1\}$ such that $a \in Z_k^{i_k}$ (further on it will be denoted by $i_k(a)$) and at least one j such that $i_j(a) = 1$.

Lemma 1
If a system of sets $Z_1, \ldots, Z_m$ satisfies conditions (3) and (4), then for every non-empty subset $\{l_1, \ldots, l_p\}$ of the set $\{1, \ldots, m\}$ and for every $(i_{l_1}, \ldots, i_{l_p}) \in 0(p)$ the set $Z_{l_1}^{i_{l_1}} \cap \ldots \cap Z_{l_p}^{i_{l_p}}$ is a cone over $\mathbb{Q}$.

Proof.

Step 1°

Consider arbitrary $\emptyset \neq \{l_1, \ldots, l_p\} \subset \{1, \ldots, m\}$ and $(i_{l_1}, \ldots, i_{l_p}) \in 0(p)$. Take $x \in Z_{l_1}^{i_{l_1}} \cap \ldots \cap Z_{l_p}^{i_{l_p}}$ and $y \in Z_{l_1}^{i_{l_1}} \cap \ldots \cap Z_{l_p}^{i_{l_p}}$. Since $(i_{l_1}, \ldots, i_{l_p}) \neq 0_p$, there exists $\nu \in \{l_1, \ldots, l_p\}$ such that $i_\nu = 1$. Obviously, $x \in Z_1^{i_1(x)} \cap \ldots \cap Z_m^{i_m(x)}$, $y \in Z_1^{i_1(y)} \cap \ldots \cap Z_m^{i_m(y)}$ and $i_{l_k}(x) = i_{l_k}(y) = i_{l_k}$ for every $k \in \{1, \ldots, p\}$. Since $i_\nu(x) = i_\nu(y) = 1$, from condition (4) it follows that

$$\begin{aligned} x+y &\in Z_1^{i_1(x)}\cap\ldots\cap Z_m^{i_m(x)} + Z_1^{i_1(y)}\cap\ldots\cap Z_m^{i_m(y)}\\ &\subset Z_1^{i_1(x)i_1(y)}\cap\ldots\cap Z_m^{i_m(x)i_m(y)}\\ &\subset Z_{l_1}^{(i_{l_1})^2}\cap\ldots\cap Z_{l_p}^{(i_{l_p})^2}\\ &= Z_{l_1}^{i_{l_1}}\cap\ldots\cap Z_{l_p}^{i_{l_p}}. \end{aligned}$$

Therefore, for every $\emptyset\neq\{l_1,\ldots,l_p\}\subset\{1,\ldots,m\}$ and every $(i_{l_1},\ldots,i_{l_p})\in 0(p)$ the set $Z_{l_1}^{i_{l_1}}\cap\ldots\cap Z_{l_p}^{i_{l_p}}$ is closed under addition.

Step 2°

Take arbitrary $\emptyset\neq\{l_1,\ldots,l_p\}\subset\{1,\ldots,m\}$ and $(i_{l_1},\ldots,i_{l_p})\in 0(p)$. Consider the set $Z_{l_1}^{i_{l_1}}\cap\ldots\cap Z_{l_p}^{i_{l_p}}$. Take $x\in Z_{l_1}^{i_{l_1}}\cap\ldots\cap Z_{l_p}^{i_{l_p}}$ and $k\in\mathbb{N}$. The set $Z_{l_1}^{i_{l_1}}\cap\ldots\cap Z_{l_p}^{i_{l_p}}$ is closed under addition, so $kx\in Z_{l_1}^{i_{l_1}}\cap\ldots\cap Z_{l_p}^{i_{l_p}}$. Consider $\frac{1}{k}x$. Obviously,

$$\frac{1}{k}x\in Z_1^{i_1\left(\frac{1}{k}x\right)}\cap\ldots\cap Z_m^{i_m\left(\frac{1}{k}x\right)}$$

and since $Z_1\cup\ldots\cup Z_m=\mathbb{R}(n)$, there exists $\nu\in\{1,\ldots,m\}$ such that $i_\nu(\frac{1}{k}x)=1$.

It follows from step 1° that the set $Z_1^{i_1\left(\frac{1}{k}x\right)}\cap\ldots\cap Z_m^{i_m\left(\frac{1}{k}x\right)}$ is closed under addition, so consequently

$$x=k\cdot\left(\frac{1}{k}x\right)\in Z_1^{i_1\left(\frac{1}{k}x\right)}\cap\ldots\cap Z_m^{i_m\left(\frac{1}{k}x\right)}.$$

As a result, for every $l\in\{1,\ldots,m\}$ we have

$$i_l(x)=i_l\Big(\frac{1}{k}x\Big),$$

which means that

$$\frac{1}{k}x\in Z_{l_1}^{i_{l_1}}\cap\ldots\cap Z_{l_p}^{i_{l_p}}.$$

We have shown that the set $Z_{l_1}^{i_{l_1}}\cap\ldots\cap Z_{l_p}^{i_{l_p}}$ is a cone over $\mathbb{Q}$.

Remark 1

Notice that for $m\neq 1$ the converse of Lemma 1 is false, and here is an example for $m=n=2$.

Define

$$\begin{aligned} Z_1&=\{(x,y)\in\mathbb{R}(2): y\le 2x\},\\ Z_2&=\{(x,y)\in\mathbb{R}(2): y\ge \tfrac{1}{2}x\}. \end{aligned}$$

It is obvious that the sets Z_1^1, Z_2^1, $Z_1^1\cap Z_2^1$, $Z_1^1\cap Z_2^0$ and $Z_1^0\cap Z_2^1$ are cones over $\mathbb{R}$ (because the sets Z_1^1, Z_2^1, Z_1^0, Z_2^0 are cones over $\mathbb{R}$). Condition (4) is not satisfied, since $(1,0)\in Z_1^1\cap Z_2^0$ and $(2,2)\in Z_1^1\cap Z_2^1$ whereas $(3,2)\notin Z_1^1\cap Z_2^0$.

Remark 2
Let us observe that if the sets $Z_1, \dots, Z_m$ satisfy conditions (3) and (4), then for $m \in \{1,2\}$ the set Z_1^0 in the case of $m=1$ and the sets Z_1^0, Z_2^0, $Z_1^0 \cap Z_2^0$ for $m=2$ are also cones over $\mathbb{Q}$. If $m=1$, then $Z_1 = \mathbb{R}(n)$, so $Z_1^0 = \emptyset$. If $m=2$, then $Z_1 \cup Z_2 = \mathbb{R}(n)$, so $\mathbb{R}(n) \setminus Z_1^1 \subset Z_2^1$ and $\mathbb{R}(n) \setminus Z_2^1 \subset Z_1^1$, and consequently $Z_1^0 \subset Z_2^1$ and $Z_2^0 \subset Z_1^1$. Therefore

$$Z_1^0 \cap Z_2^1 = Z_1^0 \qquad \text{and} \qquad Z_2^0 \cap Z_1^1 = Z_2^0\,,$$

and, by Lemma 1, Z_1^0 and Z_2^0 are cones over $\mathbb{Q}$ and the set $Z_1^0 \cap Z_2^0$ is empty.

If the sets $Z_1, \dots, Z_m$ satisfy conditions (3) and (4), then for $m > 2$ not every set

$$Z_{l_1}^0 \cap \ldots \cap Z_{l_p}^0\,,$$

where $\emptyset \neq \{l_1, \dots, l_p\}$ is necessarily a cone over $\mathbb{Q}$. Here is a suitable example for $n=2$ and $m=3$.

Define

$$\begin{aligned}
Z_1 &= \{(x,y) \in \mathbb{R}(2) :\ y \le \tfrac{1}{2}x\},\\
Z_2 &= \{(x,y) \in \mathbb{R}(2) :\ \tfrac{1}{2}x < y \le 2x\},\\
Z_3 &= \{(x,y) \in \mathbb{R}(2) :\ y > 2x\}.
\end{aligned}$$

It can be easily checked that the sets Z_1, Z_2, Z_3 satisfy conditions (3) and (4) and that the set

$$Z_2^0 = \{(x,y) \in \mathbb{R}(2) :\ y \le \tfrac{1}{2}x\} \cup \{(x,y) \in \mathbb{R}(2) :\ y > 2x\}$$

is not a cone over $\mathbb{Q}$.

3. Main result

The following theorem gives the conditions equivalent to condition (4) under the assumption of condition (3).

Theorem 1
Assume that sets $Z_1, \dots, Z_m$ satisfy condition (3). The following conditions are equivalent:

(i) *condition (4);*

(ii) *the sets $Z_1, \dots, Z_m$ are cones over $\mathbb{Q}$ for which*

$$Z_l^1 + Z_l^1 \cap Z_k^0 \subset Z_l^1 \cap Z_k^0 \tag{5}$$

for all $k, l \in \{1, \dots, m\}$ (it is enough to consider $k \neq l$);

(iii) *the sets $Z_1, \dots, Z_m$ satisfy the conditions*

$$Z_k^1 + Z_k^1 \subset Z_k^1 \,, \tag{6}$$

for every $k \in \{1, \ldots, m\}$ *and condition* (5) *for all* $k, l \in \{1, \ldots, m\}$*;*

(iv) *for all* $x, y \in \mathbb{R}(n)$ *if there exists* $\nu \in \{1, \ldots, m\}$ *such that* $x \in Z_\nu$ *and* $y \in Z_\nu$ *, then*

$$\forall\, k \in \{1, \ldots, m\}: \; x + y \in Z_k \iff x \in Z_k \text{ and } y \in Z_k \,, \tag{7}$$

(v) *for all* $k, l \in \{1, \ldots, m\}$ *the following implication holds*

$$ij \neq 0_2 \implies Z_k^{i_k} \cap Z_l^{i_l} + Z_k^{j_k} \cap Z_l^{j_l} \subset Z_k^{i_k j_k} \cap Z_l^{i_l j_l}, \tag{8}$$

where $i = (i_k, i_l)$*,* $j = (j_k, j_l) \in 0(2)$*.*

Proof. (i)⇒(ii) By Lemma 1, the sets $Z_1, \ldots, Z_m$ are cones over $\mathbb{Q}$. Assume that $z \in Z_l^1 + Z_l^1 \cap Z_k^0$ for $k, l \in \{1, \ldots, m\}$ such that $k \neq l$. It implies that there exist such $x \in Z_l^1$ and $y \in Z_l^1 \cap Z_k^0$ that $x + y = z$. Clearly, $x \in Z_1^{i_1(x)} \cap \ldots \cap Z_m^{i_m(x)}$, $y \in Z_1^{i_1(y)} \cap \ldots \cap Z_m^{i_m(y)}$ and $i_l(x) = i_l(y) = 1$. By applying condition (4) we get

$$\begin{aligned} x + y &\in Z_1^{i_1(x)} \cap \ldots \cap Z_m^{i_m(x)} + Z_1^{i_1(y)} \cap \ldots \cap Z_m^{i_m(y)} \\ &\subset Z_1^{i_1(x) i_1(y)} \cap \ldots \cap Z_m^{i_m(x) i_m(y)} \subset Z_l^{i_l(x) i_l(y)} \cap Z_k^{i_k(x) i_k(y)} \\ &= Z_l^1 \cap Z_k^0 \,. \end{aligned}$$

(ii)⇒(iii) The sets $Z_1, \ldots, Z_m$ are cones over $\mathbb{Q}$, so each of them is closed under addition, condition (6) is therefore satisfied for every $k \in \{1, \ldots, m\}$, which completes the proof.

(iii)⇒(iv) Let $x, y \in \mathbb{R}(n)$, $\nu \in \{1, \ldots, m\}$ be arbitrary and such that $x \in Z_\nu$ and $y \in Z_\nu$.

(⇐ in (7)) Lets us take an arbitrary $k \in \{1, \ldots, m\}$ such that $x \in Z_k^1$ and $y \in Z_k^1$. Condition (6) yields

$$x + y \in Z_k^1 + Z_k^1 \subset Z_k^1 \,.$$

(⇒ in (7)) Fix an arbitrary $k \in \{1, \ldots, m\}$ such that $x + y \in Z_k^1$. If $k = \nu$, then, by assumption, $x \in Z_k^1$ and $y \in Z_k^1$. If $k \neq \nu$, then suppose that $x \in Z_k^0$ or $y \in Z_k^0$. Without loss of generality we may assume that $y \in Z_k^0$. Then, by (5), we obtain

$$x + y \in Z_\nu^1 + Z_\nu^1 \cap Z_k^0 \subset Z_\nu^1 \cap Z_k^0 \subset Z_k^0 \,,$$

which is a contradiction, since the sets Z_k^1 and Z_k^0 are mutually disjoint. It means that

$$x \in Z_k^1 \qquad \text{and} \qquad y \in Z_k^1 \,,$$

which finishes the proof.

(iv)⇒(v) Let $k,l \in \{1,\dots,m\}$ and $i=(i_k,i_l)$, $j=(j_k,j_l)\in 0(2)$ be such that $ij \neq 0_2$. Consider $z \in Z_k^{i_k}\cap Z_l^{i_l} + Z_k^{j_k}\cap Z_l^{j_l}$. Then there exist $x\in Z_k^{i_k}\cap Z_l^{i_l}$ and $y \in Z_k^{j_k}\cap Z_l^{j_l}$ such that $z=x+y$. Since $ij\neq 0_2$, there exist $\nu\in\{k,l\}$ such that $i_\nu = j_\nu = 1$, and condition (7) gives

$$x+y\in Z_\nu^1 = Z_\nu^{i_\nu j_\nu}.$$

For $t\in\{k,l\}\setminus\{\nu\}$ the following cases are possible:

a) $i_t = j_t = 1$,

b) $i_t = 0$ or $j_t = 0$.

Case a). If $i_t = j_t = 1$, then $x\in Z_t^1$ and $y\in Z_t^1$, so, by condition (7), we obtain

$$x+y\in Z_t^1 = Z_t^{i_t j_t}.$$

Case b). If $i_t = 0$ or $j_t=0$, then it is not true that $x\in Z_t^1$ and $y \in Z_t^1$. Since $x\in Z_\nu^1$ and $y\in Z_\nu^1$, from condition (iv) it follows that it is not true that $x+y\in Z_t^1$, therefore

$$x+y\in Z_t^0 = Z_t^{i_t j_t}.$$

Thus

$$x+y\in Z_k^{i_k j_k}\cap Z_l^{i_l j_l}.$$

(v)⇒(i) Let $i,j\in 0(m)$ be such that $ij\neq 0_m$. Consider $z\in Z_1^{i_1}\cap\ldots\cap Z_m^{i_m} + Z_1^{j_1}\cap\ldots\cap Z_m^{j_m}$. Then there exist $x\in Z_1^{i_1}\cap\ldots\cap Z_m^{i_m}$ and $y\in Z_1^{j_1}\cap\ldots\cap Z_m^{j_m}$ such that $z=x+y$. Since $ij\neq 0_m$, there exists $k\in\{1,\dots,m\}$ such that $i_k=j_k=1$.

Let $l\in\{1,\dots,m\}$. In such a case $(i_k,i_l)\cdot(j_k,j_l)\neq 0_2$ and, by condition (v), we obtain

$$x+y\in Z_k^{i_k j_k}\cap Z_l^{i_l j_l}\subset Z_l^{i_l j_l},$$

so

$$x+y\in Z_1^{i_1 j_1}\cap\ldots\cap Z_m^{i_m j_m},$$

which completes the proof of Theorem 1.

Theorem 1 leads to the following

Corollary 1
If sets $Z_1,\dots,Z_m$ are pairwise disjoint and satisfy condition (3)*, then condition* (4) *is equivalent to the following condition: $Z_1,\dots,Z_m$ are cones over $\mathbb{Q}$.*

Proof. Assume that sets $Z_1,\dots,Z_m$ are pairwise disjoint and satisfy condition (3). Then, by Theorem 1, condition (4) is equivalent to condition (ii), namely, the sets $Z_1, \ldots, Z_m$ are cones over $\mathbb{Q}$ which satisfy condition (5) for all $k, l \in \{1, \ldots, m\}$ such that $k \neq l$. Because of the fact that for all $k, l \in \{1, \ldots, m\}$ such that $k \neq l$ the sets Z_k, Z_l are disjoint, the condition

$$Z_l^1 + Z_l^1 \cap Z_k^0 \subset Z_l^1 \cap Z_k^0$$

is reduced to the condition

$$Z_l^1 + Z_l^1 \subset Z_l^1,$$

because $Z_l^1 \subset \mathbb{R}(n) \setminus Z_k^1 = Z_k^0$ (since $Z_k^1 \cap Z_l^1 = \emptyset$), and therefore $Z_l^1 \cap Z_k^0 = Z_l^1$. Similarly, the condition

$$Z_k^1 + Z_k^1 \cap Z_l^0 \subset Z_k^1 \cap Z_l^0$$

is reduced to the condition

$$Z_k^1 + Z_k^1 \subset Z_k^1 .$$

Thus, in order to verify condition (5) for two disjoint sets contained in $\mathbb{R}(n)$ it suffices to check whether these sets are closed under addition, which completes the proof.

Remark 3

Notice that if sets $Z_1, \ldots, Z_m$ satisfy condition (3), then the fact that they fulfil condition (4) implies that they satisfy condition (5). The converse implication is not true, the assumption that the sets $Z_1, \ldots, Z_m$ are cones over $\mathbb{Q}$ cannot be omitted in condition (ii). Here is an example for $m = n = 2$.

Define

$$Z_1 = \{(x, y) \in \mathbb{R}(2) : \ x \leq 1\},$$
$$Z_2 = \mathbb{R}(2).$$

The sets Z_1, Z_2 satisfy condition (3), as well as the conditions

$$Z_1^1 + Z_1^1 \cap Z_2^0 = \emptyset = Z_1^1 \cap Z_2^0 , \quad Z_2^1 + Z_2^1 \cap Z_1^0 = \{(x, y) \in \mathbb{R}(2) : x > 1\} = Z_2^1 \cap Z_1^0.$$

Therefore, condition (5) is satisfied for $k, l \in \{1, 2\}$. Obviously, the sets Z_1, Z_2 do not satisfy condition (4), since the set Z_1 is not a cone over $\mathbb{Q}$.

Remark 4

Let us have a closer look at condition (4). Notice that the number of pairs $(i, j) \in \{0, 1\}^m \times \{0, 1\}^m$ equals 4^m. The product $ij = 0_m$ if and only if $i_k = 0$ and $j_k = 0$ or $i_k = 1$ and $j_k = 0$ or $i_k = 0$ and $j_k = 1$ for every $k \in \{1, \ldots, m\}$. Therefore the number of pairs $(i, j) \in 0(m) \times 0(m)$ satisfying the condition $ij \neq 0_m$ is equal to $4^m - 3^m$.

Observe that condition (4) is symmetrical with respect to i and j, so instead of verifying $4^m - 3^m$ conditions in order to verify condition (4) we will show that it suffices to verify $\frac{4^m-3^m+2^m-1}{2}$ conditions. The number of pairs $(i,i) \in 0(m) \times 0(m)$ is equal to $2^m - 1$ and for all remaining pairs $(i,j) \in 0(m) \times 0(m)$ such that $ij \neq 0_m$ and $i \neq j$ it suffices to verify condition (4) for half of them; that is to say, if it is verified for the pair (i,j), then there is no need to verify it for the pair (j,i) so we have

$$2^m - 1 + \frac{4^m - 3^m - (2^m - 1)}{2} = \frac{4^m - 3^m + 2^m - 1}{2}.$$

Notice that in order to verify condition (iii) of Theorem 1 (which is equivalent to condition (4) if condition (3) is assumed) it suffices to verify m^2 conditions (m in order to verify condition (6) and $m(m-1)$ to verify condition (5)).

m	1	2	3	4	5
$\frac{4^m - 3^m + 2^m - 1}{2}$	1	5	22	95	406
m^2	1	4	9	16	25

Table 1

Hence, verification of condition (iii) of Theorem 1 for $m \geq 2$ requires examining less conditions then it is the case for condition (4). Let us additionally observe that the conditions obtained from (iii) of Theorem 1 are of a simpler form than the ones obtained from (4).

Remark 5
We will show that the system of conditions obtained from condition (iii) of Theorem 1 for $n > 1$ and $m > 1$ is complete.

I. Consider the following sets:

$$Z_1 = \{(x, y, 0, \ldots, 0) \in \mathbb{R}(n) : \ y \geq x\},$$
$$Z_2 = \mathbb{R}(n)$$

and if $m \geq 3$, then

$$Z_3 = \ldots = Z_m = \emptyset.$$

Notice that condition (6) is satisfied for every $k \in \{1, \ldots, m\}$, since the sets $Z_1, \ldots, Z_m$ are cones over $\mathbb{R}$. Condition (5) is satisfied for all $(k,l) \in \{1, \ldots, m\}^2 \setminus \{(1,2)\}$, and for $k = 1$ and $l = 2$ we have

$$\begin{aligned} \mathbb{R}(n) + \mathbb{R}(n) \cap Z_1^0 &= \mathbb{R}(n) \setminus \{(0, x_2, 0, \ldots, 0) \in \mathbb{R}(n)\} \\ &\not\subset \mathbb{R}(n) \cap Z_1^0 \\ &= Z_1^0. \end{aligned}$$

II. Consider the following sets:

$$Z_1 = \{(x, x, 0, \dots, 0) \in \mathbb{R}(n)\},$$
$$Z_2 = \mathbb{R}(n) \setminus Z_1$$

and if $m \geq 3$, then

$$Z_3 = \ldots = Z_m = \emptyset.$$

Condition (5) is satisfied for all $k, l \in \{1, \dots, m\}$, since the sets $Z_1, \dots, Z_m$ are pairwise disjoint. Condition (6) is satisfied for every $k \in \{1, \dots, m\} \setminus \{2\}$, since the sets Z_k for $k \in \{1, \dots, m\} \setminus \{2\}$ are cones over $\mathbb{R}$. Condition (6) is not satisfied for $k = 2$, because $(1, 0, \dots, 0) \in Z_2$ and $(0, 1, 0, \dots, 0) \in Z_2$ whereas $(1, 1, 0, \dots, 0) \notin Z_2$.

The independence of the conditions obtained from condition (iii) of Theorem 1, which occurs even under additional assumption that the sets $Z_1, \dots, Z_m$ are cones over $\mathbb{R}$, means that when verifying condition (iii) it is necessary to consider m^2 conditions (none of them may be omitted).

Remark 6

We are going to show that if sets $Z_1, \dots, Z_m$ fulfilling condition (3) satisfy condition (iii) of Theorem 1, then, as a consequence, they satisfy the conditions

$$Z_k^1 + Z_k^1 = Z_k^1,$$
$$Z_l^1 + Z_l^1 \cap Z_k^0 = Z_l^1 \cap Z_k^0$$

for all $k, l \in \{1, \dots, m\}$.

It suffices to prove that $Z_k^1 \subset Z_k^1 + Z_k^1$ and $Z_l^1 \cap Z_k^0 \subset Z_l^1 + Z_l^1 \cap Z_k^0$.

Fix $x \in Z_k^1$ and $y \in Z_l^1 \cap Z_k^0$. On account of Theorem 1, the system $Z_1, \dots, Z_m$ satisfies condition (3), which fact, combined with Lemma 1, implies that the sets Z_k^1 and $Z_l^1 \cap Z_k^0$ are cones over $\mathbb{Q}$, so

$$\frac{1}{2}x \in Z_k^1 \quad \text{and} \quad x = \frac{1}{2}x + \frac{1}{2}x \in Z_k^1 + Z_k^1,$$

$$\frac{1}{2}y \in Z_l^1 \cap Z_k^0 \quad \text{and} \quad y = \frac{1}{2}y + \frac{1}{2}y \in Z_l^1 \cap Z_k^0 + Z_l^1 \cap Z_k^0 \subset Z_l^1 + Z_l^1 \cap Z_k^0.$$

Hence, in conditions (5) and (6) of condition (iii) of Theorem 1 the inclusion may by replaced by equality. Analogous reasoning proves that in condition (5) of condition (ii) of Theorem 1 the inclusion may by replaced by equality.

We will show that the fact that sets $Z_1, \dots, Z_m$ satisfy condition (3) and condition (i) of Theorem 1 does not necessarily imply that the condition

$$ij \neq 0_m \implies Z_1^{i_1} \cap \ldots \cap Z_m^{i_m} + Z_1^{j_1} \cap \ldots \cap Z_m^{j_m} = Z_1^{i_1 j_1} \cap \ldots \cap Z_m^{i_m j_m},$$

is satisfied (that is to say, that the inclusion in condition (4) cannot be replaced with equality). Here is an example for $m > 1$.

Put

$$Z_1 = \mathbb{R}(n),$$
$$Z_2 = \ldots = Z_m = \emptyset.$$

Notice that

$$Z_1^1 \cap Z_2^0 \cap \ldots \cap Z_m^0 + Z_1^1 \cap Z_2^1 \cap Z_3^0 \cap \ldots \cap Z_m^0$$
$$= \emptyset \subsetneq Z_1^1 \cap Z_2^0 \cap \ldots \cap Z_m^0 = \mathbb{R}(n).$$

Similarly, in condition (v) of Theorem 1 the inclusion cannot be replaced with equality. It suffices to consider the same sets as above and put $k = 1$, $l = 2$, $i = (1, 0)$, $j = (1, 1)$ and we obtain

$$Z_1^1 \cap Z_2^0 + Z_1^1 \cap Z_2^1 = \emptyset \subsetneq Z_1^1 \cap Z_2^0 = \mathbb{R}(n).$$

Remark 7
Characterizing a function $f: \mathbb{R}(2) \longrightarrow \mathbb{R}(2)$ satisfying equation (1), Z. Moszner in [9] replaces condition (4) with four conditions which are equivalent to (4) under the assumption of condition (3). These are the following conditions:

(a) $Z_1 + Z_1 \subset Z_1$,

(b) $Z_2 + Z_2 \subset Z_2$,

(c) $Z_1^0 + Z_2 \subset Z_1^0$,

(d) $Z_2^0 + Z_1 \subset Z_2^0$.

Let us compare the above conditions with the ones obtained by expanding condition (iii) of Theorem 1 for the case of $n = m = 2$. In this way, we obtain two conditions from condition (5):

(a') $Z_1^1 + Z_1^1 \subset Z_1^1$,

(b') $Z_2^1 + Z_2^1 \subset Z_2^1$

and two conditions to be verified from condition (6):

(c') $Z_1^1 + Z_1^1 \cap Z_2^0 \subset Z_1^1 \cap Z_2^0$,

(d') $Z_2^1 + Z_2^1 \cap Z_1^0 \subset Z_2^1 \cap Z_1^0$.

Clearly, conditions (a) and (a'), (b) and (b') are identical. If we assume that condition (3) holds ($Z_1 \cup Z_2 = \mathbb{R}(2)$), then we have $Z_1^1 \cap Z_2^0 = Z_2^0$ and $Z_2^1 \cap Z_1^0 = Z_1^0$, and, since addition is commutative, condition (c') corresponds precisely with condition (d), and so does (d') with (c), although they differ by notation. Therefore, if we assume that condition (3) is satisfied, then condition (iii) of Theorem 1 may be treated as a generalization of the system of conditions (a), (b), (c) and (d) from [9] for the case of n, m being arbitrarily chosen natural numbers, independent of each other.

Remark 8
It is easily seen that if in Lemma 1 and Theorem 1 we delete the assumption that

$$Z_1 \cup \ldots \cup Z_m = \mathbb{R}(n)$$

and define

$$Z_i^1 := Z_i \qquad \text{and} \qquad Z_i^0 := \left(\bigcup_{j=1}^{m} Z_j \right) \setminus Z_i$$

for $i = 1, \ldots, m$, then both Lemma 1 and Theorem 1 remain valid.

4. Another properties of the systems satisfying (3) and (4)

We start from the following

Definition 2
Let $C \subset \mathbb{R}(n)$ be a cone over $\mathbb{Q}$. Denote:

$\langle C \rangle$	–	the linear subspace of $\mathbb{R}^n$ over the field $\mathbb{R}$ generated by C;
$\overline{C}$	–	the closure of the set C in $\langle C \rangle$;
C^*	–	the interior of the set C in $\langle C \rangle$;
$\operatorname{int} C$	–	the interior of the set C in $\mathbb{R}^n$.

Now we will be investigated another properties of the systems $Z_1, \ldots, Z_m$ satisfying conditions (3) and (4).

Theorem 2
Let a system $Z_1, \ldots, Z_m$ satisfy conditions (3) *and* (4). *If there exist $k, l \in \{1, \ldots, m\}$ such that $k \neq l$ and $(Z_k \cap Z_l)^* \neq \emptyset$, then*

$$Z_k \cap \langle Z_k \cap Z_l \rangle = Z_l \cap \langle Z_k \cap Z_l \rangle.$$

Proof. Let $x \in (Z_k \cap Z_l)^*$. Then there exists $r > 0$ such that the ball

$$K(x, r) \subset Z_k \cap Z_l \subset \langle Z_k \cap Z_l \rangle.$$

For $z \in Z_l \cap \langle Z_k \cap Z_l \rangle$ there exists $q \in \mathbb{Q}_+$ such that $||qz|| < r$, so

$$x + qz \in K(x, r) \subset Z_k \cap Z_l,$$

and, by condition (iv) of Theorem 1 (the condition equivalent to (4) when condition (3) is assumed), because of the fact that $x \in Z_l^1$ and $qz \in Z_l^1$ we obtain $qz \in Z_k^1$. Since $qz \in \langle Z_k \cap Z_l \rangle$, $z \in Z_k \cap \langle Z_k \cap Z_l \rangle$ (for Z_k is a cone over $\mathbb{Q}$), which gives $Z_l \cap \langle Z_k \cap Z_l \rangle \subset Z_k \cap \langle Z_k \cap Z_l \rangle$; consequently, by symmetry, we get

$$Z_l \cap \langle Z_k \cap Z_l \rangle = Z_k \cap \langle Z_k \cap Z_l \rangle.$$

Notice that if $\operatorname{int}(Z_k \cap Z_l) \neq \emptyset$, then Theorem 2 yields $Z_k = Z_l$, which results in the following

Corollary 2

If a system $Z_1, \ldots, Z_m$ satisfies conditions (3) *and* (4)*, then for all $k, l \in \{1, \ldots, m\}$ $Z_k = Z_l$ or $Z_k \cap Z_l$ is a set with empty interior in $\mathbb{R}^n$.*

Remark 9

If the sets Z_1, Z_2 satisfy conditions (3) and (4), then $\operatorname{int}(Z_1 \cap Z_2) \neq \emptyset$ if and only if $Z_1 = Z_2 = \mathbb{R}(n)$ (the only possible division of $\mathbb{R}(n)$ into two equal sets whose union is $\mathbb{R}(n)$).

Let us make the following definition

Definition 3

For every subset $\{l_1, \ldots, l_p\} \subset \{1, \ldots, n\}$ we define the set

$$B_{l_1,\ldots,l_p} := \{(x_1, \ldots, x_n) \in \mathbb{R}(n) : \; x_{l_1} = \ldots = x_{l_p} = 0\},$$

and then we define the set

$$\mathbb{B} := \{B_{l_1,\ldots,l_p} : \; \{l_1, \ldots, l_p\} \subset \{1, \ldots, n\}\}.$$

Notice that for every $B \in \mathbb{B}$ the set $\mathbb{R}(n) \setminus B$ is a cone over $\mathbb{R}$.

Lemma 2

If a set $\emptyset \neq Z \subset \mathbb{R}(n)$ is a cone over $\mathbb{Q}$, then there exists a subset $\{l_1, \ldots, l_p\}$ of the set $\{1, \ldots, n\}$ such that $Z \subset B_{l_1,\ldots,l_p}$ and exists $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n) \in Z$ such that $\overline{x}_k > 0$ for every $k \in \{1, \ldots, n\} \setminus \{l_1, \ldots, l_p\}$.

Proof. Let K be the family of all the subsets of the set $\{1, \ldots, n\}$ which satisfy the condition

$$\forall\, \mathbf{k} = \{k_1, \ldots, k_\nu\} \in K \quad Z \subset B_{k_1,\ldots,k_\nu}\,.$$

The set K is non-empty, for $Z \subset \mathbb{R}(n) = B_\emptyset$, so $\emptyset \in K$. Obviously,

$$Z \subset B_{l_1,\ldots,l_p}\,,$$

where $L = \{l_1, \ldots, l_p\} = \bigcup_{\mathbf{k} \in K} \mathbf{k}$.

Take $x = (x_1, \ldots, x_n) \in Z$. Let $M = \{m_1, \ldots, m_t\}$ be such a subset of the set $\{1, \ldots, n\} \setminus L$ that

$$\forall\, j \in \{1, \ldots, t\} \;\; x_{m_j} = 0 \qquad \text{and} \qquad \forall\, s \in (\{1, \ldots, n\} \setminus L) \setminus M \;\; x_s > 0.$$

Observe that if the set $M \neq \emptyset$, since the set $Z \not\subset B_{m_j}$, for every $j \in \{1, \ldots, t\}$ there exists $y^{m_j} = (y_1^{m_j}, \ldots, y_n^{m_j}) \in Z$ such that $y_{m_j}^{m_j} > 0$. Define

$$\overline{x} := \begin{cases} x + \sum_{j=1}^{t} y^{m_j} & \text{if } M \neq \emptyset, \\ x & \text{if } M = \emptyset, \end{cases}$$

which finishes the proof.

Theorem 3
If a system $Z_1, \dots, Z_m$ satisfies conditions (3) and (4) and if there exists such $k \in \{1, \dots, m\}$ that $Z_k = \mathbb{R}(n)$, then $Z_i \in \mathbb{B}$ for every $i \in \{1, \dots, m\}$.

Proof. Fix an arbitrary $i \in \{1, \dots, m\}$ for which $Z_i \neq \emptyset$. Lemma 2 guarantees the existence of $\{l_1, \dots, l_p\} \subset \{1, \dots, n\}$ such that $Z_i \subset B_{l_1,\dots,l_p}$ and

$$\exists\, \overline{x} = (\overline{x}_1, \dots, \overline{x}_n) \in Z_i : \ \forall\, k \in \{1, \dots, n\} \setminus \{l_1, \dots, l_p\} \quad \overline{x}_k > 0.$$

Let $z = (z_1, \dots, z_n) \in B_{l_1,\dots,l_p}$. Then, for every $k \in \{1, \dots, n\} \setminus \{l_1, \dots, l_p\}$ there exists $q_k \in \mathbb{Q}_+$ such that

$$\overline{x}_k > q_k z_k$$

and for every $j \in \{l_1, \dots, l_p\}$

$$\overline{x}_j = z_j = 0.$$

Denote

$$q = \min\{q_j : \ j \in \{1, \dots, n\} \setminus \{l_1, \dots, l_p\}\}.$$

Then, $\overline{x} - qz \in \mathbb{R}(n) = Z_k$, $qz \in \mathbb{R}(n) = Z_k$ and $\overline{x} = (\overline{x} - qz) + qz \in Z_i$. By condition (iv) of Theorem 1 (equivalent to (4) when (3)) is assumed), we obtain $qz \in Z_i$, therefore $z \in Z_i$. We have shown that $B_{l_1,\dots,l_p} \subset Z_i$, and hence, because $Z_i \subset B_{l_1,\dots,l_p}$ we obtain

$$Z_i = B_{l_1,\dots,l_p},$$

which proves the theorem.

Remark 10
Z. Moszner in [9] (see Theorem 1) proved that every function $f = (f_1, \dots, f_p)$: $\mathbb{R}(p) \longrightarrow \mathbb{R}(p)$ satisfying condition (1) with $n = m = p$ satisfies the condition

$$\forall\, x, y \in \mathbb{R}(p) : \quad f(x + y) = f(x) \cdot f(y), \tag{9}$$

if and only if there exists $k \in \{1, \dots, p\}$ for which $f_k \neq 0$ on $\mathbb{R}(p)$, or, when we use the "language of cones", if there exists $k \in \{1, \dots, p\}$ such that $Z_k = \mathbb{R}(p)$.

More exactly, for $k = 1, \dots, p$ let M_k be subsets of $\{1, \dots, p\}$ such that M_j is empty for at least one index $j \in \{1, \dots, p\}$. Let

$$Z_k := \{x \in \mathbb{R}(p) : \ \forall i \in M_k \ \ x_i = 0\},$$

i.e. $Z_k \in \mathbb{B}$. Finally, let $a_k : \mathbb{R}^p \longrightarrow \mathbb{R}$ be additive functions. It was shown in [9], Corollary 2, that all solutions of equation (9) are of the form

$$f_k(x) = \begin{cases} \exp a_k(x) & \text{for } x \in Z_k\,, \\ 0 & \text{for } x \in \mathbb{R}(p) \setminus Z_k\,. \end{cases}$$

Thus our Theorem 3 is a natural generalization expressed in the "language of cones" of Corollary 2 in [9] to the case of functions $f\colon \mathbb{R}(n) \longrightarrow \mathbb{R}(m)$ with n and m possibly distinct.

Acknowledgment

The author wishes to express her thanks to Professor Zenon Moszner for many stimulating conversations.

References

[1] A. Bahyrycz, *The characterization of the indicator plurality function*, Wyż. Szkoła Ped. Kraków Rocznik Nauk.-Dydakt. Prace Matematyczne **15** (1998), 15-35.

[2] A. Bahyrycz, *On the problem concerning the indicator plurality function*, Opuscula Math. **21** (2001), 11-30.

[3] A. Bahyrycz, Z. Moszner, *On the indicator plurality function*, Publicationes Math. **61/3-4** (2002), 469-478.

[4] G.L. Forti, L. Paganoni, *Description of the solution of a system of functional equations related to plurality functions: the low-dimensional cases*, Results Math. **27** (1995), 346-361.

[5] G.L. Forti, L. Paganoni, *A system of functional equations related to plurality functions. A method for the construction of the solutions*, Aequationes Math. **52** (1996), 135-156.

[6] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, PWN, Uniwersytet Śląski, Warszawa – Kraków – Katowice, 1985.

[7] Z. Moszner, *Sur les fonctions de pluralité*, Aequationes Math. **47** (1994), 175-190.

[8] Z. Moszner, *Remarques sur la fonction de pluralité*, Results Math. **50** (1995), 387-394.

[9] Z. Moszner, *La fonction d'indice et la fonction exponentielle*, Annales Academiae Pedagogicae Cracoviensis Studia Mathematica **2** (2002), 23-38.

[10] F.S. Roberts, *Characterization of the plurality function*, Math. Soc. Sci. **21** (1991), 101-127.

[11] F.S. Roberts, *On the indicator function of plurality function*, Math. Soc. Sci. **22** (1991), 163-174.

[12] Z.S. Rosenbaum , *P290S2*, Aequationes Math. **46** (1993), 317-319.

Institute of Mathematics
Pedagogical University
Podchorążych 2
PL-30-084 Kraków
E-mail: bah@ap.krakow.pl

Received: 20 March 2006; final version: 15 June 2006;
available online: 11 July 2006.

Annales Academiae Paedagogicae Cracoviensis

Folia 45 Studia Mathematica VI

D.P. Patil, G. Tamone

On type sequences and Arf rings

Abstract. In this article in Section 2 we give an explicit description to compute the type sequence $\mathrm{t}_1, \ldots, \mathrm{t}_n$ of a semigroup Γ generated by an arithmetic sequence (see 2.7); we show that the i-th term t_i is equal to 1 or to the type τ_Γ, depending on its position. In Section 3, for analytically irreducible ring R with the branch sequence $R = R_0 \subsetneq R_1 \subsetneq \ldots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$, starting from a result proved in [4] we give a characterization (see 3.6) of the "Arf" property using the type sequence of R and of the rings R_j, $1 \le j \le m-1$. Further, we prove (see 3.9, 3.10) some relations among the integers $\ell^*(R)$ and $\ell^*(R_j)$, $1 \le j \le m-1$. These relations and a result of [6] allow us to obtain a new characterization (see 3.12) of semigroup rings of minimal multiplicity with $\ell^*(R) \le \tau(R)$ in terms of the Arf property, type sequences and relations between $\ell^*(R)$ and $\ell^*(R_j)$, $1 \le j \le m-1$.

0. Introduction

Let $(R, \mathfrak{m}_R)$ be a noetherian local one dimensional analytically irreducible domain, i.e., the $\mathfrak{m}$-adic completion $\hat{R}$ of R is a domain or, equivalently, the integral closure $\overline{R}$ of R in its quotient field $\mathrm{Q}(R)$ is a discrete valuation ring and a finite R-module. We further assume that R is residually rational, i.e., R and $\overline{R}$ have the same residue field. A particular important class of rings which satisfy these assumptions are semigroup rings which are coordinate rings of algebroid monomial curves.

Let $v\colon \mathrm{Q}(R) \longrightarrow \mathbb{Z} \cup \{\infty\}$ be the discrete valuation of $\overline{R}$ and let $\mathfrak{C} := \mathrm{ann}_R(\overline{R}/R) = \{x \in R \mid x\overline{R} \subseteq R\}$ be the conductor ideal of R in $\overline{R}$. Then the value semigroup $v(R) = \{v(x) \mid x \in R,\ x \neq 0\}$ is a numerical semigroup, that is, $\mathbb{N} \setminus v(R)$ is finite and therefore $v(R) = \{0 = \mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_{n-1}\} \cup \{z \in \mathbb{N} \mid z \ge c\}$, where $0 = \mathrm{v}_0 < \mathrm{v}_1 < \ldots < \mathrm{v}_{n-1} < \mathrm{v}_n := c$ are elements of $v(R)$, $n := n(R) = \ell(R/\mathfrak{C})$ and the integer $c = c(R) := \ell_{\overline{R}}(\overline{R}/\mathfrak{C})$ is also determined by $\mathfrak{C} = \{x \in \mathrm{Q}(R) \mid v(x) \ge\ c\}$ or, equivalently $\mathfrak{C} = (\mathfrak{m}_{\overline{R}})^c$.

AMS (2000) Subject Classification: Primary 13A18, Secondary 20M25.

First part of this work was done while the first author was visiting the Department of Mathematics, University of Genova, Genova, Italy. The first author thanks the colleagues in Genova for their warm hospitality.

In [11] Matsuoka studied the degree of singularity $\delta = \delta(R) := \ell(\overline{R}/R) = \mathrm{card}(\mathbb{N} \setminus v(R))$ of R by introducing the saturated chain of fractionary ideals

$$\mathfrak{C} = \mathfrak{A}_n \subsetneq \ldots \subsetneq \mathfrak{A}_1 = \mathfrak{m} \subsetneq \mathfrak{A}_0 = R \subsetneq \mathfrak{A}_1^{-1} \subsetneq \ldots \subsetneq \mathfrak{A}_n^{-1} = \overline{R},$$

where $\mathfrak{A}_i := \{x \in R \mid v(x) \geq \mathrm{v}_i\}$ and $\mathfrak{A}_i^{-1} = (R : \mathfrak{A}_i)$, $i = 0, 1, \ldots, n$. Moreover, each $\mathfrak{A}_i^{-1}$, $i = 0, \ldots, n$ is an overring of R which satisfies the assumptions that we assume for R. The sequence $\mathrm{t}_i = \mathrm{t}_i(R) := \ell(\mathfrak{A}_i^{-1}/\mathfrak{A}_{i-1}^{-1})$, $i = 1, \ldots, n$, is called the *type sequence* of R.

Various algebraic and geometric properties of the ring R are described by some numerical invariants, for example, the degree of singularity and the type sequence. Several authors have studied these numerical invariants (see for example [1], [2], [4], [5], [16]). The first term t_1 is the Cohen–Macaulay type of R and the sum $\sum_{i=1}^{n} \mathrm{t}_i$ is the degree of singularity of R. Further, the "Gorensteinness" and "almost Gorensteinness" are characterized by type sequences (see 1.2). It is worth noting here that if R is a semigroup ring, then the above properties correspond to the properties "symmetric" and "pseudo-symmetric" of numerical semigroups, respectively. These properties are of a special interest (see [7], [17]), since each numerical semigroup can be expressed as an intersection of numerical semigroups that are either symmetric or pseudo-symmetric. Furthermore, if R is analytically irreducible, then the property "Arf" can be described by its type sequence and each term t_i is related to the i-th term in the "branch sequence" of R (see § 4).

In this article we prove the following results:

(1) If Γ is a numerical semigroup generated by an arithmetic sequence, then we explicitly compute the type sequence (see 2.7) and give (see 2.9) a characterization of almost-Gorensteinness of the semigroup ring $R = K[\![\Gamma]\!]$. This is achieved by studying (see 2.6) the "holes" in Γ by using the explicit description (see 2.5) of the standard basis and the type of the numerical semigroup generated by arithmetic sequence given in [14] and [13], respectively.

(2) If R is analytically irreducible, then we relate the degree of singularity of R to the branch sequence $R = R_0 \subsetneq R_1 \subsetneq \ldots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$, starting from a result proved in [4] we give a characterization (see 3.6) of the "Arf" property using the type sequence (see 1.3) of R and of the rings R_j, $1 \leq j \leq m-1$. Further, we prove (see 3.9, 3.10) some relations among the integers $\ell^*(R)$ and $\ell^*(R_j)$, $1 \leq j \leq m-1$. These relations and a result of [6] allow us to obtain a new charaterization (see 3.12) of semigroup rings of minimal multiplicity with $\ell^*(R) \leq \tau(R)$ in terms of the Arf property, type sequences and relations between $\ell^*(R)$ and $\ell^*(R_j)$, $1 \leq j \leq m-1$.

In Section 4, we also give some illustrative examples to describe our methods.

1. Preliminaries – Assumptions and Notation

Throughout this article we make the following assumptions and notation.

1.1. Notation
Let $\mathbb{N}$ and $\mathbb{Z}$ denote the set of all natural numbers and all integers, respectively. Note that we assume $0 \in \mathbb{N}$. Further, for $a, b \in \mathbb{N}$, we denote $[a, b] := \{r \in \mathbb{N} \mid a \leq r \leq b\}$ and $\mathbb{N}_a := \{n \in \mathbb{N} \mid n \geq a\}$.

Let $(R, \mathfrak{m}_R)$ be a noetherian local one dimensional analytically irreducible domain, i.e., the integral closure $\overline{R}$ of R in its quotient field $\mathrm{Q}(R)$ is a discrete valuation ring and is a finite R-module. We further assume that R is residually rational, i.e., the residue field $k_{\overline{R}}$ of $\overline{R}$ is equal to the residue field k_R of R. A particular important class of rings which satisfy these assumptions are semigroup rings which are coordinate rings of algebroid monomial curves.

1.2. Minimal reductions and reduction number
If k_R is infinite, then for every non-zero ideal $\mathfrak{a}$ of R there exists $x \in \mathfrak{a}$ such that xR is a minimal reduction if $\mathfrak{a}$, i.e., $x\mathfrak{a}^m = \mathfrak{a}^{m+1}$ for some $m \in \mathbb{N}$. The natural number $r(\mathfrak{a}) := \min\{m \in \mathbb{N} \mid x\mathfrak{a}^m = \mathfrak{a}^{m+1}\}$ is called the *reduction number* of $\mathfrak{a}$ (see [12]). In particular, if $\mathfrak{a} = \mathfrak{m}$, then $r(\mathfrak{m})$ is called *reduction number* of R. By replacing R by the local ring $R[X]_{\mathfrak{m}[X]}$ of $R[X]$ at the prime ideal $\mathfrak{m}[X]$, we may assume that k_R is infinite and hence assume that a minimal reduction xR of $\mathfrak{m}$ exists.

We shall now recall the notions of *type sequences* and *almost Gorenstein rings.*

1.3. Type sequences — almost Gorenstein rings
Let R be as in 1.1 and let $v(R)$ be its numerical semigroup, $c = c(v(R))$ be the conductor of $v(R)$, $n = n(R) = \ell(R/\mathfrak{C}) = \mathrm{card}(v(R) \setminus \mathbb{N}_c)$ and $\delta = \delta(R) = \ell(\overline{R}/R) = \mathrm{card}(\mathbb{N} \setminus v(R))$ be the degree of singularity of R (see [11]). Let $0 = \mathrm{v}_0 < \mathrm{v}_1 < \ldots < \mathrm{v}_{n-1} < \mathrm{v}_n := c$ be elements of $v(R)$ such that $v(R) \setminus \mathbb{N}_c = \{\mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_{n-1}\}$. Note that (see [11]) $\delta(R)$ is the sum of n positive integers $\mathrm{t}_i(R) := \ell(\mathfrak{A}_i^{-1}/\mathfrak{A}_{i-1}^{-1})$, $i = 1, \ldots, n$, where $\mathfrak{A}_i := \{x \in R \mid v(x) \geq \mathrm{v}_i\}$ and $\mathfrak{A}_i^{-1} := (R : \mathfrak{A}_i) := \{x \in \mathrm{Q}(R) \mid x\mathfrak{A}_i \subseteq R\}$. The first positive integer $\mathrm{t}_1(R) = \ell(\mathfrak{m}^{-1}/R)$ is the Cohen–Macaulay type τ_R of R. The sequence $\mathrm{t}_1(R), \mathrm{t}_2(R), \ldots, \mathrm{t}_n(R)$ is called the *type sequence* of R. Several authors have studied the properties of type sequences (see [1], [5]). The term "type sequence" is chosen since the first term $\mathrm{t}_1(R) = \ell(\mathfrak{m}^{-1}/R)$ is the Cohen–Macaulay type of R. Further, we have $1 \leq \mathrm{t}_i(R) \leq \tau_R$ for every $i = 1, \ldots, n$ (see [11, § 3, Proposition 2 and Proposition 3]) and hence (see also [5, Proposition 2.1]) $\ell^*(R) \leq (\tau_R - 1)\,(\ell(R/\mathfrak{C}) - 1)$, where $\ell^*(R) := \tau_R \cdot \ell(R/\mathfrak{C}) - \ell(\overline{R}/R)$. Moreover, the equality holds if and only if $\ell(\overline{R}/R) = \tau_R + \ell(R/\mathfrak{C}) - 1$, or equivalently $\mathrm{t}_i(R) = 1$ for $i = 2, \ldots, n$.

Type sequence of a numerical semigroup Γ can also be defined analogously: Let $c = c(\Gamma) \in \mathbb{N}$ be the conductor of Γ and let $\Gamma \setminus \mathbb{N}_c = \{0 = \mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_{n-1}\}$, where $0 = \mathrm{v}_0 < \mathrm{v}_1 < \ldots < \mathrm{v}_{n-1} < \mathrm{v}_n := c$ are elements of Γ. Further, for $i = 0, \ldots, n$, let $\Gamma_i := \{h \in \Gamma \mid h \geq \mathrm{v}_i\}$, $\Gamma(i) := \{x \in \mathbb{Z} \mid x + \Gamma_i \subseteq \Gamma\}$ and let $\mathrm{t}_i = \operatorname{card}(\Gamma(i) \setminus \Gamma(i-1))$. Then $\Gamma = \Gamma(0) \subseteq \Gamma(1) \subseteq \ldots \subseteq \Gamma(n-1) \subseteq \Gamma(n) = \mathbb{N}$ and the sequence t_i, $i = 1, \ldots, n$ is called the *type sequence* of Γ. In particular, the cardinality t_1 of the set $\mathrm{T}(\Gamma) := \Gamma(1) \setminus \Gamma$ is called the *Cohen–Macaulay type* of the semigroup Γ.

The type sequence of a ring R need not be the same as the type sequence of the numerical semigroup $v(R)$ of R (see for example [5]). However, if $R = K[\![\Gamma]\!]$ is the semigroup ring of a numerical semigroup Γ over a field K, then the type sequence of R is equal to the type sequence of its semigroup $v(R) = \Gamma$.

A ring R in 1.1 is called *almost Gorenstein* if the type sequence of R is $\{\tau_R, 1, 1, \ldots, 1\}$, or equivalently, $\ell^*(R)$ attains its upper bound, i.e., $\ell(\overline{R}/R) = \tau_R - 1 + \ell(R/\mathfrak{C})$. It is clear that Gorenstein rings are almost Gorenstein but not conversely (see [16, (1.2)-(1)]).

2. The type sequence of a semigroup generated by an arithmetic sequence

Let R be as in 1.1. In addition to the notations of Section 1, we also fix the following:

2.1. Notation

Put $\Gamma := v(R)$ and let $\Gamma_i := v(\mathfrak{A}_i)$, $\Gamma(i)$ and t_i, $i = 1, \ldots, n$ be as in 1.3.

In order to compute type sequences explicitly, we need to study the "*holes*" of Γ, i.e. elements of $\mathbb{N} \setminus \Gamma$. The positions of the holes will therefore determine the type sequence of Γ. To make these things more precise first let us make the following:

2.2. Definition

An element $z \in \mathbb{Z} \setminus \Gamma$ is called a *hole of first type* (respectively, *hole of second type*) of Γ if $c - 1 - z \in \Gamma$ (respectively, if $c - 1 - z \notin \Gamma$). Then $\Gamma' := \{z \in \mathbb{Z} \setminus \Gamma \mid c - 1 - z \in \Gamma\} = \{c - 1 - h \mid h \in \Gamma\}$ is the set of holes of first type of Γ and $\Gamma'' := \{z \in \mathbb{Z} \setminus \Gamma \mid c - 1 - z \notin \Gamma\}$ is the set of holes of second type of Γ. Therefore $\mathbb{Z} = \Gamma \uplus \Gamma' \uplus \Gamma''$. Further, it is easy to see that:

$$\text{(2.2.a)} \qquad \begin{cases} \Gamma' \cap \mathbb{N} = \{c - 1 - \mathrm{v}_i \mid i \in [0, n-1]\}; \ |\Gamma' \cap \mathbb{N}| = n = c - \delta, \\ \Gamma'' \subseteq \mathbb{N} \setminus \Gamma, \ c - 1 \notin \Gamma'' \qquad \text{and} \qquad \mathrm{T}(\Gamma) \subseteq \{c-1\} \cup \Gamma''. \end{cases}$$

In particular, Γ is symmetric if and only if $\Gamma'' = \emptyset$. For this reason the cardinality of Γ'' is called the *symmetry-defect* of Γ.

The following lemma describes the holes of first type of Γ.

2.3. Lemma
$(\Gamma(i) \setminus \Gamma(i-1)) \cap \Gamma' = \{c - 1 - \mathrm{v}_{i-1}\}$ *for each* $i = 1, \ldots, n$.

Proof. Easy to verify (this essentially follows from [11, Proposition 2]).

In order to describe the holes of second type, we assume that Γ is generated by an arithmetic sequence (the description of the holes of second type in the general case is given in § 2 and § 3 of [15]). For this in addition to the notation in 2.1 and 2.2, we further fix the following notation:

2.4. Notation
Let $m, d \in \mathbb{N}$, $m \geq 2$, $d \geq 1$ be such that $\gcd(m, d) = 1$ and let p be an integer $p \geq 1$, $m_i := m + id$ for $i = 0, 1, \ldots, p+1$. Let $\Gamma := \sum_{i=0}^{p+1} \mathbb{N} m_i$ be the semigroup generated by the arithmetic sequence $m_0, m_1, \ldots, m_{p+1}$.

For any positive natural number $k \in \mathbb{N}^+$, let $q_k \in \mathbb{N}$ and $r_k \in [1, p+1]$ be the unique integers defined by the equation $k = q_k(p+1) + r_k$. We put $q := q_{m-1}$ and $r := r_{m-1} - 1$. Therefore $q \in \mathbb{N}$, $r \in [0, p]$ and $m - 2 = q(p+1) + r$.

Put $s_0 = 0$ and $s_k := m_{r_k} + q_k m_{p+1} = (1 + q_k)m + (r_k + q_k(p+1))\, d$ for $k \in [1, m-1]$. Further, we put $\mathrm{S}_1 := \{m_i + j m_{p+1} \mid i \in [1, p+1]$ and $j \in [0, q-1]\}$ and $\mathrm{S}_2 := \{m_i + q m_{p+1} \mid i \in [1, r+1]\}$. Note that $\mathrm{S}_1 = \emptyset$, if $q = 0$.

Let $0 = \mathrm{v}_0 < \mathrm{v}_1 < \cdots < \mathrm{v}_{n-1} < \mathrm{v}_n := c$ be elements of Γ such that $\Gamma \setminus \mathbb{N}_c = \{0 = \mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_{n-1}\}$. For $i \in [0, n]$, the element $\mathrm{v}_i \in \Gamma$ is called the *i-th element* of Γ.

2.5. Proposition
With the notations as in 2.4 *we have:*

(1) *The standard basis* $\mathrm{S} := S_m(\Gamma)$ *with respect to the multiplicity* $m = m_0$ *of* Γ *is*
$$\mathrm{S} = \{s_k \mid k \in [0, m-1]\} = \{0\} \cup \mathrm{S}_1 \cup \mathrm{S}_2 \,.$$

(2) *The conductor* $c := c(\Gamma)$ *and the degree of singularity* $\delta := \delta(\Gamma)$ *of* Γ *are*
$$c = (m-1)(d+q)+q+1 \quad \textit{and} \quad \delta = \left((m-1)(d+q) + (r+1)(q+1)\right)/2.$$

(3) *The set* $\mathrm{T} := T(\Gamma) = \Gamma(1) \setminus \Gamma = \{m_i + q m_{p+1} - m_0 \mid i \in [1, r+1]\} = \{c - 1 - (r - i + 1)d \mid i \in [1, r+1]\}$. *In particular, the Cohen–Macaulay type of* Γ *is* $\tau = \tau_\Gamma = r + 1$.

Proof. (1) and (3) are special cases of the general results proved in [14, (3.5)] and [13, § 5]. (2) is proved in [18, § 3, Supplement 6].

Now we give an explicit description of the positions of the holes of second type of Γ.

2.6. Lemma

With the notations as in 2.1, 2.2 *and* 2.4, *we have:*

(1) $\mathrm{card}(\Gamma'') = (q+1)r$.

(2) $\Gamma'' = \{x - jm_{p+1} \mid x \in \Gamma(1) \setminus \Gamma,\ x \neq c-1$ *and* $j \in [0,q]\}$.

(3) *For each* $j \in [0,q]$, *there exists a unique integer* $\mathrm{i}(j) \in [0, n-1]$ *such that* $jm_{p+1} = \mathrm{v}_{\mathrm{i}(j)}$ *is the* $\mathrm{i}(j)$*-th element of* Γ. *Moreover,*

$$\Gamma\left(\mathrm{i}(j)+1\right) \setminus \Gamma\left(\mathrm{i}(j)\right) = \{x - jm_{p+1} \mid x \in \Gamma(1) \setminus \Gamma\}.$$

In particular, $\mathrm{card}\left(\Gamma\left(\mathrm{i}(j)+1\right) \setminus \Gamma\left(\mathrm{i}(j)\right)\right) = \tau_\Gamma = r+1$.

Proof. (1) Immediate from 2.5-(2). (2) Easy to verify using 2.5-(3). For the proof of (3) see [15, § 2 and § 3].

Now we give an explicit description of the type sequence of a semigroup generated by an arithmetic sequence.

2.7. Theorem

Let $m, d \in \mathbb{N}$, $m \geq 3$, $d \geq 1$ *be such that* $\gcd(m,d) = 1$ *and let* p *be an integer with* $1 \leq p \leq m-2$. *Let* $\Gamma := \sum_{k=0}^{p+1} \mathbb{N}m_k$ *be the semigroup generated by the arithmetic sequence* $m_k := m + kd$, $k = 0, 1, \ldots, p+1$. *Let* $q \in \mathbb{N}$ *and* $r \in [0,p]$ *be the unique integers defined by the equation* $m - 2 = q(p+1) + r$. *Further, let* $c \in \Gamma$ *be the conductor of* Γ, $\mathbb{N}_c = \{z \in \mathbb{N} \mid z \geq c\}$ *and let* $\Gamma \setminus \mathbb{N}_c = \{0 = \mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_{n-1}\}$ *with* $\mathrm{v}_0 < \mathrm{v}_1 < \ldots < \mathrm{v}_{n-1} < \mathrm{v}_n := c$. *Then the* i*-th term* $\mathrm{t}_i = \mathrm{t}_i(\Gamma)$ *of the type sequence* $(\mathrm{t}_1, \mathrm{t}_2, \ldots, \mathrm{t}_n)$ *of* Γ *is*

$$t_i = \begin{cases} 1, & \text{if } \mathrm{v}_{i-1} \neq jm_{p+1} \text{ for every } j \in [0,q], \\ r+1, & \text{if } \mathrm{v}_{i-1} = jm_{p+1} \text{ for some } j \in [0,q]. \end{cases}$$

Proof. If $\mathrm{v}_{i-1} \neq jm_{p+1}$ for every $j \in [0,q]$, then $\Gamma(i) \setminus \Gamma(i-1) = \{c - 1 - v_{i-1}\}$ by 2.6-(1), (2), (3) and hence $\mathrm{card}\left(\Gamma(i) \setminus \Gamma(i-1)\right) = 1$. If $\mathrm{v}_{i-1} = jm_{p+1}$ for some $j \in [0,q]$, then $\mathrm{card}\left(\Gamma(i) \setminus \Gamma(i-1)\right) = r+1$ by 2.6-(3).

2.8. Corollary

In addition to the notations and assumptions as in 2.7, *further assume that* $d = 1$. *Then the* i*-th term* t_i *of the type sequence* $(\mathrm{t}_1, \mathrm{t}_2, \ldots, \mathrm{t}_n)$ *of* Γ *is*

$$t_i = \begin{cases} r+1, & \text{if } i = \binom{j+1}{2}(p+1) + j + 1 \text{ for some } j \in [0,q], \\ 1, & \text{otherwise.} \end{cases}$$

Proof. It is easy to check that for every $j \in [0, q]$, we have

$$\mathrm{i}(j) = \mathrm{card}\Big(\biguplus_{t=0}^{j} \Gamma^{(t)}\Big) = \sum_{t=0}^{j} (t(p+1)+1) = \binom{j+1}{2}(p+1) + j + 1$$

and jm_{p+1} is the $(\mathrm{i}(j)-1)$-th element $\mathrm{v}_{\mathrm{i}(j)-1}$ in Γ. Now the assertion is clear from 2.7.

2.9. Corollary
Let m, d, p, q, r and Γ be as in 2.7 and let $R := K[\![\Gamma]\!]$ be the semigroup ring of Γ over a field K. Then

(1) *R is Gorenstein if and only if $r = 0$.*

(2) *Assume that R is not Gorenstein. Then R is almost Gorenstein if and only if $m = p + 2$. Moreover, in this case we have $\tau_R = m - 1$.*

Proof. (1) Note that $\tau_R = r + 1$ by 2.5-(3). Therefore R is Gorenstein if and only if $r + 1 = \tau_R = 1$, i.e., $r = 0$.

(2) R is almost Gorenstein if and only if the type sequence of R is $\tau_R = r+1, 1, \ldots, 1$ or equivalently (by 2.7) $q = 0$, i.e. $m - 2 = r$. Now, since $m \geq p + 2$ and $r \leq p$, we have $m - 2 = r$ if and only if $m - 2 = p$.

3. Numerical invariants of analytically irreducible Arf rings

In this section we first recall some definitions and results proved in [9] on blowing-up and Arf rings. These results hold more generally, for semi-local 1-dimensional Cohen–Macaulay rings.

Let R be a semi-local Cohen–Macaulay ring of dimension 1 and let $\mathfrak{m}$ be the (Jacobson) radical of R. Let $\overline{R}$ be the integral closure of R in its total quotient ring $\mathrm{Q}(R)$. An ideal $\mathfrak{a}$ in R is called *open* if it is open in the $\mathfrak{m}$-adic topology on R, or, equivalently, $\mathfrak{m}^n \subseteq \mathfrak{a}$ for some $n \geq 1$, or, equivalently, the length $\ell(R/\mathfrak{a})$ is finite. For any two R-submodules M, N of $\overline{R}$, we put $(M : N) := \{y \in \overline{R} \mid yN \subseteq M\}$.

For an open ideal $\mathfrak{a}$ in R, let $\mathrm{B}(\mathfrak{a}) := \cup_{n\in\mathbb{N}}(\mathfrak{a}^n : \mathfrak{a}^n)$. The ring $\mathrm{B}(\mathfrak{a})$ is called the *blowing-up* of R along $\mathfrak{a}$ or the *first neighbourhood ring* of $\mathfrak{a}$.

3.1. Proposition ([9, Proposition 1.1])
For an open ideal $\mathfrak{a}$ in R, the ring $\mathrm{B}(\mathfrak{a})$ is a finitely generated R-module and $R \subseteq \mathrm{B}(\mathfrak{a}) \subseteq \overline{R}$. Moreover, if R is local and if $\mathfrak{a}$ is a $\mathfrak{m}$-primary ideal which is not principal, then $R \subsetneq \mathrm{B}(\mathfrak{a})$. In particular, if R is local and if R is not a discrete valuation ring, then $R \subsetneq \mathrm{B}(\mathfrak{m})$. Furthermore, there exists a non-zero divisor $x \in \mathfrak{a}$ such that $\mathrm{B}(\mathfrak{a}) = R[\frac{z_1}{x}, \ldots, \frac{z_r}{x}]$, where $z_1, \ldots, z_r$ is a generating set for the ideal $\mathfrak{a}$. In particular, $\mathfrak{a}\,\mathrm{B}(\mathfrak{a}) = x\,\mathrm{B}(\mathfrak{a})$.

An open ideal $\mathfrak{a}$ in R is called *stable* in R if $\mathrm{B}(\mathfrak{a}) = (\mathfrak{a} : \mathfrak{a})$, or, equivalently, $\mathfrak{a}\,\mathrm{B}(\mathfrak{a}) = \mathfrak{a}$. It is clear that if $\mathfrak{a}$ is an open ideal in R, then $\mathfrak{a}^n$ is stable for some $n > 0$ and if $\mathfrak{a}^n$ is stable, then $\mathfrak{a}^m$ is stable for every $m \geq n$.

Recall that an ideal $\mathfrak{a}$ of R is said to be *integrally closed* in R if $\mathfrak{a} = \overline{\mathfrak{a}} := \{z \in R \mid z^n + a_1 z^{n-1} + \cdots + a_n = 0 \text{ with } a_j \in \mathfrak{a}^j \text{ for every } j = 1, \ldots, n\}$.

Now we recall the definition of an *Arf ring* studied by Lipman in [9].

3.2. Branch sequence and Arf rings

Let R be a ring as above. Since $\overline{R}$ is a finite R-module, there exists a finite sequence

$$R = R_0 \subsetneq R_1 \subsetneq \ldots \subsetneq R_{m-1} \subsetneq R_m = \overline{R} \tag{3.2.1}$$

of one dimensional semi-local noetherian rings such that for each $1 \leq i \leq m$, the ring R_i is obtained from R_{i-1} by blowing up the radical of R_{i-1}. For each maximal ideal $\mathfrak{n}$ of $\overline{R}$, every local ring $R'_i := (R_i)_{\mathfrak{n} \cap R_i}$ is called *infinitely near to* R. For each $i = 0, \ldots, m$, the multiplicity and the residue field of the local ring R'_i are denoted by $e(R'_i)$ and k_i, respectively. The sequence $R'_0, R'_1, \ldots, R'_m$ is called the *branch sequence of* R *along* $\mathfrak{n}$ and the sequence of pairs $((e(R'_i), [k_i : k_0]),\ i = 0, \ldots, m$ is called the *multiplicity sequence* of R, where $[k_i : k_0]$ denotes the degree of the field extension $k_i|k_0$ (see [9, pp. 661, 669]. In particular, if R is analytically irreducible, residually rational and $R \neq \overline{R}$, then each R_i in (3.2.1) is also analytically irreducible, residually rational; if $\mathfrak{m}_i$ is the maximal ideal of R_i, then the ring R_i is obtained from R_{i-1} by blowing up $\mathfrak{m}_{i-1}$. Further, $R_i = R'_i$ for each $i = 0, \ldots, m$, since $\overline{R}$ is local and $\mathfrak{n}$ is the only maximal ideal in $\overline{R}$.

A semi-local Cohen–Macaulay ring of dimension 1 is called an *Arf ring* if every integrally closed open ideal in R is stable, or, equivalently (see [9, Theorem 2.2]), if A is any local ring infinitely near to R, then A has maximal embedding dimension, i.e., $\mathrm{embdim}(A) = e(A)$. In particular, if a local ring R is Arf, then R has maximal embedding dimension.

In the Proposition 3.3 below, we recall some conditions for a 1-dimensional Cohen–Macaulay local ring R which are equivalent to the equality $\mathrm{embdim}(R) = e(R)$.

3.3. Proposition

Let $(R, \mathfrak{m})$ be a one dimensional local Cohen–Macaulay ring and let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. Then the following statements are equivalent:

(i) $\mathrm{B}(\mathfrak{a}) = (\mathfrak{a} : \mathfrak{a})$, *i.e.,* $\mathfrak{a}$ *is stable.*

(ii) *There exists $z \in \mathfrak{a}$ such that $z\mathfrak{a} = \mathfrak{a}^2$.*

In particular, the maximal ideal $\mathfrak{m}$ is stable $\iff$ $\mathrm{embdim}(R) = e(R)$ $\iff$ $\tau_R = e(R) - 1$.

Proof. For the equivalence of (i) and (ii) see [9, 1.8] and [12, 5.1]. If $\mathfrak{a} = \mathfrak{m}$, then the equivalence: $\mathfrak{m}$ is stable $\iff$ $\operatorname{embdim}(R) = e(R)$ is proved in [9, 1.8 and 1.10]. Therefore to complete the proof is it enough to show that: $\tau_R = e_0(R) - 1 \iff x\mathfrak{m} = \mathfrak{m}^2$ for some $x \in \mathfrak{m}$. Let $x \in \mathfrak{m}$ be a minimal reduction of $\mathfrak{m}$. Then, since R is Cohen–Macaulay, $\ell(R/xR) = e(R)$ and from $xR \subseteq \ldots \subseteq (xR : \mathfrak{m}) \subseteq \ldots \subseteq \mathfrak{m} \subsetneq R$ we have $\tau_R = \ell\left((R : \mathfrak{m})/R\right) = \ell\left((xR : \mathfrak{m})/xR\right) \leq \ell(R/xR) - 1 = e(R) - 1$. Moreover, the equality $\tau_R = e(R) - 1 \iff \ell\left((xR : \mathfrak{m})/xR\right) = \ell(R/xR) - 1 \iff \ell\left(R/(xR : \mathfrak{m})\right) = 1 = \ell(R/\mathfrak{m}) \iff (xR : \mathfrak{m}) = \mathfrak{m} \iff x\mathfrak{m} = \mathfrak{m}^2$.

The following result proved in [4] (see also [5]) shows how the property Arf is described by the type sequence of its value semigroup.

3.4. Proposition ([4, Theorem 1.7-(5)])
Let $(R, \mathfrak{m})$ be a one dimensional noetherian local analytically irreducible, residually rational domain. Let v be the discrete valuation of $\overline{R}$ and let $v(R) = \{0 = \mathrm{v}_0, \mathrm{v}_1, \ldots, \mathrm{v}_{n-1}\} \cup \mathbb{N}_c$ be the value semigroup of R, where $0 = \mathrm{v}_0 < \mathrm{v}_1 < \ldots < \mathrm{v}_{n-1} < \mathrm{v}_n = c$, $\mathfrak{C}$ is the conductor of $\overline{R}$ over R, $n := n(R) = \ell(R/\mathfrak{C})$ and $c = c(R) := \ell(\overline{R}/\mathfrak{C})$. If R is an Arf ring, then $\mathrm{t}_i = \mathrm{v}_i - \mathrm{v}_{i-1} - 1$ is the i-th term in the type sequence of R.

Now we recall the following characterization of Arf rings given in [9].

3.5. Proposition ([9, Theorem 2.2 and Corollary 3.8])
Let $(R, \mathfrak{m})$ be a one dimensional noetherian local analytically irreducible ring and let $R = R_0 \subsetneq R_1 \subsetneq \ldots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ be the branch sequence of R. Then R is an Arf ring if and only if $\operatorname{embdim}(R_j) = e(R_j)$ for each $j = 0, \ldots, m$. Moreover, if R is complete with algebraically residue field k, then R is an Arf ring if and only if the value semigroup $v(R)$ of R is $\{0, e(R_0), e(R_0) + e(R_1), \ldots, e(R_0) + \cdots + e(R_{m-2})\} \cup \mathbb{N}_c$, where $c = e(R_0) + \cdots + e(R_{m-2}) + e(R_{m-1})$.

Under the assumptions of 3.5 we can characterize Arf rings using the type sequences of R and of each term in the branch sequence of R.

3.6. Theorem
Let $(R, \mathfrak{m})$ be a complete local analytically irreducible domain with algebraically closed residue field k. Let $R = R_0 \subsetneq R_1 \subsetneq \ldots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ be the branch sequence of R. For each $j = 0, \ldots, m-1$, let $\mathfrak{C}_j$ be the conductor of $\overline{R}$ over R_j, and let $n_j = n(R_j)$, $c_j = \ell(\overline{R}/\mathfrak{C}_j)$ and $\mathrm{t}_i(R_j)$ be the i-th term in the type sequence of R_j. Then: R is an Arf ring if and only if for each $j = 0, \ldots, m-1$ and $i = 1, \ldots, n_j$, we have $n_j = m - j$ and $\mathrm{t}_i(R_j) = e(R_{j+i-1}) - 1 = \mathrm{t}_{i+1}(R_{j-1})$.

Proof. (⇒): By the assumptions on R and 3.5, for each $j = 0, \dots, m-1$ we have R_j is an Arf complete domain with integral closure $\overline{R}$, the same residue field k, $R_j \subsetneq R_{j+1} \subsetneq \dots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ is the branch sequence of R_j and the value semigroup $v(R_j)$ is $\{0, \mathrm{v}_{1,j}, \mathrm{v}_{2,j}, \dots, \mathrm{v}_{m-j-1,j}\} \cup \mathbb{N}_{c_j}$, where $\mathrm{v}_{i,j} = e(R_j) + \dots + e(R_{j+i-1})$, $i = 1, \dots, m-j-1$ and $c_j = e(R_j) + \dots + e(R_{m-1})$. Therefore we have $n_j = n(R_j) = (m-j-1)+1 = m-j$. Further, for each $j = 0, \dots, m-1$, if $\{\mathrm{t}_i(R_j) \mid 1 \le i \le m-j\}$ is the type sequence of R_j, then by 3.4 we have $\mathrm{t}_i(R_j) = \mathrm{v}_{i,j} - \mathrm{v}_{i-1,j} - 1 = e(R_{j+i-1}) - 1 = \mathrm{v}_{i+1,j-1} - \mathrm{v}_{i,j-1} - 1 = \mathrm{t}_{i+1}(R_{j-1})$ for every $1 \le i \le m-j$.

(⇐): For each $j = 0, \dots, m-1$, by assumption, in particular, we have $\tau_{R_j} = \mathrm{t}_1(R_j) = e(R_j) - 1$. Therefore $\operatorname{emdim}(R_j) = e(R_j)$ by 3.3 and hence R is an Arf ring by 3.5.

In particular, for the ready reference we note the following formulas for the i-th term t_i in the type sequence of R, in terms of the types, the multiplicities and the lengths arising from the terms of the branch sequence of R.

3.7. Corollary

Let $(R, \mathfrak{m})$ be an Arf complete local domain with algebraically closed residue field k and let $R = R_0 \subsetneq R_1 \subsetneq \dots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ be the branch sequence of R. Then: $m = n = n(R)$ and for each $i = 1, \dots, n$, the i-th term t_i in the type sequence of R is given by: $\mathrm{t}_i = \tau(R_{i-1}) = e(R_{i-1}) - 1 = \ell(R_i/R_{i-1})$.

3.8. Corollary

Let $(R, \mathfrak{m})$ be an Arf complete local domain with algebraically closed residue field k and let $\mathrm{B} = \mathrm{B}(\mathfrak{m})$ be the blowing up of R along $\mathfrak{m}$. If $\mathrm{t}_1, \dots, \mathrm{t}_n$ is the type sequence of R, then $\mathrm{t}_2, \dots, \mathrm{t}_n$ is the type sequence of B.

Recall that several authors (see for example [6], [16] and references in them) have tried to characterize rings for which the inequality $\ell(\overline{R}/R) \le \tau_R \cdot \ell(R/\mathfrak{C})$ is an equality or to give a classification of the rings according to the value of the integer $\ell^*(R) := \tau_R \cdot \ell(R/\mathfrak{C}) - \ell(\overline{R}/R)$. Now, using the special properties of Arf rings and 3.6 we give some relations between $\ell^*(R)$, the terms in the type sequence of R, $\ell^*(R_j)$ and $e(R_j)$, where $R = R_0 \subsetneq R_1 \subsetneq \dots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ is the branch sequence of R. More precisely:

3.9. Theorem

Let $(R, \mathfrak{m})$ be a complete local analytically irreducible domain with algebraically closed residue field k. Let $R = R_0 \subsetneq R_1 \subsetneq \dots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ be the branch sequence of R and let $e_j = e(R_j)$ be the multiplicity of R_j, $j = 0, \dots, m$. Let $\mathrm{t}_1, \dots, \mathrm{t}_n$ be the type sequence of R. Then:

(1) *$\ell^*(R_{m-1}) = 0$ and $\ell^*(R_j) = \sum_{i=j+1}^{m-1} (m-i) \cdot (\mathrm{t}_i - \mathrm{t}_{i+1})$ for $1 \le j \le m-2$.*

(2) *For* $j = 0, \ldots, m-2$, *we have* $\ell^*(R) = \ell^*(R_j) + \sum_{i=1}^{j}(m-i)\cdot(\mathrm{t}_i - \mathrm{t}_{i+1}) = \ell^*(R_j) + \sum_{i=1}^{j}(m-i)\cdot(e_{i-1} - e_i)$.

Proof. We shall use the notation as in 3.6. Note that for every $0 \le j \le m$, $n_j = m - j$; in particular, $n = n(R) = n(R_0) = m$. Further, $\mathrm{t}_{j+1}, \ldots, \mathrm{t}_m$ is the type sequence of R_j; in particular, t_m is the type sequence of R_{m-1} and hence $n_{m-1} = n(R_{m-1}) = 1$ and $\ell^*(R_{m-1}) = 0$. Now, for $0 \le j \le m-2$, we have

$$\ell^*(R_j) = \tau(R_j)\cdot\ell(R_j/\mathfrak{C}_j) - \ell(\overline{R}/R_j) = \mathrm{t}_{j+1}\cdot n_j - \sum_{i=j+1}^{m}\ell(R_i/R_{i-1})$$
$$= \mathrm{t}_{j+1}(m-j) - \sum_{i=j+1}^{m}\mathrm{t}_i = \sum_{i=j+2}^{m}(\mathrm{t}_{j+1} - \mathrm{t}_i) = \sum_{i=j+1}^{m-1}(m-i)\cdot(\mathrm{t}_i - \mathrm{t}_{i+1}).$$

This proves (1). Now, since $\mathrm{t}_i = e(R_{i-1}) - 1 = e_{i-1} - 1$ by 3.7, we have $\mathrm{t}_i - \mathrm{t}_{i+1} = e_{i-1} - e_i$ for every $1 \le i \le m-1$ and hence by (1), we have

$$\ell^*(R) = \ell^*(R_0) = \sum_{i=1}^{m-1}(m-i)\cdot(\mathrm{t}_i - \mathrm{t}_{i+1}) = \sum_{i=1}^{j}(m-i)\cdot(\mathrm{t}_i - \mathrm{t}_{i+1}) + \ell^*(R_j)$$
$$= \sum_{i=1}^{j}(m-i)\cdot(e_{i-1} - e_i) + \ell^*(R_j).$$

This proves (2).

3.10. Corollary
With the same assumptions and notation as in 3.9, *we have:*

(1) $e_j \le e_{j-1}$ *and* $\ell^*(R_j) \le \ell^*(R)$ *for every* $j = 1, \ldots, m-1$.

(2) *For* $1 \le j \le m-2$, $\ell^*(R_j) = \ell^*(R)$ *if and only if* $e_0 = \ldots = e_{j-1} = e_j$.

Proof. Note that the inequality $e_j \le e_{j-1}$ holds for every analytically irreducible domain. Therefore by 3.9-(2) $\ell^*(R_j) \le \ell^*(R)$ for every $j = 1, \ldots, m-2$ and by 3.9-(1) $\ell^*(R_{m-1}) = 0 \le \ell^*(R)$.

(2) Since $m - i > 0$ for every $1 \le i \le j \le m-2$, by 3.9-(2) $\ell^*(R_j) = \ell^*(R)$ if and only if $e_{j-1} = e_j$ for every $j = 1, \ldots, m-2$.

Now for complete semigroup rings R such that $\ell^*(R) \le \tau_R$ and $\tau_R = e(R)-1$ using [6, Corollary 2.14], we give another characterization involving the type sequence of R and the type sequences of the rings R_j in the branch sequence of R, Arf rings, $\ell^*(R)$, $\ell^*(R_j)$, $1 \le j \le m-1$ (see 3.12 below). First we shall prove the following lemma concerning two special types of semigroup rings considered in [6, Corollary 2.14].

3.11. Lemma

Let Γ be a numerical semigroup and let $R = K[\![\Gamma]\!]$ be the semigroup ring of Γ over a field K. Let $R = R_0 \subsetneq R_1 \subsetneq \ldots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ be the branch sequence of R and let $e_j = e(R_j)$, $j = 0, \ldots, m-1$.

(1) *Suppose that Γ is generated by $e, pe+1, pe+2, \ldots, pe+(e-1)$, where e, p are positive integers with $e \geq 3$. Then $m = p$, R is an Arf ring and $e_j = e(R) = e$ for every $j = 0, \ldots, p-1$.*

(2) *Suppose that Γ is generated by $e, pe-a, pe-a+1, \ldots, pe-a+(a-1)$, where e, p, a are positive integers with $e \geq 3, p \geq 2$ and $1 \leq a \leq e-1$. Then $m = p$, R is an Arf ring, $e_j = e(R) = e$ for every $j = 0, \ldots, p-2$ and $e_{p-1} = e-a$.*

Proof. (1) It is easy to check that $\operatorname{emdim}(R) = e(R) = e$; in fact the e elements $e, pe+1, pe+2, \ldots, pe+(e-1)$ form a minimal set of generators for the semigroup Γ and $e < pe+1$. For $j = 0, \ldots, p-1$, let Γ_j be the semigroup generated by $e, (p-j)e+1, (p-j)e+2, \ldots, (p-j)e+(e-1)$ and let $\Gamma_p = \mathbb{N}$. Then it is easy to verify that the sequence $R = K[\![\Gamma_0]\!] \subsetneq K[\![\Gamma_1]\!] \subsetneq \ldots \subsetneq K[\![\Gamma_{p-1}]\!] \subsetneq K[\![\Gamma_p]\!] = \overline{R}$ is the branch sequence of R. Therefore $m = p$ and $\operatorname{emdim}(R_j) = e = e_j$ for each $j = 0, \ldots, p-1$ and hence R is Arf by 3.5.

(2) For $j = 0, \ldots, p-2$, let Γ_j be the semigroup generated by $e, (p-j)e-a$, $(p-j)e-a+1, \ldots, (p-j)e-a+(e-1)$ (note that this is a minimal set of generators for Γ_j), Γ_{p-1} generated by $e-a, e-a+1, \ldots, e, e+1, \ldots, 2e-a-1$ (note that $e-a < e$ and that $e-a, e-a+1, 2e-2a-1$ is a minimal set of generators for Γ_{p-1}) and let $\Gamma_p = \mathbb{N}$. Then it is easy to verify that the sequence $R = K[\![\Gamma_0]\!] \subsetneq K[\![\Gamma_1]\!] \subsetneq \ldots \subsetneq K[\![\Gamma_{p-2}]\!] \subsetneq K[\![\Gamma_{p-1}]\!] \subsetneq K[\![\Gamma_p]\!] = \overline{R}$ is the branch sequence of R and $\operatorname{emdim}(R_j) = e = e_j$ for each $j = 0, \ldots, p-2$, $\operatorname{emdim}(R_{p-1}) = e-a = e_{p-1}$ and hence R is Arf by 3.6.

3.12. Theorem

Let Γ be a numerical semigroup of multiplicity e and type τ_Γ . Let $R = K[\![\Gamma]\!]$ be the semigroup ring of Γ over a field K and let $R = R_0 \subsetneq R_1 \subsetneq \ldots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ be the branch sequence of R. Let $\mathrm{t}_1 = \tau_\Gamma, \mathrm{t}_2, \ldots, \mathrm{t}_n$ be the type sequence of R. For a natural number $a \leq \mathrm{t}_1$, the following statements are equivalent:

(i) *$\ell^*(R) = a$ and $\operatorname{emdim}(R) = e(R)$.*

(ii) *R is an Arf ring and*

$$\mathrm{t}_i = \begin{cases} e-1, & \text{if } 1 \leq i \leq m \text{ and } a = 0, \\ e-1, & \text{if } 1 \leq i \leq m-1 \text{ and } a > 0, \\ e-a-1, & \text{if } i = m \text{ and } a > 0. \end{cases}$$

(iii) *R is an Arf ring and*

$$\ell^*(R) = \ell^*(R_j) = \begin{cases} 0, & \text{if } 1 \leq j \leq m-1 \text{ and } a = 0, \\ a, & \text{if } 1 \leq j \leq m-2 \text{ and } a > 0, \end{cases}$$

and if $a > 0$, then $\ell^*(R_{m-1}) = 0$.

Proof. (i) ⇒ (ii): Note that by 3.3 $\operatorname{emdim}(R) = e(R) \iff \tau_R = e(R) - 1$. Therefore by [6, Corollary 2.14] the value semigroup of R is:

$$v(R) = \Gamma = \begin{cases} \mathbb{N}e + \sum_{i=1}^{e-1} \mathbb{N}(pe+i), & \text{if } a = 0 \text{ (see 3.11-(1))}, \\ \mathbb{N}e + \sum_{i=0}^{a-1} \mathbb{N}(pe-a+i), & \text{if } a > 0 \text{ (see 3.11-(2))}. \end{cases}$$

In particular, $n = n(R) = m = p$ and R is an Arf ring (see 3.11). Further, by 3.7 and 3.11, i-th term t_i in the type sequence of R is given by

$$\mathrm{t}_i = \begin{cases} e-1, & \text{if } 1 \leq i \leq m \text{ and } a = 0, \\ e-1, & \text{if } 1 \leq i \leq m-1 \text{ and } a > 0, \\ e-a-1, & \text{if } i = m \text{ and } a > 0. \end{cases}$$

(ii) ⇒ (iii): If $a = 0$, then $\ell^*(R) = 0$ and by 3.9-(2) $\ell^*(R_j) = 0$ for every $j = 1, \ldots, m-1$. If $a > 0$, then by 3.9, we have $\ell^*(R_{m-1}) = 0$ and $\ell^*(R) = \mathrm{t}_{m-1} - \mathrm{t}_m = a = \ell^*(R_j)$ for every $j = 1, \ldots, m-2$.

(iii) ⇒ (i): Clearly $\ell^*(R) = a$ by (iii) and since R is an Arf ring, we have $\operatorname{emdim}(R) = e(R)$.

4. Examples

In this section we give some examples of Arf rings and some of not Arf rings. In the following examples R denote the semigroup ring $K[\![\Gamma]\!]$ of the semigroup Γ over a field K. Note that in this case each term R_j in the branch sequence of R is also semigroup ring; in fact, if Γ is generated by $n_1, n_2, \ldots, n_p$ with $n_1 < n_2 < \ldots < n_p$, then $R_1 = K[\![\Gamma_1]\!]$, where $\Gamma_1 = v(R_1)$ is generated by $n_1, n_2 - n_1, \ldots, n_p - n_1$; by repeating this argument we get the result for R_j, $j \geq 2$.

4.1. Example

Let $e, r, r' \in \mathbb{N}$ with $e \geq 3$, $1 \leq r$, $1 \leq r'$, $r + r' \leq e-1$ and let Γ be the semigroup generated by the sequence $e, e+r, e+r+r', e+r+r'+1, \ldots, 2e+r+r'-1$. We consider the four cases (i) $r' = r = 1$; (ii) $r' = 1$, $r \geq 2$; (iii) $1 < r' \leq r$; (iv) $r < r'$ separately.

(a) We first compute the type sequence of R.

Case (i): $(r', r) = (1,1)$: This case is considered in 3.11-(1) ($p = 1$). In this case $t_1 = e-1$ is the type sequence of R.

Case (ii): $r' = 1$ and $r \geq 2$: In this case $c = e+r$ and $\Gamma \setminus \mathbb{N}_c = \{0, e\}$. Therefore $n = 2$ and $v_1 = e$. Further, $\Gamma(1) \setminus \Gamma(0) = T(\Gamma) = [r, e-1] \cup [e+1, e+r-1]$ and $\Gamma(2) \setminus \Gamma(1) = [1, r-1]$. Therefore $t_1 = \tau_R = e-1, t_2 = r-1$ and the type sequence of Γ is $e-1, r-1$. Therefore, R is almost Gorenstein if and only if $r = 2$.

Case (iii): $1 < r' \leq r$: In this case $c = e + r + r'$ and $\Gamma \setminus \mathbb{N}_c = \{0, e, e+r\}$. Therefore $n = 3$ and $v_1 = e$, $v_2 = e+r$. Further, we have

$$\Gamma(1) \setminus \Gamma(0) = T(\Gamma) = \{r\} \cup [r+r', e+r+r'-1] \setminus \{e, e+r\},$$
$$\Gamma(2) \setminus \Gamma(1) = \begin{cases} [r+1, r+r'-1], & \text{if } r = r', \\ [r', r+r'-1] \setminus \{r\}, & \text{if } r' < r, \end{cases}$$

and

$$\Gamma(3) \setminus \Gamma(2) = \begin{cases} [1, r-1], & \text{if } r' = r, \\ [1, r'-1], & \text{if } r' < r. \end{cases}$$

Therefore

$$t_1 = \tau_R = e-1, \qquad t_2 = \begin{cases} r'-1, & \text{if } r' = r, \\ r-1, & \text{if } r' < r, \end{cases} \qquad t_3 = \begin{cases} r-1, & \text{if } r' = r, \\ r'-1, & \text{if } r' < r, \end{cases}$$

and the type sequence of Γ is

$$\begin{cases} e-1, r'-1, r-1, & \text{if } r' = r, \\ e-1, r-1, r'-1, & \text{if } r' < r. \end{cases}$$

Therefore, R is almost Gorenstein if and only if $(r', r) = (2,2)$.

Case (iv): $r < r'$: In this case $c = e + r + r'$ and $\Gamma \setminus \mathbb{N}_c = \{0, e, e+r\}$. Therefore $n = 3$ and $v_1 = e$, $v_2 = e+r$. Further, we have $\Gamma(1) \setminus \Gamma(0) = T(\Gamma) = [r+r', e+r+r'-1] \setminus \{e, e+r\}$, $\Gamma(2) \setminus \Gamma(1) = [r', r+r'-1]$ and $\Gamma(3) \setminus \Gamma(2) = [1, r'-1]$. Therefore $t_1 = \tau_R = e-2$, $t_2 = r$, $t_3 = r'-1$ and the type sequence of Γ is $e-2, r, r'-1$. Therefore, R is almost Gorenstein if and only if $(r, r') = (1,2)$.

(b) Now we shall show that R is an Arf ring in cases (i), (ii), (iii) and R is not Arf in case (iv).

Case (i): $(r', r) = (1,1)$: in this case R is an Arf ring (see 3.11-(1) ($p = 1$)).

Case (ii): $r' = 1$ and $r \geq 2$: In this case, let $\Gamma_0 := \Gamma$, Γ_1 be the numerical semigroup generated by $[r, 2r-1]$, $\Gamma_2 := \mathbb{N}$ and let $R_j := K[\![\Gamma_j]\!]$ for $j = 0,1,2$. Then it is easy to see that $e(R_0) = e = \text{embdim}(R_0)$, $e(R_1) = r = \text{embdim}(R_1)$, $e(R_2) = 1 = \text{embdim}(R_2)$, $\Gamma = \Gamma_0 \subsetneq \Gamma_1 \subsetneq \Gamma_2 = \mathbb{N}$ and $R = R_0 \subsetneq R_1 \subsetneq R_2 = \overline{R}$ is the branch sequence of R. Therefore R is an Arf ring by 3.5.

CASE (iii): $1 < r' \leq r$: In this case, let $\Gamma_0 := \Gamma$, Γ_1 be the numerical semigroup generated by $\{r\} \cup [r+r', 2r+r'-1]$ (note that Γ_1 is minimally generated by $\{r\} \cup ([r+r', 2r+r'-1] \setminus \{2r\})$), Γ_2 be the numerical semigroup generated by $[r', 2r'-1]$, $\Gamma_3 := \mathbb{N}$ and let $R_j := K[\![\Gamma_j]\!]$ for $j = 0, 1, 2, 3$. Then it is easy to see that $e(R_0) = e = \text{embdim}(R_0)$, $e(R_1) = r = \text{embdim}(R_1)$, $e(R_2) = r' = \text{embdim}(R_2)$, $e(R_3) = 1 = \text{embdim}(R_3)$, $\Gamma = \Gamma_0 \subsetneq \Gamma_1 \subsetneq \Gamma_2 \subsetneq \Gamma_3 = \mathbb{N}$ and $R = R_0 \subsetneq R_1 \subsetneq R_2 \subsetneq R_3 = \overline{R}$ is the branch sequence of R. Therefore R is an Arf ring by 3.5.

CASE (iv): $1 < r' \leq r$: $r < r'$: In this case, since $e(R) = ne > e - 1 = \text{embdim}(R)$, R is not an Arf ring by 3.5.

4.2. EXAMPLE
Let $m, d, p \in \mathbb{N}$, $m \geq 2$, $p \geq 1$, $d \geq 1$, $\gcd(m, d) = 1$, Γ be the semigroup generated by an arithmetic sequence $m, m+d, \ldots, m+pd$ and let $R = K[\![\Gamma]\!]$. Let B be the blowing-up of R along the maximal ideal of R. Then (see 3.1) $\text{B} = K[\![\Gamma']\!]$, where Γ' is the semigroup generated by m, d, and so $\text{embdim(B)} = 2$. Further, by 3.5:

(i) If $d = 1$, then R is Arf if and only if $\text{embdim}(R) = m$ (in fact, in this case, $\text{B} = K[\![T]\!]$). The case $d = 1$ is also contained in Proposition 4.4 of the article [3].

(ii) If $d = 2$ or $m = 2$, then for every $j \geq 2$ the j-th term in the branch sequence of R is $R_j = K[\![\Gamma_j]\!]$, where Γ_j is the semigroup generated by $2, 2n+1$ for some integer $n \geq 1$ and so $\text{embdim}(R_j) = e(R_j)$ for every $j \geq 1$. Therefore, R is an Arf ring if and only if $\text{embdim}(R) = m$; in particular, if $m = 2$, then R is an Arf ring.

(iii) If $d \geq 3$ and $m \geq 3$, then $e(\text{B}) \geq 3$, $\text{embdim(B)} < e(\text{B})$ and hence R is not an Arf ring.

References

[1] V. Barucci, D.E. Dobbs, M. Fontana, *Maximality properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains*, Mem. Amer. Math. Soc. **125** (1997), no. 598, x+78 pp.

[2] V. Barucci, R. Fröberg, *One-dimensional almost Gorenstein rings*, J. Algebra **188** (1997), no. 2, 418-442.

[3] M. Bras-Amorós, *Acute semigroups, the order bound on the minimum distance, and the Feng–Rao improvements*, IEEE Trans. Inform. Theory **50** (2004), no. 6, 1282-1289.

[4] M. D'Anna, *Canonical module and one-dimensional analytically irreducible Arf domains*, Commutative ring theory (Fès, 1995), 215-225, Lecture Notes in Pure and Appl. Math., **185**, Dekker, New York, 1997.

[5] M. D'Anna, D. Delfino, *Integrally closed ideals and type sequences in one-dimensional local rings*, Rocky Mountain J. Math. **27** (1997), no. 4, 1065-1073.

[6] D. Delfino, *On the inequality $\lambda(\overline{R}/R) \leq t(R)\lambda(R/\mathfrak{C}$ for one-dimensional local rings*, J. Algebra **169** (1994), no. 1, 332-342.

[7] R. Fröberg, C. Gottlieb, R. Häggkvist, *On numerical semigroups*, Semigroup Forum **35** (1987), no. 1, 63-83.

[8] D. Katz, *On the number of minimal prime ideals in the completion of a local domain*, Rocky Mountain J. Math. **16** (1986), no. 3, 575-578.

[9] J. Lipman, *Stable ideals and Arf rings*, Amer. J. Math. **93** (1971), 649-685.

[10] E. Matlis, *The multiplicity and reduction number of a one-dimensional local ring*, Proc. London Math. Soc. (3) **26** (1973), 273-288.

[11] T. Matsuoka, *On the degree of singularity of one-dimensional analytically irreducible noetherian local rings*, J. Math. Kyoto Univ. **11** (1971), 485-494.

[12] A. Ooishi, *Genera and arithmetic genera of commutative rings*, Hiroshima Math. J. **17** (1987), no. 1, 47-66.

[13] D.P. Patil, I. Sengupta, *Minimal set of generators for the derivation module of certain monomial curves*, Comm. Algebra **27** (1999), no. 11, 5619-5631.

[14] D.P. Patil, B. Singh, *Generators for the derivation modules and the relation ideals of certain curves*, Manuscripta Math. **68** (1990), no. 3, 327-335.

[15] D.P. Patil, G. Tamone, *On the type sequences of one dimensional rings*, Preprint, Department of Mathematics, Indian Institute of Science, Bangalore, 2005.

[16] D.P. Patil, G. Tamone, *On the length equalities for one-dimensional rings*, J. Pure Appl. Algebra **205** (2006), 266-278.

[17] J.C. Rosales, M. B. Branco, *Numerical semigroups that can be expressed as an intersection of symmetric numerical semigroups*, J. Pure Appl. Algebra **171** (2002), no. 2-3, 303-314.

[18] G. Scheja, U. Storch, *Regular Sequences and Resultants*, Research Notes in Mathematics **8**, A K Peters, Ltd., Natick, Massachusetts 2001.

Department of Mathematics
Indian Institute of Science
Bangalore 560 012
India
E-mail: patil@math.iisc.ernet.in

Dipartimento di Matematica
Università di Genova
via Dodecaneso 35
I-16146 Genova
Italy
E-mail: tamone@dima.unige.it

Received: 16 November 2006; final version: 30 January 2007;
available online: 28 February 2007.

Annales Academiae Paedagogicae Cracoviensis

Folia 45 Studia Mathematica VI

Jadwiga Knop, Magorzata Wróbel

Some properties of i-connected sets

Abstract. The so-called i-connected sets are considered. The relationships between i-connected sets in natural topology on the plane and i-connected sets in the Hashimoto topology are studied.

1. Introduction

In order to explain where the problem comes from, let us recall that if a set is connected, then its closure is connected as well. It is obvious that in the natural topology on the straight line if a set is connected, then its interior is also connected. Note that no similar fact holds for the usual Euclidean plane. The aim of this paper is to investigate the so-called i-connected sets which are connected and have nonempty connected interiors. We present some of their properties.

Moreover, we establish the connection between i-connected sets in the Euclidean plane and i-connected sets in a stronger topology – the Hashimoto topology on the plane.

2. Some properties of i-connected sets

Let X be a nonempty set and let (X, T) stand for a topological space. For any $A \subset X$ the closure of A will be denoted by $\operatorname{cl} A$ and the interior of A by $\operatorname{int} A$.

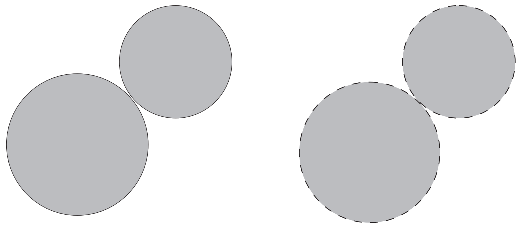
We start with the following definition.

Definition 1
Let (X, T) be a topological space. A set $A \subset X$ is said to be *i-connected* if it has a nonempty interior and both A and $\operatorname{int} A$ are connected.

Remark 1
In the natural topology on a stright line every connected set which has a nonempty interior is i-connected. Note that no similar fact holds for the Eu-

AMS (2000) Subject Classification: 28A05.

clidean plane. For instance, a set consisting of two tangent discs is connected but its interior is not (Fig. 1).



A set A is connected   The $\operatorname{int} A$ is not connected

Figure 1

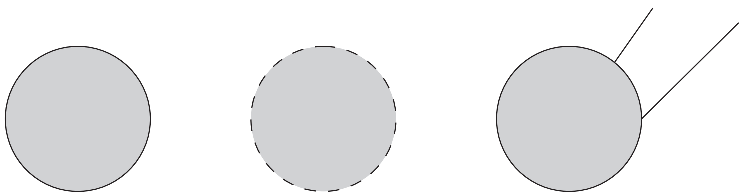
Applying the classical methods we have the following result.

Proposition
Let (X, T) be a topological space. For any set $A \subset X$ if $\operatorname{int} A$ is a connected set and $A \subset \operatorname{cl}\operatorname{int} A$, then A is i-connected.

Remark 2
For any subset $A \subset X$ in a topological space (X, T) we have the following equivalence

$$A \subset \operatorname{cl}\operatorname{int} A \iff \operatorname{cl} A = \operatorname{cl}\operatorname{int} A.$$

In particular, every regular closed set A satisfies the equality $\operatorname{cl} A = \operatorname{cl}\operatorname{int} A$.



$A = \operatorname{cl}\operatorname{int} A$ $\operatorname{cl} A = \operatorname{cl}\operatorname{int} A$ $\operatorname{cl} A \neq \operatorname{cl}\operatorname{int} A$

Figure 2

Theorem 1
Let (X, T) be a topological space and $A, B \subset X$ be fixed. If $\operatorname{int} A$ is connected, $\operatorname{cl} A = \operatorname{cl}\operatorname{int} A$ and B is a connected and boundary set such that $A \cap B \neq \emptyset$, then the sum $A \cup B$ is i-connected.

Proof. First, we prove that $\operatorname{int}(A \cup B) \subset \operatorname{cl} A$. To this end take an arbitrary $x \in \operatorname{int}(A \cup B)$. Then there exists a neighbourhood U_x of this point such that

$U_x \subset A \cup B$. Let V_x be an arbitrary neighbourhood of x. Notice that $U_x \cap V_x$ is a nonempty open set and $U_x \cap V_x \subset A \cup B$. Since B is a boundary set, we conclude that $U_x \cap V_x \not\subset B$. Therefore, $(U_x \cap V_x) \cap A \neq \emptyset$, so $V_x \cap A \neq \emptyset$, which implies that $x \in \operatorname{cl} A$. Since

$$\operatorname{int} A \subset \operatorname{int}(A \cup B) \subset \operatorname{cl} A = \operatorname{cl} \operatorname{int} A$$

and both $\operatorname{int} A$ and $\operatorname{cl} \operatorname{int} A$ are connected, the set $\operatorname{int}(A \cup B)$ is connected. Since the connectivity of $A \cup B$ is obvious, the proof is completed.

Now we are ready to introduce the definition of an i-connected topological space.

Definition 2
A topological space (X, T) is said to be *i-connected* if every connected set in (X, T) which has a nonempty interior is i-connected.

Remark 3
The straight line with a natural topology is an i-connected space but the Euclidean plane is not i-connected.

Remark 4
The property of being an "i-connected space" is a topological property.

Before stating the next theorem we have to introduce some notations.

Let (X, T) be a topological space and let a nonempty set $A \subset X$ be fixed. We denote by (A, T_A) the topological subspace of (X, T) with $T_A = \{U \cap A : U \in T\}$.

Let $\operatorname{cl}_A M$ and $\operatorname{int}_A M$ stand for the closure and the interior of $M \subset A$ in (A, T_A), respectively.

We see at once that the i-connectivity of a set in the sense of Definition 1 is not equivalent to the i-connectivity of the topological subspace induced on that set.

Now, let us quote the following lemma which is a consequence of Lemma 6.1 in [2].

Lemma 1
Let (X, T) be a topological space and let $A \subset X$. Then $M \subset A$ is connected in (A, T_A) if and only if M is connected in (X, T).

Theorem 2
Let (X, T) be a topological space and let a set $A \subset X$ be nonempty and such that $\operatorname{cl} A = \operatorname{cl} \operatorname{int} A$. If the space (X, T) is i-connected, then the subspace (A, T_A) is i-connected.

Proof. Let a set $M \subset A$ with a nonempty interior be connected in (A, T_A). According to Lemma 1, M is connected in (X, T). Now, we prove that the condition $\operatorname{int}_A M \neq \emptyset$ implies that $\operatorname{int} M \neq \emptyset$. Indeed, as $\operatorname{int}_A M \neq \emptyset$, therefore there exists an $x \in \operatorname{int}_A M$ and $U_x \in T$ such that $U_x \cap A \subset M$ and $U_x \cap A \neq \emptyset$; consequently $U_x \cap \operatorname{int} A \subset \operatorname{int} M$. Since $U_x \cap A \neq \emptyset$ and, by the assumption, $\operatorname{cl} \operatorname{int} A \subset \operatorname{cl} A$, we conclude that $U_x \cap \operatorname{int} A \neq \emptyset$ and finally $\operatorname{int} M \neq \emptyset$. From this and from the fact that (X, T) is i-connected we obtain that $\operatorname{int} M$ is connected in (X, T).

The inclusions $\operatorname{int} M \subset \operatorname{int}_A M \subset M$ imply that $\operatorname{int}_A M$ is connected in (X, T) and, by Lemma 1, connected in (A, T_A) which proves the theorem.

We can observe that the above theorem means that the i-connectivity is hereditary with respect to the property $\operatorname{cl} A = \operatorname{cl} \operatorname{int} A$, in particular, with respect to open sets.

3. The i-connected sets in the Hashimoto topology

Let (X, T) and (X, T') be topological spaces such that $T \subset T'$. It is easy to check that for any set $A \subset X$ we have:

(1) the interior of the set A in (X, T) is contained in the interior of A in (X, T');

(2) the closure of A in (X, T') is contained in the closure of A in (X, T);

(3) if $M \subset X$ is connected in the stronger topology (X, T'), then M is connected in the weaker topology (X, T).

Here and in what follows $(\mathbb{R}^2, T_d)$ denotes the Euclidean plane, i.e.,

$$T_d = \left\{U \subset \mathbb{R}^2 : \ \forall\, x \in U \ \exists\, r > 0 \quad K(x, r) \subset U\right\},$$

where $K(x, r)$ stands for the open ball centered at x and with the radius r in the Euclidean metric space.

From now on $(\mathbb{R}^2, T^*)$ stands for the Euclidean plane with the Hashimoto topology, i.e.,

$$T^* = \{U \backslash F \subset \mathbb{R}^2 : \ U \in T_d \ \text{ and } \ \mu(F) = 0\},$$

where μ denotes the Lebesgue measure in $\mathbb{R}^2$.

For simplicity we write $\operatorname{int} A$ ($\operatorname{cl} A$) for the interior (the closure) of A in the space $(\mathbb{R}^2, T_d)$ and $\operatorname{int}_* A$ ($\operatorname{cl}_* A$) for the interior (the closure) of A in $(\mathbb{R}^2, T^*)$.

Since $T_d \subset T^*$, by (1)-(3), we have:

(4) $\forall\, A \subset \mathbb{R}^2 \quad \operatorname{int} A \subset \operatorname{int}_* A$,

(5) $\forall\, A \subset \mathbb{R}^2 \quad \operatorname{cl}_* A \subset \operatorname{cl} A$,

(6) if $M \subset \mathbb{R}^2$ is connected in the Hashimoto topology $(\mathbb{R}^2, T^*)$, then M is connected in $(\mathbb{R}^2, T_d)$.

Lemma 2
If $U \subset \mathbb{R}^2$ is an open set in $(\mathbb{R}^2, T_d)$, then $\operatorname{cl} U = \operatorname{cl}_* U$.

Proof. If $U = \emptyset$, then the assertion is obvious. Assume that $U \neq \emptyset$. According to (5), it is enough to prove that $\operatorname{cl} U \subset \operatorname{cl}_* U$. Let $x \in \operatorname{cl} U$ and $V_x \in T^*$ be a neighbourhood of x. There exist a neighbourhood $U_x \in T_d$ and $F \subset \mathbb{R}^2$ such that $\mu(F) = 0$ and $V_x = U_x \backslash F$. Then $V_x \cap U = (U_x \cap U) \backslash F$. As $U_x \cap U$ is nonempty and open in $(\mathbb{R}^2, T_d)$, we have $\mu(U_x \cap U) > 0$ and finally $(U_x \cap U) \backslash F \neq \emptyset$ which proves our claim.

The main result reads now as follows.

Theorem 3
If a set $A \subset \mathbb{R}^2$ has a nonempty and connected interior in the space $(\mathbb{R}^2, T_d)$ and $\operatorname{cl} A = \operatorname{cl} \operatorname{int} A$, then A is i-connected in the space $(\mathbb{R}^2, T^*)$.

Proof. We start with the observation that if $\operatorname{int} A$ is open and connected in $(\mathbb{R}^2, T_d)$, then it is connected in $(\mathbb{R}^2, T^*)$. Indeed, suppose that $\operatorname{int} A$ is disconnected in $(\mathbb{R}^2, T^*)$. Then, according to the assumption that $\operatorname{int} A \in T^*$, there exist two nonempty sets $U_1, U_2 \in T_d$ and $F_1, F_2 \subset X$ such that $\mu(F_1) = 0 = \mu(F_2)$, $\operatorname{int} A = (U_1 \backslash F_1) \cup (U_2 \backslash F_2)$ and $(U_1 \backslash F_1) \cap (U_2 \backslash F_2) = \emptyset$. Hence $\operatorname{int} A \subset U_1 \cup U_2$ and $U_1 \cap U_2 \subset F_1 \cup F_2$. A reasoning similar to that in the proof of Lemma 2 shows that $U_1 \cap U_2 = \emptyset$ which means that $\operatorname{int} A$ is disconnected in $(\mathbb{R}^2, T_d)$, contrary to the assumption.

By (4), we have

$$\operatorname{int} A \subset \operatorname{int}_* A \subset A \subset \operatorname{cl} A.$$

Moreover, in the view of the assumption and Lemma 2, we get

$$\operatorname{cl} A = \operatorname{cl} \operatorname{int} A = \operatorname{cl}_* \operatorname{int} A$$

and, consequently,

$$\operatorname{int} A \subset \operatorname{int}_* A \subset A \subset \operatorname{cl}_* \operatorname{int} A.$$

Since $\operatorname{int} A$ and $\operatorname{cl}_* \operatorname{int} A$ are connected in $(\mathbb{R}^2, T^*)$, the sets $\operatorname{int}_* A$ and A are connected in $(\mathbb{R}^2, T^*)$, and the proof is completed.

Note that every set satisfying the assumptions of the above theorem is i-connected in $(\mathbb{R}^2, T_d)$. Therefore we obtain the following corollary.

Corollary
If a set $A \subset \mathbb{R}^2$ is i-connected in the space $(\mathbb{R}^2, T_d)$ and $\operatorname{cl} A = \operatorname{cl} \operatorname{int} A$, then A is i-connected in the space $(\mathbb{R}^2, T^*)$.

Remark 5
The assumption that $\mathrm{cl}\, A = \mathrm{cl}\,\mathrm{int}\, A$ is essential. In fact, the example below (Fig. 3) shows that there exist sets which are i-connected in $(\mathbb{R}^2, T_d)$ but are not i-connected in $(\mathbb{R}^2, T^*)$.

Note that the set from Figure 3 is not connected in the Hashimoto topology because it is the union of two nonempty disjoint and closed subsets that are represented by the disc and the line segment without one of the end points.

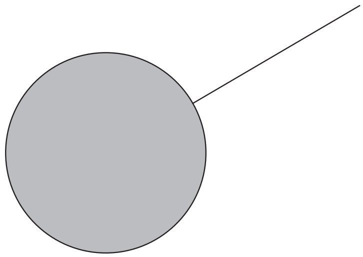


Figure 3

References

[1] H. Hashimoto, *On the* **topology and its application*, Fund. Math **91** (1976), no. 1, 5-10.

[2] J. Knop, T. Kostrzewski, M. Wróbel, *Topologia z elementami analizy matematycznej*, Wydawnictwo WSP w Częstochowie, Częstochowa, 2003.

Institute of Mathematics and Computer Science
Jan Dlugosz University in Częstochowa
Armii Krajowej 13/15
PL-42-200 Częstochowa
Poland
E-mail: j.knop@ajd.czest.pl
E-mail: m.wrobel@ajd.czest.pl

Received: 20 November 2006; final version: 22 January 2007;
available online: 5 March 2007.

Annales Academiae Paedagogicae Cracoviensis

Folia 45 Studia Mathematica VI

Sergey Novikov

Upper estimates of complexity of algorithms for multi-peg Tower of Hanoi problem

Abstract. There are proved upper explicit estimates of complexity of algorithms: for multi-peg Tower of Hanoi problem with the limited number of disks, for Reve's puzzle and for 5-peg Tower of Hanoi problem with the free number of disks.

1. Introduction

The Tower of Hanoi is a mathematical game or puzzle, which was invented by the French mathematician Edouard Lucas in 1883 under the pen-name N. Claus [1] described as an "old Indian legend".

According to the legend of the Tower of Hanoi (originally the "Tower of Brahma" in a temple in the Indian city of Benares), the temple priests are to transfer a tower consisting of 64 fragile disks of gold from one part of the temple to another, one disk at a time. The disks are arranged in order, no two of them the same size, with the largest on the bottom and the smallest on top. Because of their fragility, a larger disk may never be placed on a smaller one, and there is only one intermediate location where disks can be temporarily placed. It is said that before the priests complete their task the temple will crumble into dust and the world will vanish in a clap of thunder.

In the basic version, a stack of discs of mutually distinct sizes is arranged on one of three pegs, with the size restriction that no larger disc is atop a smaller disc. The problem is then to move the entire stack of discs to another of the three pegs by moving one disc at a time, and always maintaining the size restriction.

Let us denote by $H_3(n)$ the minimum number of moves needed to solve the puzzle with n discs for three pegs. It is known (E. Lucas)

$$H_3(n) = 2^n - 1. \tag{1}$$

AMS (2000) Subject Classification: 05-99.

The Tower of Hanoi is a well known NP problem in recreational mathematics. The problem is isomorphic to finding a Hamiltonian path on an n-hypercube.

The literature devoted to the problem is vast and enumerates at least some 200 relevant positions printed in various countries and in various languages (not counting appearances in psychological journals and textbooks in discrete mathematics [6]). Authors of [3] analyse several references from the first edition of the Bibliography of P.K. Stockmeyer, which was posted in 1997, and declare "many papers only rediscover known results".

This problem has also been widely used in the computer science as a paradigmatic teaching example for recursive solution methods. Algorithms of moving discs from one of three pegs to another peg is still used today in many computer science textbooks to demonstrate how to write a recursive algorithm or program. Also these algorithms are often proposed programming on various olympiads and competitions for informaticians. Many computer games use algorithms "The Tower of Hanoi", for example [7].

2. The multi-peg Tower of Hanoi problem

One of many possible generalizations of the Tower of Hanoi problem is to increase the number of pegs. The multi-peg Tower of Hanoi problem consists of $k > 3$ pegs $(B_1, B_2, \ldots, B_k)$ mounted on a board together with n discs of different sizes $(1, 2, \ldots, n)$. Initially these discs are placed on one peg (B_1) in order of size, with the largest (n-disc) on the bottom. The rules of the problem allow discs to be moved one at a time from one peg to another as long as a largest disc is never placed on top of a smaller disc. The goal of the problem is to transfer all the discs to another peg (B_2) with the minimum number of moves, denoted $H_k(n)$. The function $H_k(n)$ characterize the complexity of the algorithm for the solution of the multi-peg Tower of Hanoi problem.

It is surprising that an optimal solution to the k-peg version of the classic Tower of Hanoi problem is unknown for each $k \geq 4$.

Algorithms for transporting n discs from the first peg to B_2 for the case of $k > 3$ pegs were investigated by some mathematicians. A well-known result for investigation of our problem is the recurrence formula

$$S(n, k) = 2S(n - i, k) + S(i, k - 1), \tag{2}$$

where $k > 3$ is a number of pegs, n is a number of discs and $S(n, k)$ is the minimal number of moves required for transporting n discs from the first peg to B_2.

This formula was independently published in 1941 by two mathematicians Frame and Stewart [5] with the help of algorithms, which modern mathematicians name as the "Frame–Stewart algorithm scheme":

1. Recursively transport a stack of $n-i$ smallest disks from the first peg to a temporal peg, using all k pegs;
2. Transport the remaining stack of i largest discs from the first peg to the final peg, using $(k-1)$ pegs and ignoring the peg occupied by the smaller discs;
3. Recursively transport the smallest $n-i$ discs from the temporal peg to the final peg, using all k pegs.

The Frame–Stewart number denoted $S(n,k)$ or $FS(n,k)$, is the minimum number of moves needed to solve the Tower of Hanoi problem using the above Frame-Stewart algorithm scheme.

It is easily to get an other recurrence formula for the multi-peg Tower of Hanoi problem with the help of a next algorithm:

1. Move i_k smallest discs from the first peg to the peg B_k .
2. Move i_{k-1} next discs from the first peg to the peg B_{k-1} .
3. Move i_{k-2} next discs from the first peg to the peg B_{k-2} .

At last (on step $(k-3)$) move i_4 next discs from the first peg to the peg B_4 .

We sum our moves for $k-3$ steps and obtain

$$H_k(i_k) + H_{k-1}(i_{k-1}) + \cdots + H_4(i_4).$$

Step $k-2$. Move i_3 largest discs, where $i_3 = n - \sum_{j=4}^{k} i_j$, from the first peg to the peg B_2 , using three pegs ($2^{i_3}-1$ moves).

Step $k-1$. Move all discs from pegs $B_4, B_5, \ldots, B_k$ to the peg B_2 ($H_k(i_k) + H_{k-1}(i_{k-1}) + \cdots + H_4(i_4)$ moves).

We sum all moves, which needed for transporting of all n discs from the first peg to the peg B_2, and obtain

$$H_k(n) = 2\sum_{j=4}^{k} H_j(i_j) + 2^{i_3} - 1, \tag{3}$$

where $i_j \le i_{j+1}$ and $i_3 = n - \sum_{j=4}^{k} i_j$.

The recurrence formula (3) is known to mathematicians and is published in [3]. It is not comfortable for practical using as it is difficult to find optimal decomposition of the number n into $k-2$ numbers $i_3, i_4, \ldots, i_k$.

Also the following statement is known:

If $k \ge 3$ and $n \le k-1$, then

$$H_k(n) = 2n - 1. \tag{4}$$

If $n = k$, we must place on the peg B_k two smallest (2-disc and 1-disc) discs (three moves). Then we move $n-2$ discs from B_1 to $B_2, B_3, \ldots, B_{k-1}$, where on the each peg is placed one disc and n-disc is placed on B_2 ($n-2$ moves). Then we move $n-3$ discs from $B_3, \ldots, B_{k-1}$ to B_2 ($n-3$ moves). At last we move two discs from B_k to B_2 (three moves).

We sum our moves in this case and obtain the estimation

$$H_k(n) = 2n + 1. \tag{5}$$

3. The explicit estimate for multi-peg Tower of Hanoi problem with the limited number of disks

With the help of a similar algorithm we get

THEOREM 1
If $k \geq 3$ and $k \leq n \leq \frac{k(k-1)}{2}$, then

$$H_k(n) = 4n - 2k + 1. \tag{6}$$

Proof. CASE 1. We consider first a case, where $k \leq n \leq 2k-3$.
Let $n = l + k - 1$, with $l \leq k - 2$.
Then we will use the following algorithm of transferring of discs:

1. Move the $k-1$ smallest discs from the first peg to $B_2, B_3, \ldots, B_k$, so that on each peg one disc is placed and 1-disc is placed on B_2.
2. Move l smallest discs (1-disc, $2-$ disc, $\ldots$, l-disc) from temporal pegs to the peg $B*$, where we placed the $(l+1)$-disc.
3. Move l largest discs from B_1 to free pegs $B_2, B_3, \ldots, B_{l+1}$, so that on each peg one disc is placed and n-disc is placed on B_2.
4. Move $l-1$ largest discs ($(n-1)$-disc, $\ldots$, $(n-l+1)$-disc) from pegs $B_3, \ldots, B_{l+1}$ to the peg B_2.
5. Move the remaining stack of $k-2-l$ largest discs to B_2.
6. Move the remaining stack of $l+1$ smallest discs from $B*$ to B_2.

We sum our moves and obtain

$$H_k(n) = 2k + 4l - 3.$$

From $l = n - k + 1$ we have

$$H_k(n) = 2k + 4(n - k + 1) - 3 = 4n - 2k + 1. \tag{7}$$

CASE 2. In the next case we have $2k - 3 < n \leq \frac{k(k-1)}{2}$.
Let $n = \frac{k(k-1)}{2}$.
Then we will use the following algorithm to transfer the discs:

1. Move $k-1$ smallest discs from the first peg to the peg B_k ($2k-3$ moves).
2. Move $k-2$ next discs from the first peg to the peg B_{k-1} ($2k-5$ moves).

3. Move $k-3$ next discs from the first peg to the peg B_{k-2} ($2k-7$ moves).

At last on stage $k-1$ we move the last n-disc from B_1 to B_2.
We sum our moves and obtain

$$\frac{((2k-3)+1)(k-1)}{2} = (k-1)^2.$$

Then we must sum moves, which are necessary for transporting $n-1$ discs from pegs $B_3, \ldots, B_k$ to the peg B_2.

It is obvious, that this number of moves is equal to $(k-1)^2 - 1$.

Then we obtain, that the sum of moves needed for transporting $n = \frac{k(k-1)}{2}$ discs from the peg B_1 to the peg B_2 is equal to

$$H_k(n) = 2(k-1)^2 - 1. \tag{8}$$

If we have to transport $n < \frac{k(k-1)}{2}$ discs from the peg B_1 to the peg B_2 and

$$\frac{k(k-1)}{2} - n = i,$$

the number of moves in the case

$$2k - 3 < n \leq \frac{k(k-1)}{2}$$

is equal to

$$H_k(n) = 2(k-1)^2 - 1 - 4i. \tag{9}$$

From $i = \frac{k(k-1)-2n}{2}$ we have

$$\begin{aligned} H_k(n) &= 2(k-1)^2 - 1 - 2(k(k-1) - 2n) \\ &= 2(k-1)^2 - 2k(k-1) + 4n - 1 \\ &= 2(k-1)(k-1-k) + 4n - 1 \\ &= 4n - 2k + 1. \end{aligned} \tag{10}$$

This yields our statement.

4. The new estimate for Reve's puzzle

We can deduce a non-recursive (explicit) formula estimating the minimal number of moves required for transporting n discs from the peg B_1 to the peg B_2 with the help of two subsidiary pegs B_3 and B_4. The puzzle "The Tower of Hanoi" for $k = 4$ pegs is known as Reve's puzzle. There are several paper published on Reve's puzzle. For example, in [4] there are recursive algorithm computations for $H_4(20)$, $H_4(50)$, $H_4(100)$, $H_4(150)$, $H_4(200)$. However, explicit formula estimating $H_4(n)$ is not stated in that paper.

We will prove the next result.

THEOREM 2
Let n be fixed and m be an integer such that $\frac{m(m-1)}{2} < n \leq \frac{(m+1)m}{2}$. Then

$$H_4(n) = 2^{m-2}(2n - (m-2)^2 - m) + 1. \tag{11}$$

Proof. CASE 1. $n = \frac{(m+1)m}{2}$ is a triangular number.

We will apply the Frame–Stewart algorithm scheme for transporting of n discs from the B_1 to the B_4 in the following way:

1. Move i smallest discs from the first peg to the peg B_4, using all four pegs.
2. Move $n - i$ largest discs from the first peg to the peg B_2, using three pegs.
3. Move i smallest discs from the B_4 to the peg B_2, using all four pegs.

We sum our moves and obtain

$$H_4(n) = 2H_4(i) + 2^{n-i} - 1. \tag{12}$$

It is obvious, that for transporting of i discs from the B_1 to the B_4 we can transport $j < i$ smallest discs from the B_1 to the B_3, then $i - j$ largest discs from the B_1 to the peg B_4 and at last transport j smallest discs from the B_3 to the B_4. With the help of

$$H_4(i) = 2H_4(j) + 2^{i-j} - 1$$

we obtain

$$H_4(n) = 2(2H_4(j) + 2^{i-j} - 1) + 2^{n-i} - 1.$$

Using the Frame–Stewart algorithm and the formula (12) many times we obtain a next formula for calculation of the minimum number of moves needed to solve the 4-peg Tower of Hanoi problem:

$$H_4(n) = 2^m - 1 + 2(2^{m-1} - 1 + 2(2^{m-2} - 1 + \cdots + 2(2^2 - 1 + 2 \cdot 1)\cdots)), \tag{13}$$

where $m + (m-1) + \cdots + (m - (m-1)) = \frac{(m+1)m}{2} = n$ is a triangular number.

The number of moves necessary for transporting of three smallest discs from the B_1 to a temporal peg (B_3 or B_4) is described in the innest brackets.

Dropping all the brackets in the formula (13) we obtain $m - 1$ summands, which are equal to 2^m, and one, which is equal to 2^{m-1}. The others summands of the development of the formula (13) are negative integers, which sum up to

$$1 + 2 + 4 + \cdots + 2^{m-2} = 2^{m-1} - 1.$$

Then we have for triangular numbers the exact formula

$$H_4(n) = (m-1)2^m + 2^{m-1} - 2^{m-1} + 1 = (m-1)2^m + 1. \tag{14}$$

Case 2. n is nontriangular:

$$\frac{(m-1)m}{2} < n < \frac{(m+1)m}{2}.$$

We estimate the number $H_4(n)$ in the following way:

We observe from (13) that the absence of one (the smallest) disc, that is $n = \frac{(m+1)m}{2} - 1$, on the peg B_1, allows to economize 2^{m-1} moves needed for transporting discs from the peg B_1 to the peg B_2 compared to the triangular case $n = \frac{(m+1)m}{2}$.

If we have to transport $\frac{(m+1)m}{2} - n$ discs from the peg B_1 to the peg B_2 and $n > \frac{(m-1)m}{2}$, then number of "saved" moves is

$$\left(\frac{(m+1)m}{2} - n\right) 2^{m-1} = (m(m+1) - 2n)2^{m-2}.$$

Finally, we get

$$\begin{aligned} H_4(n) &= (m-1)2^m + 1 - (m(m+1) - 2n)2^{m-2} \\ &= (m-1)2^2 2^{m-2} + 1 - (m^2 + m - 2n)2^{m-2} \\ &= 2^{m-2}(4m - 4 - m^2 - m + 2n) + 1 \\ &= 2^{m-2}(2n - (m-2)^2 - m) + 1. \end{aligned}$$

Remark 1

Poole (1994) and Rangel–Mondragón [8] computed the minimum numer of moves needed to solve the Reve's puzzle by:

$$H_4(n) = 1 + \left[n - \frac{x(x-1)}{2} - 1\right] 2^x \tag{15}$$

with

$$x = \left[\frac{\sqrt{8n-7} - 1}{2}\right]$$

where [] is the Gauss bracket.

Of course, formulas (11) and (15) yield identical results for $H_4(n)$ but formula (11) seems to be more comfortable for calculation.

5. The explicit estimate for 5-peg Tower of Hanoi

With the help of formulas (2), (6) and (11) we can deduce a nonrecursive formula for an estimation of $H_5(n)$:

$$H_5(n) = H_4(n_4) + 2H_5(n_5), \tag{16}$$

where $n = n_4 + n_5$.

Corollary 1

$$H_5(n) = 2^{m-2}(2n_4 - (m-2)^2 - m) + 8n_5 - 17, \tag{17}$$

with $n_4 < n_5 \leq 10$, $n = n_4 + n_5$ *and* $\frac{(m-1)m}{2} < n_4 \leq \frac{(m+1)m}{2}$.

The formula (17) allows to estimate the function $H_5(n)$ for case $11 \leq n \leq 24$.

We can obtain another nonrecursive formula for $H_5(n)$, which applies for $n \geq 11$.

We use (2) and (16) with the following assumptions on the splitting $n = n_4 + n_5$: we take $n_4(m)$ to be the triangular number $\frac{(m+1)m}{4}$ such that $n_4 < n_5$ and the difference $n_5 - n_4$ is minimal among all decompositions of n.

Using the Frame–Stewart algorithm and the formula (16) many times we obtain a next formula for calculation of the minimum number of moves needed to solve the 5-peg Tower of Hanoi problem:

$$\begin{aligned} H_5(n) = H_4(n_4(m)) + 2(H_4(n_4(m-1)) + 2(H_4(n_4(m-2)) \\ + \cdots + 2(H_4(n_4(1)) + 2H_5(1)\cdots)), \end{aligned}$$

where $n_4 = n_4(i)$ is a triangular number and $n = \sum_{i=1}^{m} n_4(i) + 1$.

The sum of triangular numbers is called a tetrahedral number.

From (14) we deduce for a tetrahedral number

$$\begin{aligned} H_5(n) &= (m-1)2^m + 1 + 2(m-2)2^{m-1} + 2 + 2^2(m-3)2^{m-2} + 4 \\ &\quad + \cdots + 2^{m-1}(0 \cdot 2 + 1 + 2 \cdot 1)\cdots) \\ &= 2^m((m-1) + (m-2) + \cdots + 1) + (1 + 2 + \cdots + 2^m) \\ &= 2^m\left(\frac{m(m-1)}{2}\right) + 2^{m+1} - 1 = 2^{m-1}m(m-1) + 4 \cdot 2^{m-1} - 1 \\ &= 2^{m-1}(m(m-1) + 4) - 1. \end{aligned} \tag{18}$$

Theorem 3

If n *is a non-tetrahedral number, such that*

$$\sum_{i=1}^{m+1} n_4(i) \geq n > \sum_{i=1}^{m} n_4(i) + 1,$$

then

$$H_5(n) = \frac{2^{m-1}}{3}(6n - m^3 - 5m + 6) - 1. \tag{19}$$

Proof. From the formula (18) it follows that increasing the number of discs by 1 implies the increase of the number of moves ($H_5(n)$) by 2^m. Then the number of moves needed for transporting n discs, where $n > \sum_{i=1}^{m} n_4(i) + 1$ is equal to

$$H_5(n) = 2^{m-1}(m(m-1)+4) - 1 + 2^m(n - \sum_{i=1}^{m} n_4(i) - 1).$$

Since

$$\sum_{i=1}^{m} n_4(i) = \frac{(m+2)(m+1)m}{6}$$

we get

$$\begin{aligned} n - \frac{(m+2)(m+1)m}{6} - 1 &= \frac{(6n-(m+2)(m+1)m-6)}{6} \\ &= \frac{6n - m^3 - m^2 - 2m^2 - 2m - 6}{6}. \end{aligned}$$

Hence

$$\begin{aligned} H_5(n) &= 2^{m-1}(m(m-1)+4) - 1 + \frac{2^{m-1}}{3}(6n - m^3 - 3m^2 - 2m - 6) \\ &= \frac{2^{m-1}}{3}(6n - m^3 - 5m + 6) - 1. \end{aligned}$$

Remark 2
Our formula (19) allows to discover errors in results, which published in [2], where $H_5(11) - H_5(10) = 39 - 31 = 8$ and $H_5(n+1) - H_5(n) = 4$ for $n > 11$. It follows from (19), that $H_5(n+1) - H_5(n)$ cannot decrease with increasing the number of discs.

Using the Frame–Stewart algorithm scheme and the formulas (2), (6), and (19) we can obtain formulas for calculation of the minimum number of moves needed to solve the 6-peg Tower of Hanoi problem.

Corollary 2

$$H_6(n) = \frac{2^{m-1}}{3}(6n_5 - m^3 - 5m + 6) + 8n_6 - 23, \tag{20}$$

where $n_6 \leq 15$, $n = n_6 + n_5$,

$$\sum_{i=1}^{m+1} n_4(i) \geq n_5 > \sum_{i=1}^{m} n_4(i) + 1,$$

$$\frac{(m-1)m}{2} < n_4 \leq \frac{(m+1)m}{2}.$$

The formula (20) allows to estimate the function $H_6(n)$ for case $16 \leq n_6 \leq 33$.

6. Conclusion

As a conclusion we can observe with the help of our new formulas values of the function $H_k(n)$ for $n = 64$. Suppose that one move requires one second. Then it's known (E. Lucas) $H_3(n) = 2^{64} - 1$ and the puzzle takes more than 590 000 000 000 years. Then by (11) we have $H_4(64) = 18433$ (only five hours running time). We have by (19) $H_5(64) = 1535$ and $H_6(64) = 673$ and $H_7(64) = 479$ with the help of formulas (2), (6), (19) and (20) for $m = 4$, $n_4 = 10$, $n_5 = 22$, $n_6 = 21$. Next results are: $H_8(64) = 385$, $H_9(64) = 351$, $H_{10}(64) = 313$, $H_{11}(64) = 271$, which we obtain with help of formulas (2) and (6). We can easily calculate the values of $H_k(64)$ for $64 \geq k \geq 12$ with the help of one formula (6). At last $H_{65}(64) = 127$. It is obvious, that the number of moves required for solution of the puzzle in case $n = 64$, $k > 65$ stabilizes.

References

[1] N. Claus, *La Tour d'Hanoi: jeu de calcul*, Science et Nature **1** (1884), no. 8, 127-128.

[2] B. Houston, H. Masum, *Explorations in 4-peg Tower of Hanoi*, Carleton University Technical Report TR-04-10, November 2004.

[3] S. Klavžar, U. Milutinović, C. Petr, *On the Frame–Stewart algorithm for the multi-peg Tower of Hanoi problem*, Sixth Twente Workshop on Graphs and Combinatorial Optimization (Enschede, 1999), Discrete Appl. Math. **120** (2002), no. 1-3, 141-157.

[4] R.G. Rock, *Reve's Puzzle*, http://www.rit.edu/~rgr8261/Reves/RevesPuzzle.doc

[5] B.M. Stewart, J.S. Frame, *Problems and Solutions: Advanced Problems: Solutions: 3918*, Amer. Math. Monthly **48** (1941), no. 3, 216-219.

[6] P.K. Stockmeyer, *The Tower of Hanoi: A Bibliography*, September, 2005, http://www.cs.wm.edu/~pkstoc/biblio2.pdf

[7] *Game*, http://www.mazeworks.com/hanoi/

[8] *Formula*, http://mathworld.wolfram.com/TowerofHanoi.html

Institute of Computer Science
University of Podlasie
Sienkiewicza 51
PL-08-110 Siedlce
Poland
E-mail: novikov@poczta.onet.pl

Received: 20 November 2006; final version: 15 February 2007;
available online: 20 April 2007.

Annales Academiae Paedagogicae Cracoviensis
Folia 45 Studia Mathematica VI

Paweł Drygaś, Anna Król

Some remarks on the domination between conjunctions and disjunctions

Abstract. There are diverse domination properties considered in linear programming, game theory, semigroup theory and graph theory. In this paper we present a class of conjunctions which dominate each triangular conorm. Moreover, we give the characterization of such conjunctions.

1. Introduction

The notion of domination was introduced by R.M. Tardiff [11]. in the case of triangle functions. It was generalized by B. Schweizer and A. Sklar [10] to the class of associative binary operations with common domain (and common unit element) in order to construct Cartesian products of probabilistic metric spaces. The domination of t-norms is also used in construction of fuzzy equivalence relations [1] and fuzzy orderings [2]. The domination between aggregation operations is useful in investigation of aggregation procedures preserving $\mathcal{T}$-transitivity of fuzzy relations [8].

Furthermore, the characterization of the relation of domination in a class of operations is a solution of a functional inequality in the class of functions. This inequality is a natural generalization of the equation of bisymmetry. Concerning reflexivity of domination, one can obtain the equation of bisymmetry.

Some particular problems of domination were recently examined (Drewniak et al. [4], Drewniak, Król [5]). The characterization of all t-seminorms dominating each triangular conorm was given by P. Sarkoci (cf. [9]).

In this paper at first, in Section 2, we recall definitions of binary operations which will be used in the sequel. Next, we recall the notion of domination concerning two binary operations (Section 3). In Section 4 we describe the class of conjunctions which dominate each triangular conorm. Section 5 contains the characterization of such conjunctions.

AMS (2000) Subject Classification: 26D20, 39B62, 06F05.

2. Multivalued conjunctions and disjunctions

In this section we recall definitions of binary operations which will be used in the sequel.

Definition 1 (cf. [3])
A conjunction (disjunction) is any increasing binary operation

$$C\colon [0,1]^2 \longrightarrow [0,1] \qquad (\, D\colon [0,1]^2 \longrightarrow [0,1]\,)$$

fulfilling

$$C(0,0) = C(0,1) = C(1,0) = 0, \qquad C(1,1) = 1$$
$$(\, D(0,1) = D(1,0) = D(1,1) = 1, \qquad D(0,0) = 0\,).$$

Example 1
The operation $C\colon [0,1]^2 \longrightarrow [0,1]$ given by formula

$$C(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [\frac{1}{2},1]^2, \\ \min(x,y) & \text{otherwise} \end{cases}$$

is a conjunction.

Definition 2 (cf. [3])
A t-seminorm (t-semiconorm) is any increasing binary operation T (S): $[0,1]^2 \to [0,1]$ with neutral element 1 (0).

Remark 1
An operation is a t-seminorm (t-semiconorm) iff it is a conjunction (disjunction) with the neutral element 1 (0).

Remark 2
Any t-seminorm T and t-semiconorm S fulfils

$$T(x,y) \le \min(x,y); \quad S(x,y) \ge \max(x,y), \qquad x,y \in [0,1].$$

Example 2
The operation $T\colon [0,1]^2 \longrightarrow [0,1]$ given by formula

$$T(x,y) = \begin{cases} 0 & \text{if } x \in [0,\frac{1}{4}],\ y \in [0,\frac{1}{2}], \\ \min(x,y) & \text{otherwise} \end{cases}$$

is a t-seminorm.

Remark 3
The conjunction from Example 1 is not a t-seminorm.

Definition 3 ([7], Definitions 1.1, 1.13)
An associative, commutative and increasing operation T (S): $[0,1]^2 \longrightarrow [0,1]$ is called a t-norm (t-conorm), if it has the neutral element $e = 1$ $(e = 0)$.

Remark 4
An operation is a t-norm (t-conorm) iff it is an associative, commutative t-seminorm (t-semiconorm).

By an order isomorphism we can obtain a new operation from a given one.

Theorem 1 (cf. [7], p. 38)
Let us consider an increasing binary operation $F: [0,1]^2 \longrightarrow [0,1]$, *a bijection* $\varphi: [0,1] \longrightarrow [0,1]$ *and*

$$F_\varphi(x,y) = \varphi^{-1}(F(\varphi(x), \varphi(y))), \qquad x, y \in [0,1].$$

If φ is increasing and F is a t-norm (t-conorm), t-seminorm (t-semiconorm) or conjunction (disjunction) then F_φ remains a t-norm (t-conorm), t-seminorm (t-semiconorm) or conjunction (disjunction), respectively.

If φ is decreasing and F is a t-norm (t-conorm), t-seminorm (t-semiconorm) or conjunction (disjunction) then F_φ changes φ into a t-conorm (t-norm), t-semiconorm (t-seminorm) or disjunction (conjunction), respectively.

3. Notion of domination

Now we recall the notion of domination concerning two binary operations.

Definition 4 (cf. [11])
Let $F, G: [0,1]^2 \longrightarrow [0,1]$. Operation F dominates operation G $(F \gg G)$, iff

$$F(G(a,b), G(c,d)) \geq G(F(a,c), F(b,d))$$

for $a, b, c, d \in [0,1]$.

Lemma 1 ([4])
The operation $F = \min$ *dominates every increasing operation. Every increasing operation dominates* $G = \max$.

New examples of domination can be obtained from given ones by order isomorphisms.

Lemma 2 ([8], Proposition 4.2)
Let us consider increasing binary operations $F, G: [0,1]^2 \longrightarrow [0,1]$, *a bijection* $\varphi: [0,1] \longrightarrow [0,1]$ *and*

$$F_\varphi(x,y) = \varphi^{-1}(F(\varphi(x), \varphi(y))), \qquad x, y \in [0,1]. \tag{1}$$

If φ is increasing, then $F \gg G \Leftrightarrow F_\varphi \gg G_\varphi$. If φ is decreasing, then $F \gg G \Leftrightarrow F_\varphi \ll G_\varphi$.

Using the decreasing bijection $\varphi(x) = 1-x$, $x \in [0,1]$ in (1) we can consider dominations for dual operations.

Corollary 1
We have $F \gg G \Leftrightarrow F' \ll G'$, where

$$F'(x,y) = 1 - F(1-x, 1-y), \qquad x, y \in [0,1].$$

4. Domination in the class of conjunctions and disjunctions

In this section we describe the class of conjunctions which dominate each triangular conorm.

Theorem 2 ([9])
A t-seminorm C dominates the class of all t-conorms iff

$$C(x,y) \in \{0, x, y\} \qquad \text{for any } x, y \in [0,1]. \tag{2}$$

Corollary 2
If a t-seminorm C dominates every t-semiconorm then it fulfils (2).

Corollary 3
If a t-seminorm C dominates every disjunction then it fulfils (2).

In Theorem 2 one cannot replace a t-seminorm by an arbitrary conjunction. This is illustrated by the following counterexample.

Example 3
The operation given by the formula

$$C(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [\frac{1}{8}, 1]^2, \\ \min(x,y) & \text{otherwise} \end{cases}$$

is a commutative and associative conjunction with the neutral element $\frac{1}{8}$ such that $C(x,y) \in \{x,y\} \subset \{0,x,y\}$ for any $x,y \in [0,1]$. It does not dominate the t-conorm S_P, where $S_P(x,y) = x + y - xy$, $x,y \in [0,1]$. Indeed, for $x = 0.3$, $y = 0.4$, $u = 0.6$, $v = 0.2$ we have

$$\begin{aligned} C(S_P(x,y), S_P(u,v)) &= C(S_P(0.3,0.4), S_P(0.6,0.2)) = C(0.58, 0.68) \\ &= 0.68. \end{aligned}$$

On the other hand we have

$$S_P(C(x,u),C(y,v)) = S_P(C(0.3,0.6),C(0.4,0.2)) = S_P(0.6,0.4) \\ = 0.76.$$

It means, that C does not dominate the t-conorm S_P.

This is why we add an additional assumption concerning a conjunction.

Theorem 3
If $C \le \min$ is a conjunction fulfilling (2), then it dominates every t-conorm.

Proof. Let $C \le \min$ be a conjunction fulfilling (2), S be a t-conorm and $x,y,u,v \in [0,1]$. We denote

$$L = C(S(x,y),S(u,v)), \qquad R = S(C(x,u),C(y,v)).$$

If $C(x,u) = C(y,v) = 0$, then we get $R = S(0,0) = 0 \le L$. If $C(x,u) = 0$ and $C(y,v) = \min(y,v) > 0$, we have

$$R = S(0,\min(u,v)) = \min(u,v) = C(y,v) \le L.$$

Similarly in the case $C(y,v) = 0$ and $C(x,y) = \min(x,y) > 0$. Let $C(x,y) = \min(x,y) > 0$ and $C(y,v) = \min(y,v) > 0$. At first we observe that by Remark 2, $L \ge C(\min(x,u),\min(y,v)) \ge C(x,u) \ne 0$. So we have two possibilities $L = S(x,y)$ or $L = S(u,v)$. In both cases we have

$$L \ge S(\min(x,u),\min(y,v)) = R.$$

Thus C dominates every triangular conorm.

Simple computations show the following lemma.

Lemma 3
Let $C\colon [0,1]^2 \longrightarrow [0,1]$ be an increasing operation. Then $C \le \min$ and C fulfils (2) iff

$$C(x,y) \in \{0,\min(x,y)\} \qquad \textit{for any } x,y \in [0,1]. \tag{3}$$

Directly from Lemma 3 and Theorem 3 we obtain the following result.

Theorem 4
If C is a conjunction fulfilling the condition (3), then it dominates every t-conorm.

The next example shows that there exist binary operations which fulfil the assumptions of Theorem 4 but are not t-seminorms, so they do not fulfil conditions used in Theorem 2.

Example 4
By Theorem 4 the operation $C\colon [0,1]^2 \longrightarrow [0,1]$ given by formula

$$C(x,y) = \begin{cases} \min(x,y) & \text{if } x \in [\frac{1}{2},1],\ y \in [\frac{3}{4},1], \\ 0 & \text{otherwise} \end{cases}$$

dominates any t-conorm.

By duality (cf. Theorem 1, Corollary 1) we obtain analogous results for disjunctions which are dominated by any t-norm.

Theorem 5
If D is a disjunction fulfilling the condition

$$D(x,y) \in \{\max(x,y),1\} \qquad \textit{for any } x,y \in [0,1],$$

then it is dominated by every t-norm.

5. Characterization of a class of conjunction dominating any t-conorm

Conjunctions from Theorem 4 can be characterized in a way used for uninorms (cf. [6]).

Theorem 6
If C is a conjunction fulfilling (3), then there exists a decreasing function $g_C\colon [0,1] \longrightarrow [0,1]$ such that

$$C(x,y) = \begin{cases} 0 & \textit{if } y < g_C(x), \\ \min(x,y) & \textit{if } y > g_C(x), \\ 0 \textit{ or } \min(x,y) & \textit{if } y = g_C(x). \end{cases} \tag{4}$$

Moreover, for $s \in [0,1]$ let $B_s = \{x:\ g_C(x) = s\}$, $a_s = \inf B_s$, $b_s = \sup B_s$. If $a_s < b_s$, then there exists $c_s \in [a_s,b_s]$ such that

$$C(x,s) = \begin{cases} 0 & \textit{if } x < c_s\,, \\ \min(x,s) & \textit{if } x > c_s\,, \\ 0 \textit{ or } \min(x,s) & \textit{if } x = c_s\,. \end{cases} \tag{5}$$

Proof. Define $g_C(x) = \sup A_x$, where $A_x = \{y \in [0,1]:\ C(x,y) = 0\}$. Of course A_x is non-empty because $C(x,0) \le C(1,0) = 0$, so $0 \in A_x$.

Next we prove, that g_C is decreasing. First we note, that $g_C(0) = 1$, because $C(0,1) = 0$.

Let $x < y$. If $g_C(x) = 1$ then $g_C(x) \ge g_C(y)$. If $g_C(x) < 1$ then $C(x,t) = \min(x,t)$ for all $t > g_C(x)$. So, by the monotonicity of C we have $C(y,t) \ge$

$C(x,t) = \min(x,t) > 0$. Therefore $C(y,t) = \min(y,t)$ for all $t > g_C(x)$. It means, that $g_C(y) \leq g_C(x)$.

Now, let $s \in [0,1]$ be such that $a_s < b_s$. Let $c_s = \sup\{x : C(x,s) = 0\}$. We prove that $c_s \in [a_s, b_s]$. If $c_s < a_s$, then $a_s > 0$ and $s < 1$. Let $t \in (c_s, a_s)$. Then by the monotonicity of the function g_C and by definition of the set B_s we obtain $g_C(c_s) \geq g_C(t) > s$ and by (4), $C(t,s) = 0$, which leads to a contradiction. If $c_s > b_s$ then $b_s < 1$ and $s > 0$. Let $t \in (b_s, c_s)$. Then, by the monotonicity of the function g_C and by definition of the set B_s, we obtain $g_C(c_s) \leq g_C(t) < s$ and by (4), $C(t,s) = \min(t,s) > 0$, which leads to a contradiction. So, $c_s \in [a_s, b_s]$.

Directly by the definition of the point c_s and (3) we obtain (5).

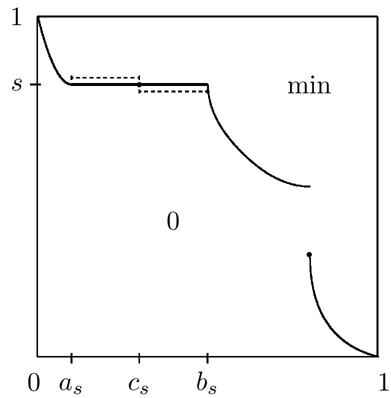


Figure. Structure of operation (4), (5) and graph of g_C

THEOREM 7
Let $g\colon [0,1] \longrightarrow [0,1]$ be a decreasing function. If $C\colon [0,1]^2 \longrightarrow [0,1]$ is the operation given by (4) *with $g_C = g$ and by* (5) *in intervals of constant values of function g and $C(1,1) = 1$, then the operation C is a conjunction fulfilling* (3).

Proof. Directly by (4) we obtain (3). Moreover, $C(1,1) = 1$ and $C(0,0) = C(0,1) = C(1,0) = 0$, because $\min(x,0) = 0$ for all $x \in [0,1]$.

Let $x, y, z \in [0,1]$ and $x < y$.

If $y \leq g(z)$, then $x < g(z)$ and $C(z,x) = 0 \leq C(z,y)$.

If $x \geq g(z)$, then $y > g(z)$ and $C(z,y) = \min(z,y) \geq \min(z,x) \geq C(z,x)$.

Otherwise, we have $C(z,y) = \min(z,y) \geq 0 = C(z,x)$.

Thus operation C is increasing with respect to the second variable. To prove the monotonicity with respect to the first variable we consider a few cases.

If $C(x,z) = \min(x,z)$, then $z \geq g(x)$. By monotonicity of g we have $z \geq g(y)$ and by (4) and (5) we have $C(y,z) = \min(y,z) \geq \min(x,z) = C(x,z)$.

Otherwise, we have $C(x,z) = 0 \leq C(y,z)$. It means that the operation C is increasing with respect to the second variable.

Corollary 4
Let $g\colon [0,1] \longrightarrow [0,1]$ *be a decreasing function. The operation* $C\colon [0,1]^2 \longrightarrow [0,1]$ *given by*

$$C(x,y) = \begin{cases} 0 & \text{if } y < g(x), \\ \min(x,y) & \text{if } y \geq g(x) \end{cases}$$

is a conjunction fulfilling (3).

Remark 5
By duality (cf. Theorem 1, Corollary 1) we may obtain a similar characterization of disjunctions which are dominated by any t-norm.

References

[1] B. De Baets, R. Mesiar, *$\mathcal{T}$-partitions*, Fuzzy Sets and Systems **97** (1998), no. 2, 211-223.

[2] U. Bodenhofer, *A similarity-based generalization of fuzzy orderings*, Universitätverlag Rudolf Trauner, Linz, 1999.

[3] G. De Cooman, E.E. Kerre, *Order norms on bounded partially ordered sets*, J. Fuzzy Math. **2** (1994), no. 2, 281-310.

[4] J. Drewniak, P. Drygaś, U. Dudziak, *Relation of domination*, in: Issues in Soft Computing Decisions and Operations Research (O. Hryniewicz, J. Kacprzyk, D. Kuchta, eds), EXIT, Warszawa 2005, 149-160.

[5] J. Drewniak, A. Król, *On the problem of domination between triangular norms and conorms*, J. Elect. Engineering, **56**, **12/s**, (2005), 59-61.

[6] J. Drewniak, P. Drygaś, *On a class of uninorms*, Operations for uncertainty modelling (Liptovsky Mikulas, 2002), Internat. J. Uncertain. Fuzziness Knowledge-Based Systems **10** (2002), suppl., 5-10.

[7] E.P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Trends in Logic—Studia Logica Library **8**, Kluwer Academic Publishers, Dordrecht, 2000.

[8] S. Saminger, R. Mesiar, U. Bodenhofer, *Domination of aggregation operators and preservation of transitivity*, Operations for uncertainty modelling (Liptovsky Mikulas, 2002), Internat. J. Uncertain. Fuzziness Knowledge-Based Systems **10** (2002), suppl., 11-35.

[9] P. Sarkoci, *Conjunctors Dominating Classes of t-conorms*, in: Abstracts of FSTA 2006, Liptovský Jan, 30.01-03.02.2006, 98.

[10] B. Schweizer, A. Sklar, *Probabilistic metric spaces*, North-Holland Series in Probability and Applied Mathematics, North-Holland Publishing Co., New York, 1983.

[11] R.M. Tardiff, *Topologies for probabilistic metric spaces*, Pacific J. Math. **65** (1976), no. 1, 233-251.

Institute of Mathematics
University of Rzeszów
al. Rejtana 16A
PL-35-310 Rzeszów
Poland
E-mail: paweldr@univ.rzeszow.pl
E-mail: annakrol@univ.rzeszow.pl

Received: 20 November 2006; final version: 21 March 2007;
available online: 15 May 2007.

Annales Academiae Paedagogicae Cracoviensis
Folia 45 Studia Mathematica VI

Paweł Solarz

Iterative roots of homeomorphisms possessing periodic points

Abstract. In this paper we give necessary and sufficient conditions for the existence of orientation-preserving iterative roots of a homeomorphism with a nonempty set of periodic points. We also give a construction method for these roots.

1. Introduction

The problem of the existence of *iterative roots* of a given function F, i.e., the solution of the following equation $G^m = F$, where $m \geq 2$ is an integer, has been considered for nearly two hundred years (see for example [1], [10], [12], [14], [15], [25]). There are also some results for some homeomorphisms of the unit circle S^1, e.g., homeomorphisms with an irrational rotation number (see [18], [24]), for the identity function (see [11]) and for some other homeomorphisms with a rational rotation number (see [16], [19], [20]). In particular, [16] relates the existence of an iterative root of F to the existence of an iterative root of $F_{|\mathrm{Per}\, F}$, where $\mathrm{Per}\, F := \{z \in S^1 \mid \exists k \in \mathbb{N} \;\; F^k(z) = z\}$. More precisely, an orientation-preserving homeomorphism $F\colon S^1 \longrightarrow S^1$ such that $F^n(z) = z$ for $z \in \mathrm{Per}\, F$, has an iterative root of order m if and only if there exists an iterative root $\psi\colon \mathrm{Per}\, F \longrightarrow \mathrm{Per}\, F$ of order m of $F_{|\mathrm{Per} F}$ such that

(i) ψ preserves orientation;

(ii) for any connected component $\overrightarrow{(u,v)}$ of $S^1 \setminus \mathrm{Per}\, F$, $\overrightarrow{(\psi(u),\psi(v))}$ and $\overrightarrow{(u,v)}$ are both increasing (or both decreasing) arcs of F^n.

Recall that an arc $\overrightarrow{(u,v)}$, where $u, v \in \mathrm{Per}\, F$ and $\overrightarrow{(u,v)} \cap \mathrm{Per}\, F = \emptyset$, is called increasing (resp. decreasing) arc of F^n if there is an $x \in \overrightarrow{(u,v)}$ such that $F^n(x) \in \overrightarrow{(x,v)}$ (resp. $F^n(x) \in \overrightarrow{(u,x)}$).

This paper answers the question when iterative roots of the function $F_{|\mathrm{Per} F}$ exist and generalizes results from [20]. For this purpose we apply the method

AMS (2000) Subject Classification: Primary 39B12, Secondary 26A18.

which is used for the construction of the iterative roots of a homeomorphism with an irrational rotation number (i.e., the method that uses a solution of some Schröder equation, see [18]).

2. Preliminaries

We begin with recalling some definitions and notations. For any $u, w, z \in S^1$ there exist unique $t_1, t_2 \in \langle 0, 1)$ such that $we^{2\pi i t_1} = z$, $we^{2\pi i t_2} = u$. Define

$$u \prec w \prec z \quad \text{if and only if} \quad 0 < t_1 < t_2$$

(see [2]). Some properties of this relation can be found in [3], [4] and [5].

We say that a function $F\colon A \longrightarrow S^1$, where $A \subset S^1$, *preserves orientation* if for any $u, w, z \in A$ such that $u \prec w \prec z$ we have $F(u) \prec F(w) \prec F(z)$.

For every orientation-preserving homeomorphism $F\colon S^1 \longrightarrow S^1$ there exists a unique (up to translation by an integer) homeomorphism $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, called the *lift* of F, such that $F\left(e^{2\pi i x}\right) = e^{2\pi i f(x)}$ and $f(x+1) = f(x) + 1$ for all $x \in \mathbb{R}$. Moreover, the limit

$$\alpha(F) := \lim_{n\to\infty} \frac{f^n(x)}{n} \pmod 1, \qquad x \in \mathbb{R}$$

always exists and does not depend on x and the choice of f. This number is called the *rotation number* of F (see [9]). It appears that a homeomorphism $F\colon S^1 \longrightarrow S^1$ preserves orientation if and only if f is a strictly increasing function (see for example [4]). Moreover, $\alpha(F)$ is a rational number if and only if $\operatorname{Per} F \neq \emptyset$ (see for example [9]).

Let us introduce a classification of orientation-preserving homeomorphisms. Namely, for $n \in \mathbb{N}$ and $q \in \{0, 1 \ldots, n-1\}$ such that $\gcd(q, n) = 1$ denote by $\mathcal{F}_{q,n}$ the set of all orientation-preserving homeomorphisms F of the circle with $\alpha(F) = \frac{q}{n}$. From now on writing $F \in \mathcal{F}_{q,n}$ without any additional assumptions on q and n, we mean that the numbers q and n are such that $n \in \mathbb{N}$, $q \in \{0, \ldots, n-1\}$ and $\gcd(q, n) = 1$.

Finally, for any distinct $u, z \in S^1$ put $\overrightarrow{(u, z)} := \{w \in S^1 \mid u \prec w \prec z\}$ (such a set is said to be an *open arc*) and $\overrightarrow{\langle u, z)} := \overrightarrow{(u, z)} \cup \{u\}$.

Remark 1
If $F \in \mathcal{F}_{q,n}$, then $\operatorname{Per} F = \{z \in S^1 \mid F^n(z) = z\}$ and n is the minimal number such that $F^n(z) = z$ for $z \in \operatorname{Per} F$. In fact, notice that $\alpha(F^n) = n\alpha(F) \pmod 1 = 0$. Therefore F^n has a fixed point (see [9], Ch. 3, § 3). The assertion follows from the fact that every two periodic points of an orientation-preserving homeomorphism have the same period (see for example [17], p. 16). Now suppose that $F^m(z) = z$ for an $m \in \{1, \ldots, n-1\}$ and a $z \in \operatorname{Per} F$. Then $m\frac{q}{n} \pmod 1 = 0$. Thus n divides m, a contradiction.

For any $F \in \mathcal{F}_{q,n}$ define the following set

$$\mathcal{M}_F := \{u \in \operatorname{Per} F \mid \exists w \in \operatorname{Per} F,\ w \neq u :\ \overrightarrow{(u,w)} \cap \operatorname{Per} F = \emptyset\}.$$

Such a set is F-*invariant* (i.e., $F(\mathcal{M}_F) = \mathcal{M}_F$). It may happen that $\mathcal{M}_F = \emptyset$ (if $\operatorname{Per} F = S^1$), $\mathcal{M}_F = \operatorname{Per} F$ (for example, if $\operatorname{Per} F$ is finite) or $\emptyset \subsetneq \mathcal{M}_F \subsetneq \operatorname{Per} F$ (for example, if $\operatorname{int}(\operatorname{Per} F) \neq \emptyset$). Moreover, if $\mathcal{M}_F \neq \emptyset$, then $S^1 \setminus \operatorname{Per} F \neq \emptyset$. Since $\operatorname{Per} F$ is closed, we have that $S^1 \setminus \operatorname{Per} F$ is a sum of pairwise disjoint open arcs. Denote the family of these arcs by $\mathcal{A}_F$. For every $\overrightarrow{(u,w)} \in \mathcal{A}_F$, where $u, w \in \operatorname{Per} F$, put $l\left(\overrightarrow{(u,w)}\right) := u$ and observe that l maps bijectively $\mathcal{A}_F$ onto $\mathcal{M}_F$. Setting $I_u := l^{-1}(u)$ for $u \in \mathcal{M}_F$ we have

$$S^1 \setminus \operatorname{Per} F = \bigcup_{u \in \mathcal{M}_F} I_u .$$

For the convenience of the reader we recall the relevant, slightly modified material from [21].

Proposition 1
Let $F \in \mathcal{F}_{q,n}$ be such that $\operatorname{Per} F \neq S^1$ and let $I \in \mathcal{A}_F$. Then $\overrightarrow{(z, F^n(z))} \subset I$ for every $z \in I$ or $\overrightarrow{(F^n(z), z)} \subset I$ for every $z \in I$.

Moreover, if $\overrightarrow{(z, F^n(z))} \subset I$ (resp. $\overrightarrow{(F^n(z), z)} \subset I$) for a $z \in I$, then $\overrightarrow{(z_1, F^n(z_1))} \subset F(I)$ (resp. $\overrightarrow{(F^n(z_1), z_1)} \subset F(I)$) for all $z_1 \in F(I)$.

We also recall a sketch of the proof. Assume $z \in I \in \mathcal{A}_F$. Then $F^n(z) \in I$ and $z \neq F^n(z)$. Therefore $\overrightarrow{(z, F^n(z))} \subset I$ or $\overrightarrow{(F^n(z), z)} \subset I$. Suppose that $\overrightarrow{(z, F^n(z))} \subset I$. Since F preserves orientation we have

$$\overrightarrow{(F^{ln}(z), F^{n(l+1)}(z))} \subset I \qquad \text{for all } l \in \mathbb{Z}.$$

Moreover, $\bigcup_{l \in \mathbb{Z}} \overrightarrow{\langle F^{ln}(z), F^{(l+1)n}(z))} = I$. Now fix $u \in I$. We may assume $u \neq F^{ln}(z)$ for $l \in \mathbb{Z}$. Then $u \in \overrightarrow{(F^{nj}(z), F^{n(j+1)}(z))}$ for some $j \in \mathbb{Z}$. Hence $F^n(u) \in \overrightarrow{(F^{n(j+1)}(z), F^{n(j+2)}(z))}$, as F preserves orientation. This gives $\overrightarrow{(u, F^n(u))} \subset I$.

For the second assertion suppose that $\overrightarrow{(z, F^n(z))} \subset I$ for an $z \in I$. Let $z_1 \in F(I)$ be fixed. Then there exists a $z_0 \in I$ such that $F(z_0) = z_1$ and $\overrightarrow{(z_0, F^n(z_0))} \subset I$. Hence $\overrightarrow{(z_1, F^n(z_1))} = F\left(\overrightarrow{(z_0, F^n(z_0))}\right) \subset F(I)$. This ends the sketch of the proof.

Now we present some results concerning the Schröder equation

$$\psi \circ F = s\psi, \tag{1}$$

where $s \in S^1$ and $F: S^1 \longrightarrow S^1$ is an orientation-preserving homeomorphism with a rational rotation number. It is a known fact (see for example [9], [17] or [22]) that if F is a homeomorphism with an irrational rotation number and $s = e^{2\pi i\alpha(F)}$, then (1) has a continuous solution $\psi: S^1 \longrightarrow S^1$. If F is a homeomorphism with a rational rotation number and such that $\operatorname{card}(\operatorname{Per} F) \leq \aleph_0$, then the only continuous solutions of (1) are constant functions. Of course, in this case $s = 1$ (see Theorem 4.1 in [7]). On the other hand, it follows from Theorem 4.2 in [7] that, if F is an orientation-preserving homeomorphism such that $\operatorname{Per} F = S^1$ and $F \neq \mathrm{id}_{S^1}$, then there exists a constant $s \neq 1$ for which (1) has a homeomorphic and orientation-preserving solution $\psi: S^1 \longrightarrow S^1$. The following theorem generalizes the results from Theorem 4.2 in [7].

Theorem 1
Let $n > 1$ and $F \in \mathcal{F}_{q,n}$. There exists an orientation-preserving continuous mapping $\psi: \operatorname{Per} F \longrightarrow S^1$ such that

$$\psi(F(z)) = e^{2\pi i\alpha(F)}\psi(z), \qquad z \in \operatorname{Per} F. \tag{2}$$

The solution of (2) depends on an arbitrary function.

The proof of the above theorem is based on Theorem 4.2 from [7] and the following observation.

Lemma 1
For any $F \in \mathcal{F}_{q,n}$, where $n > 1$, with $\operatorname{Per} F \neq S^1$ there exist infinitely many homeomorphisms $\hat{F} \in \mathcal{F}_{q,n}$ such that $\operatorname{Per} \hat{F} = S^1$ and $\hat{F}(z) = F(z)$ for $z \in \operatorname{Per} F$.

Proof. Fix $F \in \mathcal{F}_{q,n}$ such that $\operatorname{Per} F \neq S^1$. Define the equivalence relation on $\mathcal{M}_F$:

$$p \sim q \iff \exists k \in \mathbb{Z} \;\; p = F^k(q).$$

By $E_\sim$ denote the set of class representatives. In other words, we decompose $\mathcal{M}_F$ onto cycles of F. Let $\phi_{p,k}: I_{F^k(p)} \longrightarrow I_{F^{k+1}(p)}$ for all $p \in E_\sim$ and $k \in \{0, \ldots, n-2\}$ be arbitrary orientation-preserving homeomorphisms. Put

$$\phi_{p,n-1}(z) := \phi_{p,0}^{-1} \circ \phi_{p,1}^{-1} \circ \ldots \circ \phi_{p,n-2}^{-1}(z), \qquad z \in I_{F^{n-1}(p)}. \tag{3}$$

It is easy to see that $\phi_{p,n-1}: I_{F^{n-1}(p)} \longrightarrow I_p$ for $p \in E_\sim$ are orientation-preserving homeomorphisms. Let $z \in S^1 \setminus \operatorname{Per} F$. There exist a unique $p \in E_\sim$ and $k \in \{0, \ldots, n-1\}$ such that $z \in I_{F^k(p)}$. Set

$$\phi(z) := \phi_{p,k}(z).$$

and observe that ϕ maps $S^1 \setminus \operatorname{Per} F$ onto $S^1 \setminus \operatorname{Per} F$ and

$$\phi^n(z) = \begin{cases} \phi_{p,n-1} \circ \ldots \circ \phi_{p,0}(z), & k = 0, \\ \phi_{p,k-1} \circ \ldots \circ \phi_{p,0} \circ \phi_{p,n-1} \circ \ldots \circ \phi_{p,k}(z), & k \neq 0. \end{cases}$$

This and (3) give $\phi^n(z) = z$ for $z \in S^1 \setminus \operatorname{Per} F$.

Now we show that ϕ preserves orientation. To do this, observe that for every $z \in I_p$, where $p \in \mathcal{M}_F$, we have $\phi(z) \in I_{F(p)}$. Fix $u, w, z \in S^1 \setminus \operatorname{Per} F$ such that $u \prec w \prec z$. Notice that if $\{u, w, z\} \subset I_p$ for a $p \in \mathcal{M}_F$, then the definition of ϕ gives $\phi(u) \prec \phi(w) \prec \phi(z)$. Now assume that there exist distinct $p, q \in \mathcal{M}_F$ such that exactly one element from the set $\{u, w, z\}$ belongs to I_p and the rest of them belong to I_q. In view of Lemma 2 in [4], it is sufficient to consider only the case: $\overrightarrow{(z, u)} \subset I_p$ and $w \in I_q$. Hence $\overrightarrow{(\phi(z), \phi(u))} \subset I_{F(p)}$ and $\phi(w) \in I_{F(q)}$. Since $I_{F(q)} \cap I_{F(p)} = \emptyset$, we have $\phi(u) \prec \phi(w) \prec \phi(z)$. Finally, let $\operatorname{card}(\mathcal{M}_F) \geq 3$ and let $u \in I_p, w \in I_q$ and $z \in I_t$, where $p, q, t \in \mathcal{M}_F$ are such that $p \neq q \neq t \neq p$. The arcs I_p, I_q and I_t are pairwise disjoint, so we have $p \prec q \prec t$. Hence $F(p) \prec F(q) \prec F(t)$. On the other hand, $\phi(u) \in I_{F(p)}$, $\phi(w) \in I_{F(q)}$ and $\phi(z) \in I_{F(t)}$. Thus $\phi(u) \prec \phi(w) \prec \phi(z)$, as $I_{F(p)}$, $I_{F(q)}$ and $I_{F(t)}$ are pairwise disjoint arcs.

Define the function $\hat{F} \colon S^1 \longrightarrow S^1$ as follows:

$$\hat{F}(z) := \begin{cases} F(z), & z \in \operatorname{Per} F, \\ \phi(z), & z \in S^1 \setminus \operatorname{Per} F. \end{cases}$$

Clearly, $\hat{F}$ is a surjection. To show that $\hat{F}$ is an orientation-preserving homeomorphism it is sufficient to prove that it preserves orientation. Similarly as above fix $u, w, z \in S^1$ such that $u \prec w \prec z$. By virtue of Lemma 2 in [4] it is enough to consider three cases:

(i) $\operatorname{card}(\operatorname{Per} F) \geq 3$ and $u, w, z \in \operatorname{Per} F$ or $u, w, z \in S^1 \setminus \operatorname{Per} F$ (this one is clear).

(ii) $u, z \in \operatorname{Per} F$ and $w \in S^1 \setminus \operatorname{Per} F$. There exists a $p \in \mathcal{M}_F \cap \overrightarrow{\langle u, z \rangle}$ such that $w \in I_p$ and $\hat{F}(w) = \phi(w) \in I_{F(p)}$. Thus $F(p) \in \overrightarrow{\langle F(u), F(z) \rangle}$. Consequently, $I_{F(p)} \subset \overrightarrow{(F(u), F(z))}$. Finally, $\hat{F}(u) \prec \hat{F}(w) \prec \hat{F}(z)$, as $\hat{F}_{|\operatorname{Per} F} = F$.

(iii) $u, z \in S^1 \setminus \operatorname{Per} F$ and $w \in \operatorname{Per} F$. In this case it may happen that $u, z \in I_p$ for a $p \in M_F$ or $u \in I_p$ and $z \in I_q$ for some $p, q \in \mathcal{M}_F$, $p \neq q$. Suppose that $u, z \in I_p$ for a $p \in M_F$. Then $\overrightarrow{(z, u)} \subset I_p$ and $w \notin I_p$. Hence $\overrightarrow{(\hat{F}(z), \hat{F}(u))} = \overrightarrow{(\phi(z), \phi(u))} \subset I_{F(p)}$ and $\hat{F}(w) = F(w) \notin I_{F(p)}$. Thus $\hat{F}(u) \prec \hat{F}(w) \prec \hat{F}(z)$. Now suppose that $u \in I_p$ and $z \in I_q$ for some $p, q \in \mathcal{M}_F, p \neq q$. Then $p \prec u \prec w$ and $w \prec z \prec p$. A similar reasoning to this in (ii) yields $\hat{F}(p) \prec \hat{F}(u) \prec \hat{F}(w)$ and $\hat{F}(w) \prec \hat{F}(z) \prec \hat{F}(p)$. Hence, by Lemma 1 in [3], we obtain $\hat{F}(u) \prec \hat{F}(w) \prec \hat{F}(z)$.

Finally, notice that $\hat{F}_{|O(z)} = F_{|O(z)}$, where $O(z) := \{z, F(z), \dots, F^{n-1}(z)\}$ for $z \in \operatorname{Per} F$. Thus $\alpha(F) = \alpha(\hat{F})$. Consequently, $\hat{F} \in \mathcal{F}_{q,n}$, and the proof is completed.

Now we give the *proof of Theorem 1.* To do this fix $F \in \mathcal{F}_{q,n}$, where $n > 1$. Notice that if $\operatorname{Per} F = S^1$, then, in view of Theorem 4.2 in [7], there exist an orientation-preserving homeomorphism (depending on an arbitrary function) $\psi\colon S^1 \longrightarrow S^1$ and a $q' \in \{1, \dots, n-1\}$ with $\gcd(q', n) = 1$ such that

$$\psi(F(z)) = e^{2\pi i \frac{q'}{n}} \psi(z), \qquad z \in S^1.$$

The equality $\alpha(F) = \frac{q'}{n}$ follows from the fact that the homeomorphism ψ conjugates F and the rotation $R(z) = e^{2\pi i \frac{q'}{n}} z$ and ψ is an orientation-preserving homeomorphism (see Theorem 1 in [8]). Henceforth assume that $\operatorname{Per} F \neq S^1$. Let $\hat{F}$ be an orientation-preserving homeomorphism, which exists by Lemma 1, and let $\hat{\psi}\colon S^1 \longrightarrow S^1$ be an orientation-preserving homeomorphic solution of

$$\hat{\psi}(\hat{F}(z)) = e^{2\pi i \alpha(F)} \hat{\psi}(z), \qquad z \in S^1.$$

Put $\psi := \hat{\psi}_{|\operatorname{Per} F}$. Observe that $\psi\colon \operatorname{Per} F \longrightarrow S^1$ is the desired solution of (2).

Definition 1

Given $F \in \mathcal{F}_{q,n}$ put

$$\mathcal{M}_F^+ := \left\{ p \in \mathcal{M}_F \mid \overrightarrow{(z, F^n(z))} \subset I_p \text{ for } z \in I_p \right\}$$

and

$$\mathcal{M}_F^- := \left\{ p \in \mathcal{M}_F \mid \overrightarrow{(F^n(z), z)} \subset I_p \text{ for } z \in I_p \right\}.$$

Notice that $\mathcal{M}_F^+ \cap \mathcal{M}_F^- = \emptyset$. Indeed, if $p \in \mathcal{M}_F^+ \cap \mathcal{M}_F^-$, then for any $z \in I_p$ we would have $\overrightarrow{(F^n(z), z)} \subset I_p$ and $\overrightarrow{(z, F^n(z))} \subset I_p$. Hence $S^1 = I_p$, a contradiction.

Remark 2

From Proposition 1 we get $\mathcal{M}_F^+ \cup \mathcal{M}_F^- = \mathcal{M}_F$ and $F(\mathcal{M}_F^+) \subset \mathcal{M}_F^+$. This inclusion and the fact that $\mathcal{M}_F^+ \subset \operatorname{Per} F$ yield

$$\mathcal{M}_F^+ = F^{n-1}(F(\mathcal{M}_F^+)) \subset F(\mathcal{M}_F^+).$$

Thus for any $F \in \mathcal{F}_{q,n}$, we have $\mathcal{M}_F^+ \cup \mathcal{M}_F^- = \mathcal{M}_F$ and $F(\mathcal{M}_F^+) = \mathcal{M}_F^+$.

Since for all $F \in \mathcal{F}_{q,n}$ the sets $\operatorname{Per} F$, $\mathcal{M}_F$, $\mathcal{M}_F^+$ and $\mathcal{M}_F^-$ are invariant sets of F we have the following result.

Remark 3
Let $F \in \mathcal{F}_{q,n}$, $n > 1$, $\psi\colon \operatorname{Per} F \longrightarrow S^1$ be an orientation-preserving continuous solution of (2) and let $X \in \{\operatorname{Per} F, \mathcal{M}_F, \mathcal{M}_F^+, \mathcal{M}_F^-\}$. Then

$$\psi(X) = e^{2\pi i\alpha(F)}\psi(X).$$

3. Main results

Here we give necessary and sufficient conditions for the existence of orientation-preserving continuous iterative roots of order $m > 2$ of a mapping $F \in \mathcal{F}_{q,n}$. Throughout this section we will assume that $n > 1$. We begin with the following observation.

Lemma 2
Let $m \geq 2$ be an integer and let $F \in \mathcal{F}_{q,n}$. Suppose that the equation

$$G^m(z) = F(z), \qquad z \in S^1 \tag{4}$$

has an orientation-preserving continuous solution. Then there are an orientation-preserving continuous solution of (2) and a $j \in \{0, \dots, m-1\}$ such that

$$e^{2\pi i \frac{\alpha(F)+j}{m}}\psi(X) = \psi(X), \tag{5}$$

where $X \in \{\operatorname{Per} F, \mathcal{M}_F, \mathcal{M}_F^+, \mathcal{M}_F^-\}$.

Proof. Since G satisfies (4), we have $\alpha(F) = m\alpha(G) \pmod 1$. This yields $\frac{\alpha(F)+j}{m} = \alpha(G)$ for a $j \in \{0, \dots, m-1\}$. Theorem 1 implies the existence of an orientation-preserving continuous solution of the following equation

$$\psi(G(z)) = e^{2\pi i \frac{\alpha(F)+j}{m}}\psi(z), \qquad z \in \operatorname{Per} G. \tag{6}$$

Thus

$$\psi(G^m(z)) = \psi(F(z)) = e^{2\pi i\alpha(F)}\psi(z), \qquad z \in \operatorname{Per} G.$$

Hence and from the fact that $\operatorname{Per} F = \operatorname{Per} G$ implies $\mathcal{M}_F = \mathcal{M}_G$, we get that ψ is a solution of (2) satisfying (5) for $X \in \{\operatorname{Per} F, \mathcal{M}_F\}$. Moreover, $\alpha(G) = \frac{\alpha(F)+j}{m} = \frac{q'}{nl}$, where $q' := \frac{q+jn}{\gcd(q+jn,m)}$, $l := \frac{m}{\gcd(q+jn,m)}$ and $\gcd(q', nl) = 1$, so $G \in \mathcal{F}_{q'nl}$. Hence, if $\operatorname{Per} F \neq S^1$, then $p \in \mathcal{M}_G^+$ gives $\overrightarrow{(z, G^{nl}(z))} \subset I_p$ for every $z \in I_p$. Since

$$G^{kln}(z) \in I_p \quad \text{and} \quad \overrightarrow{(G^{kln}(z), G^{(k+1)nl}(z))} \subset I_p \qquad \text{for } k \in \mathbb{Z},$$

we have $\overrightarrow{(z, G^{nm}(z))} \subset I_p$. Consequently, $p \in \mathcal{M}_F^+$. Whence $\mathcal{M}_G^+ \subset \mathcal{M}_F^+$. Similarly, $\mathcal{M}_G^- \subset \mathcal{M}_F^-$, so $\mathcal{M}_F \setminus \mathcal{M}_F^- = \mathcal{M}_F^+ \subset \mathcal{M}_G^+ = \mathcal{M}_G \setminus \mathcal{M}_G^-$. Finally, $\mathcal{M}_F^+ = \mathcal{M}_G^+$ and $\mathcal{M}_F^- = \mathcal{M}_G^-$. In view of the above facts and Remark 3 equality (5) holds for $X \in \{\operatorname{Per} F, \mathcal{M}_F, \mathcal{M}_F^+, \mathcal{M}_F^-\}$.

Corollary 1

Let $F \in \mathcal{F}_{q,n}$. If $G: S^1 \longrightarrow S^1$ is an orientation-preserving homeomorphism satisfying (4) *for an integer $m \geq 2$, then $\mathcal{M}_F = \mathcal{M}_G$, $\mathcal{M}_F^+ = \mathcal{M}_G^+$ and $\mathcal{M}_F^- = \mathcal{M}_G^-$.*

Now suppose that $F \in \mathcal{F}_{q,n}$ is such that $\operatorname{Per} F \neq S^1$, $m > 1$ is an integer and $\psi: \operatorname{Per} F \longrightarrow S^1$ is an orientation-preserving continuous solution of (2) satisfying (5) for $X = \operatorname{Per} F$ and a $j \in \{0, \ldots, m-1\}$. This fact yields equality (5) for $X = \mathcal{M}_F$. Indeed, put

$$h_\psi(z) := \psi^{-1}\left(e^{2\pi i \frac{\alpha(F)+j}{m}} \psi(z)\right), \qquad z \in \operatorname{Per} F. \tag{7}$$

It is easy to see that $h_\psi: \operatorname{Per} F \longrightarrow \operatorname{Per} F$ is an orientation-preserving homeomorphism. Notice that $z \in \operatorname{Per} F \setminus \mathcal{M}_F \neq \emptyset$ if and only if there exist a $w \in \operatorname{Per} F \setminus \{z\}$ and $z_n \in \overrightarrow{(z,w)} \cap \operatorname{Per} F$ for $n \in \mathbb{N}$ such that $z_n \to z$ as $n \to \infty$. This is equivalent to $h_\psi^{-1}(z_n) \to h_\psi^{-1}(z)$ as $n \to \infty$ and $h_\psi^{-1}(z_n) \in \overrightarrow{(h_\psi^{-1}(z), h_\psi^{-1}(w))} \cap \operatorname{Per} F$, which gives $h_\psi^{-1}(z) \in \operatorname{Per} F \setminus \mathcal{M}_F$ or equivalently $z \in h_\psi(\operatorname{Per} F \setminus \mathcal{M}_F)$. Hence $h_\psi(\mathcal{M}_F) = \mathcal{M}_F$.

However, (5) with $X = \operatorname{Per} F$ does not imply (5) for $X \in \{\mathcal{M}_F^+, \mathcal{M}_F^-\}$. An example of a function $F \in \mathcal{F}_{1,2}$ such that $\operatorname{Per} F = \mathcal{M}_F = \{1, i, -1, -i\}$, $\mathcal{M}_F^+ = \{1, -1\}$ may be given. Put $\psi(z) = z$ for $z \in \operatorname{Per} F$. Then ψ is a solution of (2) satisfying (5) for $m = 2$, $j = 0$ and $X \in \{\operatorname{Per} F, \mathcal{M}_F\}$, but $e^{2\pi i \frac{1}{4}} \mathcal{M}_F^+ \neq \mathcal{M}_F^+$. Therefore assume subsidiarily that (5) holds for $X = \mathcal{M}_F^+$ and introduce the equivalence relation ρ on $\mathcal{M}_F$:

$$(p,q) \in \rho \iff \exists k \in \mathbb{Z} \;\; q = H_\psi^k(p), \qquad p, q \in \mathcal{M}_F, \tag{8}$$

where $H_\psi := h_{\psi|\mathcal{M}_F}$ and h_ψ is given by (7). Let W_ρ be the set of class representatives of ρ.

Notice that (5) with $X = \mathcal{M}_F^+$ yields $[p]_\rho \subset \mathcal{M}_F^+$ or $[p]_\rho \subset \mathcal{M}_F^-$ for all $p \in W_\rho$.

Definition 2

Let $F \in \mathcal{F}_{q,n}$ be such that $\operatorname{Per} F \neq S^1$, $m > 1$ be an integer, $\psi: \operatorname{Per} F \longrightarrow S^1$ be an orientation-preserving continuous solution of (2) satisfying (5) for $X \in \{\operatorname{Per} F, \mathcal{M}_F^+\}$ and a $j \in \{0, \ldots, m-1\}$ and let W_ρ be the set of class representatives of the relation ρ given by (8). Put

$$m' := \gcd(q + jn, m), \quad l := \frac{m}{m'} \quad \text{and} \quad n' := nl. \tag{9}$$

Let $(z_{p,k})_{k\in\mathbb{Z}}$ for $p \in W_\rho$ be sequences such that the points $z_{p,dn'+r} \in I_{H_\psi^r(p)}$ for $r \in \{0, \ldots, l-1\}$ and $d \in \{0, \ldots, m'-1\}$ are arbitrary fixed and such that

$$\begin{aligned} &H_\psi^r(p) \prec z_{p,r} \prec z_{p,n'+r} \prec \ldots \prec z_{p,(m'-1)n'+r} \prec F^n(z_{p,r}), \quad \text{if } p \in \mathcal{M}_F^+ \\ &\text{or} \\ &H_\psi^r(p) \prec F^n(z_{p,r}) \prec z_{p,(m'-1)n'+r} \prec \ldots \prec z_{p,n'+r} \prec z_{p,r}, \quad \text{if } p \in \mathcal{M}_F^- \end{aligned} \tag{10}$$

and the remaining points are given by

$$z_{p,k+m} := F(z_{p,k}), \qquad k \in \mathbb{Z},\ p \in W_\rho\,. \tag{11}$$

Now we show that the above sequences are well defined and we prove some of their properties.

Lemma 3

Under assumptions of Definition 2, for all $i \in \mathbb{Z}$ and $p \in W_\rho$ there exist unique $s \in \{0,\dots,m'-1\}$, $r' \in \{0,\dots,l-1\}$ and $k \in \mathbb{Z}$ such that $z_{p,i} = F^k(z_{p,sn'+r'})$. Moreover,

$$\{z_{p,dn'+r}\}_{d\in\mathbb{Z}} \subset I_{H^r_\psi(p)}, \qquad p \in W_\rho\,,\ r \in \{0,\dots,n'-1\}, \tag{12}$$

and for any $p \in W_\rho$, $[p]_\rho \subset \mathcal{M}_F^+$ (resp. $[p]_\rho \subset \mathcal{M}_F^-$) if and only if

$$z_{p,an'+r} \prec z_{p,bn'+r} \prec z_{p,cn'+r} \quad (\textit{resp. } z_{p,cn'+r} \prec z_{p,bn'+r} \prec z_{p,an'+r}) \tag{13}$$

for any $r \in \{0,\dots,n'-1\}$ and all $a,b,c \in \mathbb{Z}$ such that $a<b<c$.

Proof. Fix $p \in W_\rho$ and $i \in \mathbb{Z}$. Write $i = dn'+r$, where $d \in \mathbb{Z}$ and $r \in \{0,\dots,n'-1\}$. If $d \in \{0,\dots,m'-1\}$ and $r \in \{0,\dots,l-1\}$, then by Definition 2, $s=d$, $r'=r$, $k=0$ and obviously $z_{p,dn'+r} \in I_{H^r_\psi(p)}$.

Suppose that $d \in \mathbb{Z}\setminus\{0,\dots,m'-1\}$ and $r \in \{0,\dots,l-1\}$. Put $t = \left[\frac{d}{m'}\right]$ ($[x]$ denotes the integer part of x), $k=tn$, $s = d-tm'$ and $r'=r$. Notice that $d = tm'+s$, $s \in \{0,\dots,m'-1\}$ and by (11),

$$F^{tn}(z_{p,sn'+r}) = z_{p,sn'+r+mtn} = z_{p,(tm'+s)n'+r} = z_{p,dn'+r}\,. \tag{14}$$

Since $F^{tn}(I_u) = I_u$ for $u \in \mathcal{M}_F$ and $z_{p,sn'+r} \in I_{H^r_\psi(p)}$, by (14) we have $z_{p,dn'+r} \in I_{H^r_\psi(p)}$.

Finally assume that $d \in \mathbb{Z}$ and $r \in \{l,\dots,n'-1\}$. As $\gcd(q,n)=1$ and $m' = \gcd(q+jn,m)$ we have $\gcd(m',n)=1$. Hence there exists a unique $b \in \{1,\dots,n-1\}$ such that $m'b = 1 \,(\mathrm{mod}\ n)$. Set $a_r := \left[\frac{r}{l}\right]$, $r' = r - a_r l$ and $k_r := a_r b \,(\mathrm{mod}\ n)$. Thus $m'k_r = a_r \,(\mathrm{mod}\ n)$ which, in view of the fact that $r = a_r l + r'$, gives $mk_r + r' = r (\mathrm{mod}\ n')$ and, in consequence,

$$mk_r + r' = xn' + r \qquad \text{for some } x \in \mathbb{Z}. \tag{15}$$

This time put $t_r := \left[\frac{d-x}{m'}\right]$, $k = k_r + t_r n$ and $s = d - x - t_r m'$. Then

$$F^{k_r+t_rn}(z_{p,sn'+r'}) = z_{p,\left(d-\frac{k_rm+r'-r}{n'}\right)n'+r'+k_rm} = z_{p,dn'+r}\,.$$

Since $r' \in \{0,\dots,l-1\}$ and $d-x \in \mathbb{Z}$, we obtain $z_{p,(d-x)n'+r'} \in I_{H^{r'}_\psi(p)}$. To prove $z_{p,dn'+r} \in I_{H^r_\psi(p)}$ it is enough to show that $F^{k_r}(H^{r'}_\psi(p)) = H^r_\psi(p)$. Notice that from (7),

$$H^m_\psi(z) = \psi^{-1}(e^{2\pi \mathrm{i}\frac{q}{n}}\psi(z)) = F(z), \qquad z \in \mathcal{M}_F .$$

This, (15) and the fact that $H^{xn'}_\psi(p) = p$ yield

$$F^{k_r}\left(H^{r'}_\psi(p)\right) = H^{mk_r+r'}_\psi(p) = H^{xn'+r}_\psi(p) = H^r_\psi(p).$$

The proof of the remaining part of the lemma runs in the same way as the proof of the second assertion of Lemma 7 in [20] (it is enough to take $H^r_\psi(p)$, r_1, and k_r instead of $a_{R_{N_F}(i+rk'q')}$, $R_l(r)$ and p_r, respectively).

Let $(z_{p,k})_{k\in\mathbb{Z}}$, where $p \in W_\rho$, be the family of sequences given by (10) and (11). Define the following families of arcs:

$$L_{p,k} := \begin{cases} \overrightarrow{\langle z_{p,k}, z_{p,k+n'}\rangle}, & p \in \mathcal{M}^+_F, \\ \overrightarrow{\langle z_{p,k+n'}, z_{p,k}\rangle}, & p \in \mathcal{M}^-_F \end{cases} \qquad \text{for } k \in \mathbb{Z},\ p \in W_\rho . \tag{16}$$

From Lemma 3 it follows that

$$F(L_{p,k}) = L_{p,k+m}, \qquad k \in \mathbb{Z},\ p \in W_\rho .$$

Lemma 4

Under assumptions of Definition 2 if for any $p \in W_\rho$ the sequences $(z_{p,k})_{k\in\mathbb{Z}}$ are given by (10) *and* (11) *and $\{L_{p,k}\}_{k\in\mathbb{Z}}$ are the families of arcs defined by* (16)*, then*

$$\bigcup_{d\in\mathbb{Z}} L_{p,dn'+r} = I_{H^r_\psi(p)}, \qquad r \in \{0,\ldots,n'-1\}. \tag{17}$$

Proof. Fix $r \in \{0,\ldots,n'-1\}$ and suppose that $p \in W_\rho \cap \mathcal{M}^+_F$. From (13) we have $z_{p,dn'+r} \in \overrightarrow{\langle z_{p,(d-1)n'+r}, z_{p,(d+1)n'+r}\rangle}$ for $d \in \mathbb{Z}$. Hence by (12) and (16),

$$L_{p,dn'+r} \subset \overrightarrow{\langle z_{p,(d-1)n'+r}, z_{p,(d+1)n'+r}\rangle} \subset I_{H^r_\psi(p)}, \qquad d \in \mathbb{Z}.$$

Thus

$$\bigcup_{d\in\mathbb{Z}} L_{p,dn'+r} \subset I_{H^r_\psi(p)} .$$

To prove the converse inclusion fix $z \in I_{H^r_\psi(p)}$. By Lemma 4 in [21] (see also Remark 3 in [20]) we have

$$I_{H^r_\psi(p)} = \bigcup_{k\in\mathbb{Z}} \overrightarrow{\langle F^{kn}(z_{p,r}), F^{(k+1)n}(z_{p,r})\rangle} .$$

Hence $z \in \overrightarrow{\langle F^{k_0 n}(z_{p,r}), F^{(k_0+1)n}(z_{p,r})\rangle}$ for a $k_0 \in \mathbb{Z}$. On the other hand, by (11) and (13),

$$\begin{aligned}\overrightarrow{\langle F^{k_0 n}(z_{p,r}), F^{(k_0+1)n}(z_{p,r})\rangle} &= \overrightarrow{\langle z_{p,k_0 nm+r}, z_{p,(k_0+1)nm+r}\rangle} \\ &= \bigcup_{s=0}^{m'} L_{p,k_0 nm+sn'+r} \\ &\subset \bigcup_{k\in\mathbb{Z}} L_{p,kn'+r}\,.\end{aligned}$$

This ends the proof.

Theorem 2

Let $F \in \mathcal{F}_{q,n}$ be such that $\operatorname{Per} F \neq S^1$, $m \geq 2$ be an integer and let $\psi\colon \operatorname{Per} F \longrightarrow S^1$ be an orientation-preserving continuous solution of (2) satisfying (5) for $X \in \{\operatorname{Per} F, \mathcal{M}_F^+\}$ and a $j \in \{0, \dots, m-1\}$. Suppose that W_ρ is the selector of ρ given by (8), $(z_{p,k})_{k\in\mathbb{Z}}$ for $p \in W_\rho$ are the families of sequences given by (10) and (11) and $\{L_{p,k}\}_{k\in\mathbb{Z}}$ for $p \in W_\rho$ are the families of arcs defined by (16). If $G_{p,k}\colon L_{p,k} \longrightarrow L_{p,k+1}$ for $k \in \{0,1,\dots,m-2\}$ and $p \in W_\rho$ are orientation-preserving surjections, then there exists a unique orientation-preserving homeomorphism $G : S^1 \longrightarrow S^1$ satisfying (4) and such that

$$G_{|L_{p,k}} = G_{p,k} \qquad \text{for } p \in W_\rho \text{ and } k \in \{0,1,\dots,m-2\}.$$

Moreover, $\alpha(G) = \frac{q+jn}{nm}$.

Proof. Some parts of the proof of this theorem are similar to the proof of Theorem 5 from [20]. Here we give only the sketch of these parts. For the details we refer the reader to [20]. Fix $p \in W_\rho$ and orientation-preserving surjections $G_{p,k}\colon L_{p,k} \longrightarrow L_{p,k+1}$ for $k \in \{0,1,\dots,m-2\}$. Put

$$G_{p,m-1} := F \circ G_{p,0}^{-1} \circ G_{p,1}^{-1} \circ \ldots \circ G_{p,m-2}^{-1}\,. \tag{18}$$

For the remaining integers k there exist unique $d \in \mathbb{Z}\setminus\{0\}$ and an $r \in \{0,1,\dots,m-1\}$ such that $k = md + r$. For such k's define

$$G_{p,k} = G_{p,md+r} := F^d \circ G_{p,r} \circ F^{-d}_{|L_{p,k}}. \tag{19}$$

It might be shown that $G_{p,k}(L_{p,k}) = L_{p,k+1}$ for $k \in \mathbb{Z}$ and $G_{p,k}\colon L_{p,k} \longrightarrow L_{p,k+1}$ for $k \in \mathbb{Z}$ are orientation-preserving surjections.

Now fix $z \in S^1 \setminus \operatorname{Per} F$. There exist a $p \in W_\rho$ and an $r \in \{0,\dots,n'-1\}$, where n' is determined by (9), such that $z \in I_{H_\psi^r(p)}$. By (17), $z \in L_{p,dn'+r}$ for some $d \in \mathbb{Z}$. Notice that such a d is unique. Indeed, the assumption

$L_{p,cn'+r} \cap L_{p,dn'+r} \neq \emptyset$ for some $c, d \in \mathbb{Z}$, $c \neq d$, contradicts (13). Define a function $\widetilde{G}: S^1 \setminus \operatorname{Per} F \longrightarrow S^1 \setminus \operatorname{Per} F$ as follows:

$$\widetilde{G}(z) := G_{p,dn'+r}(z), \quad z \in L_{p,dn'+r}\,,\ p \in W_\rho\,,\ d \in \mathbb{Z},\ r \in \{0, \dots, n'-1\}. \tag{20}$$

Notice that for every $u \in \mathcal{M}_F$ there exist unique $p \in W_\rho$ and $r \in \{0, \dots, n'-1\}$ such that $u = H_\psi^r(p)$. Therefore by (20), (17) and the properties of $G_{p,k}$ we have

$$\begin{aligned}\widetilde{G}(I_u) &= \widetilde{G}\big(I_{H_\psi^r(p)}\big) = \widetilde{G}\left(\bigcup_{d\in\mathbb{Z}} L_{p,dn'+r}\right) = \bigcup_{d\in\mathbb{Z}} L_{p,dn'+r+1} = I_{H_\psi^{r+1}(p)} \\ &= I_{H_\psi(u)}\end{aligned}$$

(if $r+1 = n'$ we use the equality $H_\psi^{n'}(p) = p$).

It is easy to see that $\widetilde{G}: S^1 \setminus \operatorname{Per} F \longrightarrow S^1 \setminus \operatorname{Per} F$ is a surjection. By induction it can be proved that $\widetilde{G}$ satisfies

$$\widetilde{G}^m(z) = F(z), \qquad z \in S^1 \setminus \operatorname{Per} F. \tag{21}$$

Moreover, using the same method as in the proof of Theorem 5 in [20] (the proof of 1^o) it can be shown that $\widetilde{G}$ preserves orientation on every I_p for $p \in \mathcal{M}_F$.

We are now in a position to define the solution of (4). Namely, put

$$G(z) = \begin{cases} \widetilde{G}(z), & z \in S^1 \setminus \operatorname{Per} F, \\ h_\psi(z), & z \in \operatorname{Per} F, \end{cases} \tag{22}$$

where h_ψ is defined by (7). It is easy to see that G maps S^1 onto itself. Furthermore, setting $F = h_\psi$ and $\phi = \widetilde{G}$ and repeating the same argument as in the proof of Lemma 1 (i.e., the proof of the fact that $\hat{F}$ preserves orientation) one can obtain that G preserves orientation. Since S^1 is a closed set, it follows that G is an orientation-preserving homeomorphism. Moreover, (7) and (21) imply that G satisfies (4).

It remains to show that $\alpha(G) = \frac{q+jn}{nm}$. From Lemma 1 there exists an orientation-preserving homeomorphism $\hat{G}$ such that $\alpha(\hat{G}) = \alpha(G)$, $\hat{G}(z) = G(z)$ for $z \in \operatorname{Per} F = \operatorname{Per} G$ and $\operatorname{Per} \hat{G} = S^1$. From Theorem 4.2 in [7] it follows that $\hat{G}$ is conjugated to a rotation. On the other hand, by (22), $\hat{G}(z) = h_\psi(z)$ for $z \in \operatorname{Per} F$. By (7) we get that $\hat{G}$ is conjugated to $R(z) = e^{2\pi i \frac{q+jn}{mn}} z$, $z \in S^1$. Hence $\alpha(\hat{G}) = \frac{q+jn}{mn}$ (see Theorem 1 in [8]), and the assertion follows.

Remark 4

Suppose that $F \in \mathcal{F}_{q,n}$ is such that $\operatorname{Per} F \neq S^1$. Then every continuous and orientation-preserving solution G of (4) with $\alpha(G) = \frac{\alpha(F)+jn}{mn}$, where $j \in$

$\{0,\dots,m-1\}$, may be obtained by the method described in the proof of Theorem 2. Indeed, suppose that $G\colon S^1 \longrightarrow S^1$ is a solution of (4) for an integer $m \geq 2$. Then $\alpha(G) = \frac{\alpha(F)+jn}{mn}$ for a $j \in \{0,\dots,m-1\}$. Furthermore, by (4), $\operatorname{Per} F = \operatorname{Per} G$, $\mathcal{A}_F = \mathcal{A}_G$ and, by Corollary 1, $\mathcal{M}_F = \mathcal{M}_G$, $\mathcal{M}_F^+ = \mathcal{M}_G^+$ and $\mathcal{M}_F^- = \mathcal{M}_G^-$. Lemma 2 implies that there exists an orientation-preserving continuous mapping $\psi\colon \operatorname{Per} F \longrightarrow S^1$ satisfying (6). Put $h_\psi := G_{|\operatorname{Per} G}$ and $H_\psi := G_{|\mathcal{M}_G}$. By (6), h_ψ satisfies (7) and $H_\psi = h_{\psi|\mathcal{M}_G}$. Notice that

$$G(I_p) = I_{G(p)} = I_{H_\psi(p)}\,, \qquad p \in \mathcal{M}_G\,. \tag{23}$$

Let ρ be the relation on $\mathcal{M}_G = \mathcal{M}_F$ given by (8) and let W_ρ be its selector. Fix $p \in W_\rho$, $z_{p,0} \in I_p$ and put

$$z_{p,k} := G^k(z_{p,0}), \qquad k \in \mathbb{Z} \setminus \{0\}. \tag{24}$$

Obviously, $(z_{p,k})_{k\in\mathbb{Z}}$ satisfies (11). Moreover, (23) and the fact that $H^{n'} = \mathrm{id}_{\mathcal{M}_F}$, where n' is given in (9), yield

$$\begin{aligned} z_{p,dn'+r} = G^{dn'+r}(z_{p,0}) \in I_{H_\psi^{dn'+r}(p)} = I_{H_\psi^r(p)},\\ d\in\mathbb{Z},\ r \in \{0,\dots,n'-1\}. \end{aligned} \tag{25}$$

By Definition 1, since n' is the minimal number such that $G^{n'}(z) = z$ for $z \in \operatorname{Per} G$ and $\mathcal{M}_F^+ = \mathcal{M}_G^+$, we have $\overrightarrow{\langle z_{p,0}, z_{p,n'}\rangle} \subset I_p$, if $p \in \mathcal{M}_G^+$ and $\overrightarrow{\langle z_{p,n'}, z_{p,0}\rangle} \subset I_p$, if $p \in \mathcal{M}_G^-$. Hence in view of (24), (25) and the fact that G preserves orientation we get

$$\overrightarrow{\langle z_{p,(d+1)n'+r}, z_{p,dn'+r}\rangle} \subset I_{H_\psi^r(p)}\,, \qquad (\text{resp. } \overrightarrow{\langle z_{p,dn'+r}, z_{p,(d+1)n'+r}\rangle} \subset I_{H_\psi^r(p)})$$

for $d \in \mathbb{Z}$, $r \in \{0,\dots,n'-1\}$ and $p \in \mathcal{M}_G^+$ (resp. $p \in \mathcal{M}_G^-$). Consequently,

$$H_\psi^r(p) \prec z_{p,r} \prec z_{p,n'+r} \prec \dots \prec z_{p,(m'-1)n'+r} \prec G^{m'n'}(z_{p,r}) = F^n(z_{p,r})$$

$$(\text{resp. } H_\psi^r(p) \prec F^n(z_{p,r}) = G^{m'n'}(z_{p,r}) \prec z_{p,(m'-1)n'+r} \prec \dots \prec z_{p,n'+r} \prec z_{p,r}).$$

Let $\{L_{p,k}\}_{k\in\mathbb{Z}}$ be defined by (16). Notice that

$$G(L_{p,k}) = L_{p,k+1}\,, \qquad k \in \mathbb{Z}. \tag{26}$$

Now put

$$G_{p,k} := G_{|L_{p,k}}\,, \qquad p \in W_\rho\,,\ k \in \mathbb{Z}. \tag{27}$$

From (4), (26) and (27) we have

$$F_{|L_{p,0}} = G_{p,m-1} \circ G_{p,m-2} \circ \dots \circ G_{p,1} \circ G_{p,0}\,, \qquad p \in W_\rho\,,$$

thus (18) holds. Furthermore, (4) implies $G \circ F = F \circ G$. Thus $G \circ F^k = F^k \circ G$ for any $k \in \mathbb{Z}$. From this, (26) and (27) we get (19).

Theorem 2 and Remark 4 solve the problem of the existence of iterative roots of homeomorphisms having the set of periodic points different from the whole circle. Notice that if $F \in \mathcal{F}_{q,n}$ is such that $\operatorname{Per} F = S^1$, then taking $G := h_\psi$, where h_ψ is defined by (7), we get $G^m = F$. To sum up, we have obtained the following result.

Theorem 3

Let $m \geq 2$ be an integer and let $F \in \mathcal{F}_{q,n}$. Equation (4) *has orientation-preserving and continuous solution if and only if an orientation-preserving continuous solution $\psi\colon \operatorname{Per} F \longrightarrow S^1$ of* (2) *satisfies* (5) *for $X \in \{\operatorname{Per} F, \mathcal{M}_F^+\}$ and for a $j \in \{0, \dots, m-1\}$. Moreover, if $\operatorname{Per} F \neq S^1$, then for all ψ and j satisfying* (5) *for $X \in \{\operatorname{Per} F, M_F^+\}$ there exist infinitely many solutions of* (4)*.*

The following remark results from the above theorem. It answers the question of the existence of the iterative roots of the mapping $F_{|\operatorname{Per} F}$, where $F\colon S^1 \longrightarrow S^1$ is an orientation-preserving homeomorphism having periodic points.

Remark 5

Let $m \geq 2$ be an integer and let $F \in \mathcal{F}_{q,n}$. The mapping $F_{|\operatorname{Per} F}\colon \operatorname{Per} F \longrightarrow \operatorname{Per} F$ has continuous and orientation-preserving iterative roots of order m if and only if some orientation-preserving continuous solution $\psi\colon \operatorname{Per} F \longrightarrow S^1$ of (2) satisfies

$$e^{2\pi \mathrm{i} \frac{\alpha(F)+j}{m}} \psi(\operatorname{Per} F) = \psi(\operatorname{Per} F)$$

for some $j \in \{0, \dots, m-1\}$.

We conclude with an observation concerning homeomorphisms with a finite and non-empty set of periodic points.

Theorem 4

Suppose that $F \in \mathcal{F}_{q,n}$ is such that $1 < \operatorname{card}(\operatorname{Per} F) =: N_F < \infty$ and $m \geq 2$ is an integer. Let moreover ψ_1 and ψ_2 be orientation-preserving continuous solutions of (2) *satisfying* (5) *for $X \in \{\operatorname{Per} F, \mathcal{M}_F^+\}$ and a $j \in \{0, \dots, m-1\}$ and let $h_{\psi_1}, h_{\psi_2}\colon \operatorname{Per} F \longrightarrow \operatorname{Per} F$ be defined by* (7)*. Then $h_{\psi_1}(z) = h_{\psi_2}(z)$ for $z \in \operatorname{Per} F$.*

In the proof of Theorem 4 we will use the following proposition, which is a slightly modified Theorem 3 from [21] (see also Theorem 2 in [20]).

Proposition 2

Suppose that $F\colon S^1 \longrightarrow S^1$ is an orientation-preserving homeomorphism such that $1 < \operatorname{card}(\operatorname{Per} F) =: N_F < \infty$. Let $z_0 \in \operatorname{Per} F$ be an arbitrary element and let $z_1, \dots, z_{N_F - 1} \in \operatorname{Per} F$ satisfy the following condition:

$$\operatorname{Arg} \frac{z_p}{z_0} < \operatorname{Arg} \frac{z_{p+1}}{z_0}, \qquad p \in \{0, \dots, N_F - 2\}.$$

Then $\alpha(F) = \frac{q}{n}$, *where* $0 \le q < n$ *and* $\gcd(q,n) = 1$, *if and only if*

$$F(z_p) = z_{(p+k_F q) \pmod{N_F}}, \qquad p \in \{0, \ldots, N_F - 1\},$$

where $k_F := \frac{N_F}{n}$.

Proof of Theorem 4. In view of Theorem 2 there exist orientation-preserving homeomorphisms G_1 and G_2 such that $\operatorname{Per} G_i = \operatorname{Per} F$, $G_i^m = F$ and $\alpha(G_i) = \frac{q+jn}{mn} = \frac{q'}{n'}$ for $i \in \{1,2\}$, where $q' := \frac{q+jn}{m'}$ and m', n' are given in (9). Moreover, $G_i(z) = h_{\psi_i}(z)$ for $z \in \operatorname{Per} F$ and $i \in \{1,2\}$. Let $z_0, \ldots, z_{N_F-1} \in \operatorname{Per} F$ be defined as in Proposition 2 and let $K := \frac{N_F}{n'} = k_{G_1} = k_{G_2}$. By Proposition 2 we have

$$\begin{aligned} h_{\psi_1}(z_p) = G_1(z_p) &= z_{(p+Kq') \pmod{N_F}} = G_2(z_p) \\ &= h_{\psi_2}(z_p) \end{aligned}$$

for every $p \in \{0, \ldots, N_F - 1\}$. Thus the assertion follows.

The property described in Theorem 4 does not have to occur for homeomorphisms with infinitely many periodic points. For example, let $F(z) = e^{\pi \mathrm{i}} z$ for $z \in S^1$ and let $m = 2$. Then $F \in \mathcal{F}_{1,2}$, $\mathcal{M}_F^+ = \emptyset$ and $\operatorname{Per} F = S^1$. Put $\psi_1(z) = z$ for $z \in S^1$ and $\psi_2(e^{2\pi \mathrm{i} x}) = e^{2\pi \mathrm{i} d(x)}$ for $x \in \langle 0, 1)$, where

$$d(x) = \begin{cases} -2x^2 + 2x, & x \in \langle 0, \frac{1}{2}), \\ -2\left(x - \frac{1}{2}\right)^2 + 2\left(x - \frac{1}{2}\right) + \frac{1}{2}, & x \in \langle \frac{1}{2}, 1). \end{cases}$$

Notice that ψ_1 and ψ_2 satisfy (2) and (5) for $X \in \{\operatorname{Per} F, \mathcal{M}_F^+\}$ and $j = 0$, but $h_{\psi_1} \neq h_{\psi_2}$.

References

[1] Ch. Babbage, *Essay towards the calculus of functions,* Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. (1815), 389-423; II, ibid. (1816), 179-256.

[2] M. Bajger, *On the structure of some flows on the unit circle*, Aequationes Math. **55** (1998), 106-121.

[3] K. Ciepliński, *General construction of non-dense disjoint iteration groups on the circle*, Czechoslovak Math. J. **55** (2005), 1079-1088.

[4] K. Ciepliński, *On the embeddability of a homeomorphism of the unit circle in disjoint iteration groups*, Publ. Math. Debrecen **55** (1999), no. 3-4, 363-383.

[5] K. Ciepliński, *On conjugacy of disjoint iteration groups on the unit circle*, European Conference on Iteration Theory (Muszyna Złockie,1998), Ann. Math. Sil. **13** (1999), 103-118.

[6] K. Ciepliński, *The rotation number of the composition of homeomorphisms*, Rocznik Nauk.-Dydakt. Prace Mat. **17** (2000), 83-87.

[7] K. Ciepliński, M.C. Zdun, *On a system of Schröder equations on the circle*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **13** (2003), 1883-1888.

[8] K. Ciepliński, M.C. Zdun, *On semi-conjugacy equation for homeomorphisms of the circle*, in: Functional Equations – Results and Advances, Adv. Math. (Dordr.) **3**, Kluwer Academic Publishers, Dordrecht 2002, 135-158.

[9] I.P. Cornfeld, S.V. Fomin, Y.G. Sinai, *Ergodic theory*, Grundlehren Math. Wiss. **245**, Spirnger-Verlag, Berlin, Heidelberg, New York, 1982.

[10] R. Isaacs, *Iterates of fractional order,* Canad. J. Math. **2** (1950), 409-416.

[11] W. Jarczyk, *Babagge equation on the circle,* Publ. Math. Debrecen **63** (2003), no. 3, 389-400.

[12] M. Kuczma, *On the functional equation* $\varphi^n(x) = g(x)$, Ann. Polon. Math. **11** (1961), 161-175.

[13] M. Kuczma, *Functional Equations in a single variable*, Monografie Mat. **46** Polish Scientific Publishers, Warszawa, 1968.

[14] M. Kuczma, B. Choczewski, R. Ger, *Iterative functional equations*, Encyclopaedia Math. Appl. **32**, Cambridge Univ. Press, Cambridge, 1990.

[15] S. Łojasiewicz, *Solution générale de l'équation fonctionnelle* $f(f \ldots f(x) \ldots) = g(x)$*,* Ann. Soc. Polon. Math. **24** (1951), 88-91.

[16] J.H. Mai, *Conditions for the existence of* N*th iterative roots of self-homeomorphisms of the circle*, Acta Math. Sinica (Chin. Ser.) **30** (1987), no. 2, 280-283.

[17] W. de Melo, S. van Strein, *One-dimensional dynamics*, Ergeb. Math. Grenzgeb. (3), [Results in Mathematics and Related Areas (3)] **25**, Springer-Verlag, Berlin, 1993.

[18] P. Solarz, *On iterative roots of a homeomorphism of the circle with an irrational rotation number*, Math. Pannon. **13** (2002), no. 1, 137-145.

[19] P. Solarz, *On some iterative roots*, Publ. Math. Debrecen **63** (2003), no. 4, 677-692.

[20] P. Solarz, *Iterative roots of some homeomorphism with a rational rotation number*, Aequationes Math. **72** (2006), no. 1-2, 152-171.

[21] P. Solarz, *On some propreties of orientation-preserving surjections of the circle*, (to appear).

[22] P. Walters, *An Introduction to Ergodic Theory*, Grad. Texts in Math. **79**, Springer-Verlag, New York, Berlin, 1982.

[23] M.C. Zdun, *On embedding of homeomorphisms of the circle in continuous flow*, Iteration theory and its functional equations, (Lochau 1984), Lecture Notes in Math. **1163**, Springer-Verlag, Berlin, New York, 1985, 218-231.

[24] M.C. Zdun, *On iterative roots of homeomorphisms of the circle*, Bull. Pol. Acad. Sci. Math. **48** (2000), no. 2, 203-213.

[25] G. Zimmermann, *Über die Existenz iterativer Wurzeln von Abbildungen*, Doctoral dissertation, University of Marburg 1978.

Institute of Mathematics
Pedagogical University
ul. Podchorążych 2
PL-30-084 Kraków
Poland
E-mail: psolarz@ap.krakow.pl

Received: 18 January 2007; final version: 11 April 2007;
available online: 6 June 2007.

Annales Academiae Paedagogicae Cracoviensis

Folia 45 Studia Mathematica VI

Jurij Povstenko

Fractional calculus and diffusive stress

Abstract. The main ideas of fractional calculus are recalled. A quasi-static uncoupled theory of diffusive stresses based on the anomalous diffusion equation with fractional derivatives is formulated.

It is well known that integrating by parts $n-1$ times the calculation of the n-fold primitive of a function $f(t)$ can be reduced to the calculation of a single integral of the convolution type

$$I^n f(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau)\,\mathrm{d}\tau, \tag{1}$$

where n is a positive integer.

The Laplace transform rule for an integral (1) can be found in every textbook on this subject

$$\mathcal{L}\{I^n f(t)\} = \frac{1}{s^n}\mathcal{L}\{f(t)\},$$

where s is the transform variable.

The Riemann–Liouville fractional integral is introduced as a natural generalization of the convolution type form (1):

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \qquad \alpha > 0,$$

where $\Gamma(\alpha)$ is the gamma function. The Laplace transform rule for the fractional integral reads

$$\mathcal{L}\{I^\alpha f(t)\} = \frac{1}{s^\alpha}\mathcal{L}\{f(t)\}.$$

The Riemann–Liouville derivative of the fractional order α is defined as left-inverse to the fractional integral I^α [15, 6]

AMS (2000) Subject Classification: 26A33.

$$D_{RL}^{\alpha} f(t) = D^n I^{n-\alpha} f(t)$$
$$= \begin{cases} \dfrac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau\right], & n-1 < \alpha < n, \\ \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} f(t), & \alpha = n \end{cases}$$

and for its Laplace transform it requires the knowledge of the initial values of the fractional integral $I^{n-\alpha} f(t)$ and its derivatives of the order $k = 1, 2, \ldots, n-1$:

$$\mathcal{L}\{D_{RL}^{\alpha} f(t)\} = s^{\alpha} \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} D^k I^{n-\alpha} f(0^+) s^{n-1-k}, \qquad n-1 < \alpha < n.$$

An alternative definition of the fractional derivative was proposed by Caputo [2, 3]:

$$D_C^{\alpha} f(t) = I^{n-\alpha} D^n f(t) =$$
$$= \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \dfrac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}\,\mathrm{d}\tau, & n-1 < \alpha < n, \\ \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} f(t), & \alpha = n. \end{cases}$$

For its Laplace transform rule the Caputo fractional derivative requires the knowledge of the initial values of the function $f(t)$ and its integer derivatives of order $k = 1, 2, \ldots, n-1$:

$$\mathcal{L}\{D_C^{\alpha} f(t)\} = s^{\alpha} \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} D^k f(0^+) s^{\alpha-1-k}, \qquad n-1 < \alpha < n.$$

The Caputo fractional derivative is a regularization in the time origin for the Riemann–Liouville fractional derivative by incorporating the relevant initial conditions [5]. In this paper we shall use the Caputo fractional derivative omitting the index C. The major utility of this type fractional derivative is caused by the treatment of differential equations of fractional order for physical applications, where the initial conditions are usually expressed in terms of a given function and its derivatives of integer (not fractional) order, even if the govering equation is of fractional order [11].

The definitions of space-fractional differential operators can be found in [15, 6, 1]. The cumbersome aspects of these operators disappear when one computes their Fourier transforms.

The space-fractional derivative of order β is defined as a pseudo-differential operator with the following rule for the Fourier transform [8]:

$$\mathcal{F}\left\{\frac{\mathrm{d}^\beta f(x)}{\mathrm{d}|x|^\beta}\right\} = -|\xi|^\beta \mathcal{F}\left\{f(x)\right\},$$

where ξ is the transform variable. For the Fourier transform of fractional Laplacian one has [15]

$$\mathcal{F}\big\{(-\Delta)^{\frac{\beta}{2}} f(\mathbf{x})\big\} = |\xi|^\beta \mathcal{F}\left\{f(\mathbf{x})\right\}.$$

If the function $f(\mathbf{x})$ depends only on the radial coordinate $r = |\mathbf{x}|$, then in the two-dimensional case we can obtain the corresponding formula for the Hankel transform

$$\mathcal{H}\big\{(-\Delta)^{\frac{\beta}{2}} f(r)\big\} = |\xi|^\beta \mathcal{H}\left\{f(r)\right\}.$$

A quasi-static uncoupled theory of diffusive stress is governed by the equilibrium equation in terms of displacements [12]

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\mathrm{grad}\,\mathrm{div}\,\mathbf{u} = \beta_c K \,\mathrm{grad}\, c,$$

the stress-strain-concentration relation

$$\boldsymbol{\sigma} = 2\mu\mathbf{e} + (\lambda\,\mathrm{tr}\,\mathbf{e} - \beta_c K c)\mathbf{I},$$

and the time-fractional diffusion equation

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a\,\Delta\, c, \qquad 0 \le \alpha \le 2, \tag{2}$$

where $\mathbf{u}$ is the displacement vector, $\boldsymbol{\sigma}$ the stress tensor, $\mathbf{e}$ the linear strain tensor, c the concentration, a the diffusivity coefficient, λ and μ are Lamé constants, $K = \lambda + \frac{2}{3}\mu$, β_c is the diffusion coefficient of volumetric expansion, $\mathbf{I}$ denotes the unit tensor.

If a bounded solid is considered, the corresponding boundary conditions should be given; for unbounded medium

$$\lim_{|\mathbf{x}|\to\infty} \mathbf{u}(\mathbf{x},t) = 0,$$
$$\lim_{|\mathbf{x}|\to\infty} c(\mathbf{x},t) = 0.$$

Equation (2) should also be subject to initial conditions

$$t=0: \qquad c = P(\mathbf{x}), \qquad 0 < \alpha \le 2,$$
$$t=0: \qquad \frac{\partial c}{\partial t} = W(\mathbf{x}), \qquad 1 < \alpha \le 2.$$

The most suitable method to solve the obtained system of equations is the method of integral transforms. For classical domains the exponential, sine and cosine Fourier transforms and the Hankel transform with respect to spatial coordinates can be used. The Laplace transform with respect to time is also extensively employed.

In a two-dimensional medium in an axially symmetric case equation (2) has the form

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a\left(\frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r}\right).$$

We choose the initial conditions

$$t = 0: \qquad c = p\,\frac{\delta(r)}{2\pi r}, \qquad 0 < \alpha \le 2,$$

$$t = 0: \qquad \frac{\partial c}{\partial t} = 0, \qquad 1 < \alpha \le 2,$$

where $\delta(r)$ is the Dirac delta function.

The nonzero components of the stress tensor expressed in terms of displacement potential read [9]

$$\sigma_{rr} + \sigma_{\theta\theta} = -2\mu\Delta\Phi,$$

$$\sigma_{rr} - \sigma_{\theta\theta} = 2\mu\left(\frac{\partial^2\Phi}{\partial r^2} - \frac{1}{r}\frac{\partial\Phi}{\partial r}\right).$$

The displacement potential is determined from the equation

$$\Delta\,\Phi = mc, \qquad m = \frac{1+\nu}{1-\nu}\frac{\beta_c}{3},$$

where ν is the Poisson ratio.

We present results corresponding to $\alpha = \frac{1}{2}$ and obtained using the Laplace transform with respect to time t and the Hankel transform with respect to polar coordinate r:

$$c = \frac{p}{4\pi^{\frac{3}{2}} a\sqrt{t}} \int_0^\infty \exp\left(-u^2 - \frac{\rho^2}{8u}\right)\frac{\mathrm{d}u}{u},$$

$$\sigma_{rr} = -2\mu m \frac{p}{\pi^{\frac{3}{2}} a\sqrt{t}\rho^2} \int_0^\infty \mathrm{e}^{-u^2}\left[1 - \exp\left(-\frac{\rho^2}{8u}\right)\right]\mathrm{d}u,$$

$$\sigma_{\theta\theta} = -2\mu m \frac{p}{\pi^{\frac{3}{2}} a\sqrt{t}} \int_0^\infty \mathrm{e}^{-u^2}\left[\left(\frac{1}{4u} + \frac{1}{\rho^2}\right)\exp\left(-\frac{\rho^2}{8u}\right) - \frac{1}{\rho^2}\right]\mathrm{d}u,$$

where the similarity variable

$$\rho = \frac{r}{\sqrt{a}\,t^{\frac{\alpha}{2}}}$$

has been chosen.

It should be noted that in two-dimensional case the fundamental solution to a Cauchy problem for equation (2) in the case $\alpha = \frac{1}{2}$ has the logarithmical singularity at the origin (in contrast to nonsingular solution of classical diffusion equation in the case $\alpha = 1$).

Additional insights into applications of fractional calculus in continuum mechanics and physics as well as extensive literature on the subject can be found in [10, 13, 4, 7, 14].

References

[1] P.L. Butzer, U. Westphal, *An introduction to fractional calculus*, in: R. Hilfer (Ed.), Applications of Fractional Calculus in Physics, World Sci. Publ., River Edge, 2000, 1-85.

[2] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent, Part II*, Geophysical Journal of the Royal Astronomical Society, **13**, 1967, 529-539.

[3] M. Caputo, *Elasticità e Dissipazione*, Zanichelli, Bologna, 1969. (In Italian).

[4] R. Gorenflo, F. Mainardi, *Fractional calculus: integral and differential equations of fractional order*, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics (Udine, 1996), CISM Courses and Lectures **378**, Springer, Vienna, 1997, 223-276,

[5] R. Gorenflo, F. Mainardi, *Fractional calculus and stable probability distributions*, Fourth Meeting on Current Ideas in Mechanics and Related Fields (Kraków, 1997), Arch. Mech. (Arch. Mech. Stos.) **50** (1998), no. 3, 377-388.

[6] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, **204**, Elsevier Science B.V., Amsterdam, 2006.

[7] F. Mainardi, *Fractional calculus: some basic problems in continuum and statistical mechanics*, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics (Udine, 1996), CISM Courses and Lectures **378**, Springer, Vienna, 1997, 291-348.

[8] F. Mainardi, Y. Luchko, G. Pagnini, *The fundamental solution of the space-time fractional diffusion equation*, Fract. Calc. Appl. Anal. **4** (2001), no. 2, 153-192.

[9] W. Nowacki, *Thermoelasticity*, Translated from the Polish by Henryk Zorski, Second edition, Pergamon Press, Oxford and PWN—Polish Scientific Publishers, Warsaw, 1986.

[10] K.B. Oldham, J. Spanier, *The Fractional Calculus*, Mathematics in Science and Engineering, Vol. **111**, Academic Press, New York-London, 1974.

[11] I. Podlubny, *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and some of Their Applications*, Mathematics in Science and Engineering **198**, Academic Press, Inc., San Diego, 1999.

[12] Y.Z. Povstenko, *Stresses exerted by a source of diffusion in a case of a non-parabolic diffusion equation*, Internat. J. Engrg. Sci. **43** (2005), no. 11-12, 977-991.

[13] Y.Z. Povstenko, *Fractional heat conduction equation and associated thermal stress*, J. Thermal Stresses **28** (2005), no. 1, 83-102.

[14] Y.A. Rossikhin, M.V. Shitikova, *Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids*, Applied Mechanics Reviews, **50** (1997), 15-67.

[15] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach Science Publishers, Yverdon, 1993.

Institute of Mathematics
and Computer Science
Jan Długosz University
al. Armii Krajowej 13/15
PL-42-200 Częstochowa
Poland
E-mail: j.povstenko@ajd.czest.pl

Received: 20 November 2006; final version: 9 May 2007;
available online: 6 June 2007.

Annales Academiae Paedagogicae Cracoviensis

Folia 45 Studia Mathematica VI

Wioletta Syzdek

Submaximal Riemann-Roch expected curves and symplectic packing

Abstract. We study Riemann–Roch expected curves on $\mathbb{P}^1 \times \mathbb{P}^1$ in the context of the Nagata–Biran conjecture. This conjecture predicts that for a sufficiently large number of points multiple points Seshadri constants of an ample line bundle on algebraic surface are maximal. Biran gives an effective lower bound N_0 . We construct examples verifying to the effect that the assertions of the Nagata–Biran conjecture can not hold for small number of points. We discuss cases where our construction fails. We observe also that there exists a strong relation between Riemann–Roch expected curves on $\mathbb{P}^1 \times \mathbb{P}^1$ and the symplectic packing problem. Biran relates the packing problem to the existence of solutions of certain Diophantine equations. We construct such solutions for any ample line bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ and a relatively small number of points. The solutions geometrically correspond to Riemann–Roch expected curves. Finally we discuss in how far the Biran number N_0 is optimal in the case of $\mathbb{P}^1 \times \mathbb{P}^1$. In fact, we conjecture that it can be replaced by a lower number and we provide an evidence justifying this conjecture.

Introduction

The aim of this paper is to prove that for the surface $X = \mathbb{P}^1 \times \mathbb{P}^1$ with the (a, b) polarization there exists a constant $R_0 = R_0(a, b)$ such that for $r \geq R_0$ there are no Riemann–Roch expected submaximal curves through r general points (Theorem 3.5).

This fact has consequences for the symplectic packing problem which is strongly connected to the existence of Riemann–Roch expected submaximal curves. More precisely, Biran relates the packing problem to the existence of solutions of certain Diophantine equations ([Bi1] Theorem 6.1.A 2) but the solutions geometrically correspond exactly to Riemann–Roch expected submaximal curves. In particular, Theorem 3.5 implies that for $N \geq R_0$ the surface $\mathbb{P}^1 \times \mathbb{P}^1$ with the polarization (a, b) admits full symplectic packing by N equal

AMS (2000) Subject Classification: 14C20, 14J26.

balls (Theorem 3.18). This improves the result of Biran [[Bi2] Theorem 1.A] as in almost all cases our number R_0 is smaller than Biran's bound N_0. We conjecture that the Biran number N_0 in the Nagata–Biran Conjecture 1.2 can be replaced by R_0.

On the other hand, to complete Theorem 3.5, we are looking for Riemann–Roch expected submaximal curves for $r \leq R_0$ points in general position. We observe that for $r \leq 2 \cdot \lfloor \frac{a}{b} \rfloor + 5$ points we can write down such curves (Proposition 3.7). The cases $r \geq 2 \cdot \lfloor \frac{a}{b} \rfloor + 6$ are more complicated. Only for some polarizations (a, b) we can find Riemann–Roch expected submaximal curves. More precisely, to find them, we construct first a sequence of Riemann–Roch expected curves (Proposition 3.9) and next we compute their submaximality areas, i.e., we estimate polarizations for which curves are submaximal (Lemma 3.10). For $r = 2 \cdot \lfloor \frac{a}{b} \rfloor + 6$ in Proposition 3.11 we give an algorithm producing Riemann-Roch expected submaximal curves. For the number of points r in the range between $2 \cdot \lfloor \frac{a}{b} \rfloor + 6$ and R_0, the situation seems to be hard to control (see Examples 3.13 and 3.14). For such r's for which we found Riemann–Roch expected submaximal curves, we compute Seshadri quotients. If we know that a curve is irreducible, then this quotient is already the Seshadri constant, if not (i.e., a curve can be reducible) then we found only an upper-bound for this constant (Theorem 3.17).

Notation: The symbol $\mathbb{Z}_{\geq 0}$ denotes the set of non-negative integers. For a given real number x we denote by $\lfloor x \rfloor$ its round-down. We work throughout over the field $\mathbb{C}$ of complex numbers. By a polarized variety we mean a pair (X, L) consisting a smooth variety X and an ample line bundle L on it. For a coherent sheaf $\mathcal{F}$ on X we write by $H^i(\mathcal{F}) = H^i(X, \mathcal{F})$ and $h^i(\mathcal{F}) = \dim_{\mathbb{C}} H^i(\mathcal{F})$.

1. Seshadri constants and the Nagata--Biran conjecture

The concept of Seshadri constant was introduced by Demailly in [De]. He associated a real number $\varepsilon(L; x)$ to an ample line bundle L and a point x of an algebraic variety. This number in effect measures how much of positivity of L is concetrated in x. In general, Seshadri constants are very hard to control and their exact value is known only in few examples. In this paper we use a generalized definition of Seshadri constant (see also [Xu]) on a surface X.

Definition 1.1
Let L be a nef line bundle on a smooth projective surface X. The Seshadri constant of L at $x_1, \ldots, x_r \in X$ is the real number

$$\varepsilon(L; x_1, \ldots, x_r) = \inf_{D \cap \{x_1, \ldots, x_r\} \neq \emptyset} \frac{L.D}{\sum_{i=1}^{r} \operatorname{mult}_{x_i} D},$$

where the infimum is taken over all reduced and irreducible curves D passing through at least one of the points $x_1, \ldots, x_r$ and $\operatorname{mult} x_i\, D$ is the multiplicity of the curve D at x_i.

It follows from Kleiman's nefness criterion that $\varepsilon(L; x_1, \ldots, x_r) \leq \sqrt{\frac{L^2}{r}}$. If the value of $\varepsilon(L; x_1, \ldots, x_r)$ is less than the upper bound, then we say that the Seshadri constant of L at $x_1, \ldots, x_r$ is *submaximal*. If $\varepsilon(L; x_1, \ldots, x_r) = \frac{L.D}{\sum_{i=1}^{r} \operatorname{mult}_{x_i} D}$, then we say that the curve D *computes* the Seshadri constant and we call such a curve a *Seshadri curve*. By the *Seshadri quotient* of a curve G at $x_1, \ldots, x_r$ we mean $\frac{L.G}{\sum_{i=1}^{r} \operatorname{mult}_{x_i} G}$.

As a function on X^r the Seshadri constant $\varepsilon(L; \cdot, \ldots, \cdot)$ is semi-continuous and it attains the greatest value at a very general point of X^r (i.e., on the complement of a union of at most countably many Zariski closed subsets). For more details see [Og]. We denote by $\varepsilon(L; r)$ this greatest value. It is conjectured that for r sufficiently large $\varepsilon(L; r)$ has the maximal possible value which is $\varepsilon_{max}(L; r) = \sqrt{\frac{L^2}{r}}$. In fact there are effective predictions of what the least number r should be.

NAGATA-BIRAN CONJECTURE 1.2
Let (X, L) be a polarized surface. Let k_0 be the smallest integer such that in the linear system $|k_0 L|$ there exists a smooth non-rational curve and let $N_0 := k_0^2 L^2$. With the above assumptions

$$\varepsilon(L; x_1, \ldots, x_r) = \sqrt{\frac{L^2}{r}}$$

for general $x_1, \ldots, x_r \in X$ and $r \geq N_0$.

We should note that in the case when the Seshadri constant $\varepsilon(L; x_1, \ldots, x_r)$ is submaximal, the number of Seshadri curves is bounded.

PROPOSITION 1.3
Let (X, L) be a polarized surface with the Picard number ϱ and let $x_1, \ldots, x_r$ be points in X such that $\varepsilon = \varepsilon(L; x_1, \ldots, x_r)$ is submaximal. There are at most $\varrho + r - 1$ irreducible and reduced Seshadri curves.

Proof. Let $\pi\colon Y \longrightarrow X$ be the blowing up of X at $x_1, \ldots, x_r$ with exceptional divisors $E_1, \ldots, E_r$ and let $H := \pi^* L$. Suppose that $C_1, \ldots, C_s$ are irreducible and reduced curves computing ε and $\widetilde{C_1}, \ldots, \widetilde{C_s}$ are their proper transforms. The $\mathbb{Q}$-divisor $M := H - \varepsilon \sum_{i=1}^{r} E_i$ is nef and big and for arbitrary $\lambda_i \geq 0$ we have

$$
\begin{aligned}
M.\Big(\sum_{i=1}^{s}\lambda_i\widetilde{C_i}\Big)
&= \sum_{i=1}^{s}\lambda_i\cdot\big(M.\widetilde{C_i}\big) = \sum_{i=1}^{s}\lambda_i\Big(\pi^*L.\widetilde{C_i} - \varepsilon\sum_{j=1}^{r}E_j.\widetilde{C_i}\Big)\\
&= \sum_{i=1}^{s}\lambda_i\Big(\pi^*L.\pi^*C_i - \sum_{k=1}^{r}\operatorname{mult}_{x_k}C_i\cdot(\pi^*L.E_k) - \varepsilon\sum_{j=1}^{r}\operatorname{mult}_{x_j}C_i\Big)\\
&= \sum_{i=1}^{s}\lambda_i\,(L.C_i - 0 - L.C_i) = 0.
\end{aligned}
$$

The Hodge Index Theorem implies that the intersection matrix of $\widetilde{C_1},\dots,\widetilde{C_s}$ is negative definite. Since $\varrho(Y)=\varrho+r$, it must be $s\le\varrho+r-1$.

We also observe that this upper bound is optimal.

Example 1.4
Let $X=\mathbb{P}^2$ and the number of points be $r=7$. In this case $\varrho=1$ and from the previous proposition we have that the number of irreducible and reduced Seshadri curves is at most 7. On the other hand for any $i\in\{1,\dots,7\}$ there exists an irreducible cubic D_i with $\operatorname{mult}_{x_i}D_i=2$ and $\operatorname{mult}_{x_j}D_i=1$ for $j\ne i$. Every D_i computes the Seshadri constant $\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1);x_1,\dots,x_7)=\frac{3}{8}<\sqrt{\frac{1}{7}}$. So there are exactly 7 submaximal curves in this case.

Let D be a curve on a surface X passing through $x_1,\dots,x_r$ with multiplicities $m_1:=\operatorname{mult}_{x_1}D,\dots,m_r:=\operatorname{mult}_{x_r}D$ respectively. To the curve D we assign its *multiplicity vector* $M_D:=(m_1,\dots,m_r)\in\mathbb{Z}^r$.

Definition 1.5
A curve D is *almost-homogeneous* if all but at most one of the coordinates of its multiplicity vector M_D are equal. In this case we can also say that the multiplicity vector is almost-homogeneous.

Now using the same arguments as in [Sz] Corollary 4.6, after some elementary calculations we can prove the following

Proposition 1.6
Let (X,L) be a polarized surface with the Picard number ϱ and let $x_1,\dots,x_r$ be general points on X. If ϱ is equal one or two and the Seshadri constant $\varepsilon(L;x_1,\dots,x_r)$ is submaximal, then any irreducible and reduced Seshadri curve is almost-homogeneous.

Proof. A submaximal Seshadri constant $\varepsilon(L; x_1, \dots, x_r)$ implies by the real valued Nakai–Moishezon criterion [CP] that there exists a computing curve.

Let D be an irreducible and reduced Seshadri curve with the multiplicity vector $M_D = (m_1, \dots, m_r)$. Since the points are general, the monodromy group acts as the full symmetric group S_r , i.e., for $\sigma \in S_r$ there exists a curve D_σ with the multiplicity vector $M_{D_\sigma} = (m_{\sigma(1)}, \dots, m_{\sigma(r)})$ which is also an irreducible Seshadri curve.

If the curve were not be almost-homogeneous, then it is easy to check that we would get too many Seshadri curves contradicting Proposition 1.3.

2. Seshadri submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$

By the (a, b) *polarization* or by a *curve of type* (a, b) in the product $\mathbb{P}^1 \times \mathbb{P}^1$ we mean a curve of bidegree a, b.

Definition 2.7
Let $D \subset X$ be a curve passing through points $x_1, \dots, x_r$ with multiplicities $m_1, \dots, m_r$, respectively. We say that D is Riemann–Roch expected (for short R-R expected) if

$$h^0(\mathcal{O}_X(D)) - \sum_{i=1}^{r} \binom{m_i + 1}{2} > 0.$$

This simply means that a curve D is R-R expected if its existence follows from the naive dimension count (note that it takes at most $\binom{m+1}{2}$ independent linear conditions on a linear system to pass through a given point with multiplicity at least m).

Remark 2.8

(1) On $(\mathbb{P}^2, \mathcal{O}(1))$ we have $N_0 = 9$ and curves computing Seshadri constants for $r \le N_0$ points are R-R expected.

(2) On $\mathbb{P}^1 \times \mathbb{P}^1$ with the $(1, 1)$ polarization, we have $N_0 = 8$ and again all curves computing Seshadri constant for at most 8 points are R-R expected.

This implies that in these two examples the number N_0 suggested by Biran cannot be lowered. However, there are cases (e.g. $(1, 2)$ polarization, see [S1]) suggesting that the Biran number N_0 might not be optimal even in the simple case of $\mathbb{P}^1 \times \mathbb{P}^1$. We address this question in this article. Before proceeding, we need some more notation. For a vector $M = (m_1, \dots, m_r) \in \mathbb{Z}^r_{\ge 0}$ with non-negative entries we define

$$|M| := \sum_{i=1}^{r} m_i,$$
$$\alpha(M) := \max\{|m_i - m_j| : \ i,j = 1,\dots,r\},$$
$$\mathbf{l}(M) := \sum_{i=1}^{r} \binom{m_i+1}{2}.$$

Lemma 2.9
Let $M_1 = (m,\dots,m,m+\delta) \in \mathbb{Z}^r_{\geq 0}$ with $r \geq 2$ and an integer δ. If $|\delta| = c\cdot r + q$, with $c \in \mathbb{N}$, $0 \leq q < r$ and

$$M_2 = (\underbrace{m+\operatorname{sgn}(\delta)\cdot c,\dots,m+\operatorname{sgn}(\delta)\cdot c}_{r-q}, \underbrace{m+\operatorname{sgn}(\delta)\cdot(c+1),\dots,m+\operatorname{sgn}(\delta)\cdot(c+1)}_{q}),$$

then $\mathbf{l}(M_2) \leq \mathbf{l}(M_1)$ and the equality holds if and only if $|\delta| = 0$ or $|\delta| = 1$.

Proof. This is a simple computation.

An obvious consequence of this lemma is

Corollary 2.10
Let $\mathcal{M}_p = \{M \in \mathbb{Z}^r_{\geq 0} : \ |M| = p\}$. Let M_0 be an element in $\mathcal{M}_p$ imposing the least theoretical number of conditions i.e. $\mathbf{l}(M_0) = \min\{\mathbf{l}(M) \mid M \in \mathcal{M}_p\}$. Then either $\alpha(M_0) = 0$, or if this is not the case, then $\alpha(M_0) = 1$.

We have also

Corollary 2.11
Let (X,L) be a polarized surface with the Picard number $\varrho \leq 2$ and let $x_1,\dots,$ $x_r \in X$ be fixed general points. If $M_D = (m_1,\dots,m_r) \in \mathbb{Z}^r_{\geq 0}$ is a multiplicity vector of a R-R expected submaximal irreducible and reduced curve D at $x_1,\dots,x_r$, and $r \geq 3$, then up to permutation M_D is of the form

$$M_D = (m,\dots,m,m+\delta) \qquad \text{with } \delta \in \{-1,0,1\}.$$

Proof. Since the Picard number $\varrho \leq 2$ and D is a reduced and irreducible submaximal curve, by Proposition 1.6 its multiplicity vector M_D, up to permutation, is of the form

$$M_D = (m,\dots,m,m+\delta).$$

Suppose that $|\delta| \geq 2$. Then as the points are general, we have r different submaximal curves. By Lemma 2.9 there exists also a R-R expected submaximal curve D' with $\alpha(D') \leq 1$. This implies, again by generality of the points $x_1,\dots,x_r$, the existence of at least $\frac{1}{2}(r-1)r$ additional submaximal curves which contradicts Proposition 1.3. Hence $|\delta| \leq 1$.

3. Symplectic packing and the Nagata--Biran conjecture

Let (M, ω) be a closed symplectic 4-manifold and $B(\lambda_q, \omega_{std})$ be the standard closed 4-ball of radius λ_q with $\omega_{std} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ the standard symplectic form of $\mathbb{R}^4$. Consider a symplectic packing φ_λ of (M, ω) by N equal balls of radii λ (i.e., $\varphi_\lambda = \coprod_{q=1}^N \varphi_q \colon \coprod_{q=1}^N B(\lambda, \omega_{std}) \longrightarrow (M, \omega)$ is an embedding and for all q we have that a restriction to the q-th ball coincides with $\varphi_q \colon B(\lambda, \omega_{std}) \longrightarrow (M, \omega)$ and $\varphi_q^* \omega = \omega_{std}$). For a symplectic manifold of finite volume McDuff and Polterovich in [MP] introduced

$$v_N(M, \Omega) = \sup_\lambda \frac{\mathrm{Vol}(\mathrm{Image}\, \varphi_\lambda)}{\mathrm{Vol}(M, \Omega)},$$

where the supremum we take over all $\lambda \in \mathbb{R}_+$ such that φ_λ exists. If $v_N(M, \Omega) = 1$ then there exists a *full filling*, in the other case, i.e., $v_N(M, \Omega) < 1$, there is a *packing obstruction*.

In [Bi1] Biran proved the following theorem.

THEOREM 3.1 ([Bi1] Theorem 6.1.A 2)
On $\mathbb{P}^1 \times \mathbb{P}^1$ with the (a, b) polarization we have

$$v_N = \min \left\{ 1, \frac{N}{2ab} \cdot \inf_{(\alpha, \beta) \in D_N} \left(\frac{a\alpha + b\beta}{2\alpha + 2\beta - 1} \right)^2 \right\},$$

where D_N is the set of all non-negative solutions $\alpha, \beta, m_1, \ldots, m_N \geq 0$ of the system of Diophantine equations:

$$\begin{cases} 2\alpha\beta = \sum_{q=1}^N m_q^2 - 1, \\ 2\alpha + 2\beta = \sum_{q=1}^N m_q + 1. \end{cases}$$

In particular on $\mathbb{P}^1 \times \mathbb{P}^1$ with the $(1,1)$ polarization we have: $v_1 = \frac{1}{2}$, $v_2 = 1$, $v_3 = \frac{2}{3}$, $v_4 = \frac{8}{9}$, $v_5 = \frac{9}{10}$, $v_6 = \frac{48}{49}$, $v_7 = \frac{224}{225}$ and $v_N = 1$ for any $N \geq 8$ ([Bi1]).

For $\mathbb{P}^2$ there is a similar picture obtained by McDuff and Polterovich in [MP].

Later Biran proved that for a polarized surface (X, L) there exists N_0 such that for all $N \geq N_0$ we have $v_N = 1$. More precisely, if we denote by k_0 the smallest integer such that in the linear system $|k_0 L|$ there exists a smooth non-rational curve, then $N_0 = k_0^2 L^2$ (see [Bi2] Theorem 1.A.).

Now we want to study the surface $\mathbb{P}^1 \times \mathbb{P}^1$ in the context of Theorem 3.1. More precisely we want to find a relation between the number v_N and the existence of R-R expected submaximal curves at N points.

First we introduce the following definition.

Definition 3.2
For the (a, b) polarization L on $\mathbb{P}^1 \times \mathbb{P}^1$ we define the following constants:

(1) $$N_0 := \begin{cases} 8ab & \text{for } a = 1 \text{ or } b = 1, \\ 2ab & \text{for } a \geq 2 \text{and } b \geq 2, \end{cases}$$

(2) $$R_0 := \frac{3a^2 + 2ab + 3b^2}{2ab} + \frac{(a+b)\sqrt{2(a^2+b^2)}}{ab},$$

(3) $$r_0 := \left\lfloor \frac{2(a+b)^2}{ab} \right\rfloor.$$

Lemma 3.3
For every positive integers a and b one has

$$r_0 \leq R_0 \tag{3.3.1}$$

and the equality holds only for $a = b$. Moreover we have

$$R_0 \leq N_0, \tag{3.3.2}$$

and the equality holds if and only if $a = 1$ and $b = 1$ or $a = 2$ and $b = 2$.

Proof. Straightforward calculations.

Since the conditions in the last definition are symmetric, we can assume without loss of generality that $a \geq b$. We can write a in the unique way as

$$a = k \cdot b + j, \qquad \text{with } k \geq 1 \text{ and } j \in \{0, \ldots, b-1\}. \tag{3.3.3}$$

We keep this notation for the rest of this article.

Now we compute the value of r_0 .

Lemma 3.4
For any (a, b) polarization we have

$$r_0 = \begin{cases} 2k+4 & \textit{for } j \in \left\langle 0, \frac{\sqrt{4k^2+4k-15}-2k+1}{4}\, b\right) \cap \mathbb{N}, \\ 2k+5 & \textit{for } j \in \left\langle \frac{\sqrt{4k^2+4k-15}-2k+1}{4}\, b, \frac{1+\sqrt{k^2+2k-3}-k}{2}\, b\right) \cap \mathbb{N}, \\ 2k+6 & \textit{for } j \in \left\langle \frac{1+\sqrt{k^2+2k-3}-k}{2}\, b, b-1\right\rangle \cap \mathbb{N}. \end{cases}$$

Proof. Since $a = k \cdot b + j$, from the Definition 3.2 it follows that

$$r_0 = 2k + 4 + \left\lfloor \frac{2kbj + 2b^2 + 2j^2}{b^2k + bj} \right\rfloor .$$

To prove our claim it is enough to show that for all $w \in \langle 0, b-1 \rangle$

$$\frac{2kbw + 2b^2 + 2w^2}{b^2k + bw} < 3, \tag{3.4.1}$$

or equivalently

$$2w^2 + b(2k-3)w - (3k-2)b^2 < 0.$$

This is an elementary calculation.

Now we are in a good position to formulate the following result,

THEOREM 3.5
Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. If L is the (a,b) polarization, then there are no R-R expected submaximal curves on X through $r \geq R_0 = R_0(a,b)$ general points.

Proof. Fix $r \geq R_0$ and suppose to the contrary that $D \subset X$ of type (α, β) is R-R expected and submaximal. We can assume that the multiplicity vector of D is $M_D = (m, \ldots, m, m+\delta)$, where $\delta \in \{-1, 0, 1\}$ and m is a non-negative integer (by Corollary 2.11). Hence the number of independent conditions imposed by M_D is

$$\mathbf{l}(M) = (r-1)\binom{m+1}{2} + \binom{m+\delta+1}{2} = \frac{1}{2}\left(rm^2 + rm + 2m\delta + \delta^2 + \delta\right).$$

Since $h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\alpha, \beta)) = \alpha\beta + \alpha + \beta + 1$ and D is R-R expected, and by Proposition 1.3 there is no continuous family of submaximal curves, we must have

$$\alpha\beta + \alpha + \beta = \frac{1}{2}\left(rm^2 + rm + 2m\delta + \delta^2 + \delta\right),$$

or equivalently

$$\beta = \frac{rm^2 + rm + 2m\delta + \delta^2 + \delta - 2\alpha}{2(\alpha+1)}. \tag{3.5.1}$$

The submaximality of D means that

$$\frac{a\beta + \alpha b}{rm + \delta} < \sqrt{\frac{2ab}{r}}. \tag{3.5.2}$$

Substituting $t := \sqrt{r}$, conditions (3.5.1) and (3.5.2) give us the inequality

$$\begin{aligned} &2tb\alpha^2 - (2\sqrt{2ab}t^2m + 2ta - 2tb + 2\sqrt{2ab}\delta)\alpha \\ &\quad + (at^2m + a\delta^2 + at^2m^2 + 2am\delta + a\delta - 2\sqrt{2ab}tm)t - 2\sqrt{2ab}\delta < 0. \end{aligned}$$

We consider it as a quadratic inequality in the variable α. We know that the set of solutions is non-empty, hence

$$\begin{aligned} &-2abt^3\left(t-\frac{2(a+b)}{\sqrt{2ab}}\right)m+((a-b)^2 \\ &-2ab(1+\delta)\delta)t^2+2\sqrt{2ab}(a+b)\delta t+2ab\delta^2>0. \end{aligned} \tag{3.5.3}$$

If we assume that $t > \frac{2(a+b)}{\sqrt{2ab}} = \sqrt{r_0}$, then (3.5.3) is equivalent to

$$m < \frac{((a-b)^2 - 2ab(1+\delta)\delta)t^2 + 2\sqrt{2ab}(a+b)\delta t + 2ab\delta^2}{2t^3(abt - \sqrt{2ab}(a+b))}. \tag{3.5.4}$$

In the case $\delta = 0$ the inequality (3.5.4) is equivalent to

$$m < \frac{(a-b)^2}{2abt^2 - 2\sqrt{2ab}(a+b)t}. \tag{3.5.5}$$

If $t \geq \sqrt{R_0}$ then the right side of (3.5.5) is smaller than 1 and it must be $m = 0$, but this contradicts the definition of the multiple point Seshadri constant.

In the case $\delta = -1$ the inequality (3.5.4) is equivalent to

$$m < \frac{(a-b)^2t^2 - 2\sqrt{2ab}(a+b)t + 2ab}{2t^3(abt - \sqrt{2ab}(a+b))}. \tag{3.5.6}$$

Since $\sqrt{r_0} \geq 1$, $t \geq \sqrt{r_0}$ implies also $t \geq \dfrac{1}{\sqrt{r_0}}$ and

$$(a-b)^2t^2 - 2\sqrt{2ab}(a+b)t + 2ab \leq (a-b)^2t^2.$$

Applying the last inequality to (3.5.6) we obtain the condition (3.5.5) and we reduce our problem to the previous one.

In the case $\delta = 1$, the inequality (3.5.4) is equivalent to

$$m < \frac{((a-b)^2 - 4ab)t^2 + 2\sqrt{2ab}(a+b)t + 2ab}{2t^3(abt - \sqrt{2ab}(a+b))}. \tag{3.5.7}$$

Since our condition is still symmetric, then without loss of generality we may use notation (3.3.3). We observe that for $t \geq \sqrt{k+4}$ there is the inequality:

$$\frac{((a-b)^2 - 4ab)t^2 + 2\sqrt{2ab}(a+b)t + 2ab}{2t^3(abt - \sqrt{2ab}(a+b))} \leq \frac{(a-b)^2}{2abt^2 - 2\sqrt{2ab}(a+b)t}. \tag{3.5.8}$$

If $t \geq \sqrt{R_0}$ then (3.5.8) holds and

$$\frac{(a-b)^2}{2abt^2 - 2\sqrt{2ab}(a+b)t} < 1.$$

In this case it can happen that (3.5.7) has a solution, namely $m = 0$. Since D is R-R expected, (3.5.1) holds and we obtain that only a fiber through one of the points $x_1, \ldots, x_r$ comes into consideration. It is easy to see that the Seshadri quotient given by the fiber is submaximal for at most $2k + 2 - \frac{2}{b}$ points, which by Lemmas 3.3 and 3.4 gives a contradiction with our assumption $t \geq \sqrt{R_0}$.

To complete the picture, we should find R-R expected submaximal curves for $r < R_0$ points. Before we begin, we make an obvious

Observation 3.6
Let (X, L) be a polarized surface. Let $D \subset X$ be a curve which at r points gives the Seshadri quotient at most $\sqrt{\frac{L^2}{r}}$. If $\sqrt{\frac{L^2}{r}}$ is non-rational then D is submaximal.

Proof. Let $M_D = (m_1, \ldots, m_r)$ be a multiplicity vector for D. By assumption $\frac{L.D}{\sum_{i=1}^r m_i} \leq \sqrt{\frac{L^2}{r}}$. Since the number on the left side is always rational, then the equality can hold only in the case when $\sqrt{\frac{L^2}{r}}$ is rational.

It means only that in practice we will be looking for R-R expected curves which at r points give a Seshadri quotient at most $\sqrt{\frac{L^2}{r}}$.

Analyzing the value of the formula in (3.5.3), for $r \leq 2k + 5$ we find R-R expected curves which give Seshadri quotients at most $\sqrt{\frac{L^2}{r}}$. We observe that these curves depend on k and sometimes on j.

Proposition 3.7
Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ with the (a, b) polarization. If $r \leq 2k + 5$, then R-R curves which give Seshadri quotients at most $\sqrt{\frac{L^2}{r}}$ arelike in the following tables:

(a) *Table 1 in the case $k = 1$,*

(b) *Table 2 in the case $k \geq 2$.*

Proof. Since all curves from the tables fulfill the condition (3.5.1), they are R-R expected. One can also check that for appropriate j we have

$$\frac{L.D}{\sum_{i=1}^r m_i} \leq \sqrt{\frac{L^2}{r}}.$$

As we observed, R-R expected submaximal curves depend sometimes on j. We see also that only in one case it can happen that for some $r \leq 2k + 5$ and for some polarization we obtain two different types of submaximal curves.

r	Type of curve	$\ldots \leq j \leq \ldots$		m	δ	The Seshadri quotient	$\sqrt{\frac{L^2}{r}}$
1	$(1,0)$	0	$b-1$	0	1	b	$\sqrt{2(b+j)b}$
2	$(1,0)$	0	$b-1$	0	1	b	$\sqrt{(b+j)b}$
3	$(1,1)$	0	$b-1$	1	0	$\frac{2b+j}{3}$	$\sqrt{\frac{2(b+j)b}{3}}$
4	$(1,1)$	0	$b-1$	1	-1	$\frac{2b+j}{3}$	$\sqrt{\frac{(b+j)b}{2}}$
5	$(2,1)$	0	$b-1$	1	0	$\frac{3b+j}{5}$	$\sqrt{\frac{2(b+j)b}{5}}$
6	$(2,2)$	0	$\frac{1}{3}b$	1	1	$\frac{4b+2j}{7}$	$\sqrt{\frac{(b+j)b}{3}}$
	$(2,1)$	$\frac{1}{3}b$	$b-1$	1	-1	$\frac{3b+j}{5}$	
7	$(4,4)$	0	$\frac{1}{7}b$	2	1	$\frac{8b+4j}{15}$	$\sqrt{\frac{2(b+j)b}{7}}$
	$(4,3)$	$\frac{1}{7}b$	$\frac{5}{9}b$	2	-1	$\frac{7b+3j}{13}$	
	$(3,1)$	$(3-\sqrt{7})b$	$b-1$	1	0	$\frac{4b+j}{7}$	

Table 1

r	Type of curve	$\ldots \leq j \leq \ldots$		m	δ	The Seshadri quotient	$\sqrt{\frac{L^2}{r}}$
1	$(1,0)$	0	$b-1$	0	1	b	$\sqrt{2(kb+j)b}$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$2k$	$(1,0)$	0	$b-1$	0	1	b	$\sqrt{\frac{(kb+j)b}{k}}$
$2k+1$	$(k,1)$	0	$b-1$	1	0	$\frac{2kb+j}{2k+1}$	$\sqrt{\frac{2(kb+j)b}{2k+1}}$
$2k+2$	$(k,1)$	0	$b-1$	1	-1	$\frac{2kb+j}{2k+1}$	$\sqrt{\frac{(kb+j)b}{k+1}}$
$2k+3$	$(k+1,1)$	0	$b-1$	1	0	$\frac{(2k+1)b+j}{2k+3}$	$\sqrt{\frac{2(kb+j)b}{2k+3}}$
$2k+4$	$(k^2+k,k+1)$	0	$\frac{1}{k+2}b$	k	1	$\frac{(k+1)(2kb+j)}{2k^2+4k+1}$	$\sqrt{\frac{(kb+j)b}{k+2}}$
	$(k+1,1)$	$\frac{1}{k+2}b$	$b-1$	1	-1	$\frac{(2k+1)b+j}{2k+3}$	
$2k+5$	$(k+2,1)$	0	$b-1$	1	0	$\frac{2(k+1)b+j}{2k+5}$	$\sqrt{\frac{2(kb+j)b}{2k+5}}$

Table 2

Remark 3.8
In the case $k = 1$, if we take b such that

$$\big((3-\sqrt{7})b,\ \tfrac{5}{9}b\big) \cap \mathbb{N} \neq \emptyset$$

then for $r = 7$ points and $(3-\sqrt{7})b < j < \frac{5}{9}b$ we have two types of R-R expected submaximal curves coming from type $(3,1)$ and $(4,3)$. The number of submaximal curves is altogether 14. Since we can have at most 8 reduced, irreducible and submaximal, it means that at least one of them is reducible. We see that the curve of type $(3,1)$ is a component of a curve of type $(4,3)$. Moreover, we observe that if $j \leq \frac{3}{8}b$ then $\frac{7b+3j}{13} \leq \frac{4b+j}{7}$.

In all other cases, if different types of R-R expected curves give the same Seshadri quotient, then this quotient is equal to $\sqrt{\frac{L^2}{r}}$, but it means that it is no longer submaximal.

Now we want to show that for $r = 2k+6$ points there exist R-R expected submaximal curves at least for (a,b) such that $r_0 = 2k+6$ (see Lemma 3.4). We observe that R-R expected submaximal curves still depend on k and j and in general case we can not write an explicit form, as we could for $r \leq 2k+5$ points.

We construct now a sequence of curves for which we compute their *submaximality area*, i.e., we estimate polarizations for which our curves are submaximal.

Proposition 3.9
Let $l \in \mathbb{Z}_+$ be a positive integer. Given the following sequences:

$$\alpha_1 := (l+1)(l+2), \qquad \beta_1 := l+2,$$
$$m_1 := l+1, \qquad \delta_1 := 1$$

and for $n \geq 2$

$$\alpha_{n+1} := (l+1)\alpha_n - \beta_n + 1, \qquad \beta_{n+1} := \alpha_n,$$
$$m_{n+1} := \frac{(2l+4)\alpha_n - 2\beta_n + 1 + \delta_n}{2l+6}, \qquad \delta_{n+1} := -\delta_n$$

we have:

(1) *for every positive integer $n \in \mathbb{Z}_+$ we have $\frac{\alpha_n}{\beta_n} > l$;*

(2) *if $r = 2l+6$ and D_n, with $n \in \mathbb{Z}_+$, is a curve of type (α_n, β_n) with a multiplicity vector $M_{D_n} = (m_n, \ldots, m_n, m_n + \delta_n)$ at r points, then*

$$h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\alpha_n, \beta_n)) = \mathbf{l}(M_{D_n}) + 1. \tag{3.9.1}$$

The condition (3.9.1) *in particular means that the curve D_n is R-R expected.*

Proof. (1) Easy induction on n.

(2) One can check that for all positive integers $n \in \mathbb{Z}_+$ we have

$$2\alpha_n + 2\beta_n - 2(l+3)m_n - \delta_n - 1 = 0. \tag{3.9.2}$$

Due to this equality we observe that for $n \geq 2$ it holds:

$$\alpha_{n+1} = \beta_n + (l+3)\alpha_n - (2l+6)m_n - \delta_n$$

and

$$m_{n+1} = \alpha_n - m_n\,.$$

Using the induction on n we can prove that for all $n \in \mathbb{Z}_+$ the condition (3.9.1) is true.

Now we want to compute submaximality areas for curves $\{D_n\}_{n\in\mathbb{Z}_+}$ from Proposition 3.9. In order to do this, first we prove the following

Lemma 3.10
Let $l, c, z \in \mathbb{Z}_+$ be positive integers with $z \in \langle 0, c-1\rangle$. Let z_n be the smaller solution of the following equation in z

$$\frac{(lc+z)\beta_n + c\alpha_n}{(2l+6)m_n + \delta_n} = \sqrt{\frac{(lc+z)c}{l+3}} \tag{3.10.1}$$

with z as the indeterminate. Then the sequence $\{z_n\}_{n\in\mathbb{N}} \subset \langle 0, c-1\rangle$ is strictly decreasing and

$$\lim_{n\to\infty} z_n = \frac{1+\sqrt{l^2+2l-3}-l}{2}\,c.$$

Proof. Let $\widetilde{z_1}$ and $\widetilde{z_2}$ be solutions of the equation

$$\frac{(lc+z)\beta_{n+1} + c\alpha_{n+1}}{(2l+6)m_{n+1} + \delta_{n+1}} = \sqrt{\frac{(lc+z)c}{l+3}} \tag{3.10.2}$$

with z as the indeterminate. We may assume without loss of generality that $\widetilde{z_1} < \widetilde{z_2}$. By definition we have $z_{n+1} = \widetilde{z_1}$. Using direct calculations we can show that $\widetilde{z_2} = z_n$. Since $\widetilde{z_1} < \widetilde{z_2}$ and n was arbitrary, then the sequence $\{z_n\}_{n\in\mathbb{N}}$ is strongly decreasing.

On the other hand, for every positive integer n we have

$$z_n + z_{n-1} = \frac{(2l+6)[-\alpha_n\beta_n - l\beta_n^2 + (2l+6)m_n^2 + 2m_n\delta_n] + 1}{(l+3)\beta_n^2}\,c.$$

Then from Proposition 3.9 (2) we obtain that

$$z_n + z_{n-1} = \frac{(2l+6)[-\alpha_n\beta_n - l\beta_n^2 + 2\alpha_n\beta_n + 2\alpha_n + 2\beta_n - (2l+6)m_n - \delta_n - 1] + 1}{(l+3)\beta_n^2}\,c.$$

By (3.9.2) it follows

$$z_n + z_{n-1} = \frac{(2l+6)\beta_n(\alpha_n - l\beta_n) + 1}{(l+3)\beta_n^2}\, c > 0,$$

where the inequality follows from Proposition 3.9 (1). Since $\{z_n\}_{n\in\mathbb{Z}_+}$ is strongly decreasing, then for every $n \in \mathbb{Z}_+$ we have $z_n > 0$. In particular it means that the sequence $\{z_n\}_{n\in\mathbb{Z}_+}$ is convergent. If this is the case, then

$$\lim_{n\to\infty} z_n = \frac{1}{2}\lim_{n\to\infty}(z_n + z_{n-1}) = \frac{1}{2}c\lim_{n\to\infty}\left(2\frac{\alpha_n}{\beta_n} - 2l + \frac{1}{(l+3)\beta_n^2}\right).$$

From Proposition 3.9 it follows that

$$\lim_{n\to\infty}\beta_n = +\infty \qquad \text{and} \qquad \lim_{n\to\infty}\frac{\alpha_n}{\beta_n} \geq l \geq 1. \tag{3.10.3}$$

Since $\lim_{n\to\infty}\frac{1}{(l+3)\beta_n^2} = 0$ and $\lim_{n\to\infty} z_n$ exists, then also $\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}$ exists. Let $g := \lim_{n\to\infty}\frac{\alpha_n}{\beta_n}$. We obtain that

$$\begin{aligned} g = \lim_{n\to\infty}\frac{\alpha_{n+1}}{\beta_{n+1}} &= \lim_{n\to\infty}\frac{(l+1)\alpha_n - \beta_n + 1}{\alpha_n} \\ &= \lim_{n\to\infty}\left((l+1) - \frac{\beta_n}{\alpha_n} + \frac{1}{\alpha}\right). \end{aligned} \tag{3.10.4}$$

On the other hand by (3.10.3) we have $g \geq 1$ and hence there exists

$$\lim_{n\to\infty}\frac{\beta_n}{\alpha_n} = \frac{1}{g}.$$

Combining this fact with (3.10.4) we obtain our assertion.

As a simple consequence of the previous lemma we obtain the following

Proposition 3.11
Let $r = 2k+6$ be the number of points on $(\mathbb{P}^1\times\mathbb{P}^1, L)$ with the (a,b) polarization L. Let $\{z_n\}_{n\in\mathbb{N}}$ and (α_n,β_n) with m_n and δ_n be like in Lemma 3.10 and Proposition 3.9 respectively. If for some n_0 there is $z_{n_0} < j < z_{n_0-1}$, then the curve D_{n_0} of type $(\alpha_{n_0},\beta_{n_0})$ with the multiplicity vector $M_{D_{n_0}} = (m_{n_0},\dots,m_{n_0},m_{n_0}+\delta_{n_0})$ is R-R expected submaximal at r points. If $j = z_{n_0}$ or $j = z_{n_0-1}$ then $\sqrt{\frac{L^2}{r}}$ is rational and D_{n_0} computes this quotient.

Proof. Since $z_{n_0} < j < z_{n_0-1}$, then by Lemma 3.10 we have

$$\frac{(kb+j)\beta_{n_0} + b\alpha_{n_0}}{(2k+6)m_{n_0} + \delta_{n_0}} < \sqrt{\frac{(kb+j)b}{k+3}}.$$

This inequality means that the curve D_{n_0} of type $(\alpha_{n_0}, \beta_{n_0})$ with the multiplicity vector $M_{D_{n_0}} = (m_{n_0}, \ldots, m_{n_0}, m_{n_0} + \delta_{n_0})$ is submaximal. By Proposition 3.9 (2) the curve D_{n_0} is also R-R expected.

If $j = z_{n_0}$ or $j = z_{n_0 - 1}$ then

$$\frac{(kb+j)\beta_{n_0} + b\alpha_{n_0}}{(2k+6)m_{n_0} + \delta_{n_0}} = \sqrt{\frac{(kb+j)b}{k+3}}$$

and $\sqrt{\frac{(kb+j)b}{k+3}} = \sqrt{\frac{L^2}{r}}$ must be rational. The previous equality also means that the curve of type $(\alpha_{n_0}, \beta_{n_0})$ computes the quotient $\sqrt{\frac{L^2}{r}}$.

In this way we obtain the following:

Theorem 3.12
Consider the surface $X = \mathbb{P}^1 \times \mathbb{P}^1$ with the (a, b) polarization. If $r \leq r_0$ (see Definition 3.2 and Lemma 3.4) and $\sqrt{\frac{L^2}{r}}$ is non-rational, then there exist R-R expected submaximal curves at r points.

Proof. If $r_0 \leq 2k+5$ then expected curves are given in Proposition 3.7.

If $r_0 = 2k+6$ then by Lemma 3.4 it must be

$$j \in \left\langle \tfrac{1+\sqrt{k^2+2k-3}-k}{2}\, b, b-1 \right\rangle \cap \mathbb{N} \qquad \text{with } k \leq b - 1 + \tfrac{1}{b}.$$

We observe that $\frac{1+\sqrt{k^2+2k-3}-k}{2}\, b$ is an integer only for $k = 1$. In this special case the number $\sqrt{\frac{L^2}{r}}$ is rational.

We should also observe that the sequence from Lemma 3.10 is in fact a partition of the interval

$$\left\langle \tfrac{1+\sqrt{k^2+2k-3}-k}{2}\, b, b-1 \right\rangle.$$

The rest of the proof follows from Proposition 3.11.

Assume that the number of points r is at least $r_0 + 1$ but smaller than R_0. We observe that in this case the situation seems to be out of control. We have conditions (3.5.5), (3.5.6) and (3.5.7) which should eliminate the most of multiplicities m. On the other hand, for r from the neighborhood of r_0 functions on the right side can obtain very high values. We observe that sometimes for $r_0 < r < R_0$ there are no R-R expected submaximal curves.

Example 3.13
Let L be $(9, 5)$ polarization. In this case $k = 1$, $b = 5$, $j = 4$ and $R_0 = \frac{68}{15} + \frac{28}{45}\sqrt{53} \approx 9.063$. Analyzing conditions (3.5.5), (3.5.6) and (3.5.7) we obtain

(1) $m < 0$, which is absurd, or (2) $m = 1$, for $\delta = 0$. Since now $r = 9$, in the last case we have only one possibility: $\alpha = 4$ and $\beta = 1$. We see that this curve gives the quotient

$$\frac{L.D}{\sum m_i} = \frac{29}{9} > \sqrt{10} = \sqrt{\frac{L^2}{r}},$$

which is not submaximal.

On the other hand we have:

EXAMPLE 3.14
Let L be $(3,1)$ polarization. We have $k = 3$, $b = 1$, $j = 0$, $R_0 = 6 + \frac{8}{3}\sqrt{2} \approx 11.962$ and hence $r = 11$. Analyzing the same conditions as in the Example 3.13 we obtain (1) $m < 0$, which is absurd, or (2) $m \leq 3$, for $\delta = 0$. We see that curve D of type $(5,1)$ with $m = 1$ gives the quotient

$$\frac{L.D}{\sum_{i=1}^{11} m_i} = \frac{8}{11} < \sqrt{\frac{6}{11}} = \sqrt{\frac{L^2}{r}},$$

which is submaximal.

These examples show that in general for r in the range between r_0 and R_0 it is difficult to prove for which number of points there are R-R expected submaximal curves. We can only generalized Example 3.14.

PROPOSITION 3.15
Let L be the (a,b) polarization. Let $r = 2k + 2n + 1$ with non-negative integer n. If $(1 + n - \sqrt{2k + 2n + 1})b \leq j \leq b - 1$, then a curve of type $(k + n, 1)$ with $m = 1$ and $\delta = 0$ gives the Seshadri quotient at most $\sqrt{\frac{L^2}{r}}$.

Proof. By D we denote a curve of type $(k + n, 1)$. D has the multiplicity vector $M_D = (1, \ldots, 1)$. Since $h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(k+n, 1)) = 2k+2n+2 = \mathbf{l}(M_D) - 1$, then D is R-R expected. We compute that

$$\frac{L.D}{\sum_{i=1}^{r} m_i} = \frac{(2k + n)b + j}{2k + 2n + 1},$$

hence $\frac{L.D}{\sum_{i=1}^{r} m_i}$ is at most $\sqrt{\frac{L^2}{r}}$ if and only if

$$(1 + n - \sqrt{2k + 2n + 1})b \leq j \leq b - 1.$$

The Seshadri quotient given by D is submaximal for $j \neq (1+n-\sqrt{2k + 2n + 1})b$.

In this place we should note that $\langle (1+n-\sqrt{2k+2n+1})b, b-1\rangle \cap \mathbb{N} \neq \emptyset$ only for

$$0 \leq n < \frac{b+\sqrt{2(k+1)b^2-2b}-1}{b}.$$

Now we are in a good position to formulate the following lemma.

LEMMA 3.16
Let D_h be a R-R expected curve of type $(h,1)$ through r points in general position. If $r \geq 2h+1$ and the multiplicity vector $M_{D_h} = (1,\dots,1)$ then D_h is irreducible.

Proof. We prove this lemma by induction on h.

Step 1. For $h=1$ we have that D_1 is of type $(1,1)$ with the multiplicity vector $M_{D_1} = (1,1,1)$. If D_1 is reducible then D_1 decomposes in the sum of two fibers. Since points are in general position, then the sum of two fibers gives the multiplicity vector $(1,1,0) \neq M_{D_1}$, a contradiction.

Step 2. We assume our thesis for $h < h_0$. We want to show that a curve D_{h_0} of type $(h_0,1)$ through $r \geq 2h_0+1$ points with the multiplicity vector $M_{D_{h_0}} = (1,\dots,1)$ is irreducible.

We assume to the contrary that D_{h_0} is reducible. Then we take the decomposition on irreducible components. There are two possibilities:

(1) D_{h_0} is the sum of curves of type $(1,0)$ and $(h,1)$ with $h < h_0$, or

(2) D_{h_0} is the sum of curves of type $(1,0)$ and $(0,1)$.

In the first case we have

$$D_{h_0} = (h_0-h)\cdot(1,0)+(h,1).$$

Since points are in general position, then multiplicity vector for a curve $(1,0)$ at r points is $(0,\dots,0,1)$. Since $(h,1)$ is irreducible, then by the inductive assumption we have that it goes through at least $2h+1$ points with multiplicities 1. Finely we obtain that curves $(h_0-h)\cdot(1,0)$ and $(h,1)$ go through at least $(h_0-h)+(2h+1) = h_0+h+1$ points. Since $h_0+h+1 < 2h_0+1$, the multiplicity vector of the sum of curves $(h_0-h)\cdot(1,0)$ and $(h,1)$ is different from $M_{D_{h_0}}$, a contradiction.

In the second case we have

$$D_{h_0} = h_0\cdot(1,0)+(0,1).$$

Since points are in general position, a multiplicity vector of the sum $h_0\cdot(1,0)+(0,1)$ at r points is $(\underbrace{1,\dots,1}_{h_0+1},\underbrace{0,\dots,0}_{r-h_0-1}) \neq M_{h_0}$, a contradiction.

According to this lemma we can say more about Seshadri constants on $\mathbb{P}^1 \times \mathbb{P}^1$. More precisely we have the following theorem.

THEOREM 3.17
For $(\mathbb{P}^1 \times \mathbb{P}^1, L)$ with L of type (a, b) Seshadri constants are like in the following tables

(1) *Table 3 for $k = 1$,*

(2) *Table 4 for $k \geq 2$.*

r	Type of curve	The submaximality area $\ldots \leq j \leq \ldots$		m	δ	$\varepsilon(L; r)$	$\sqrt{\frac{L^2}{r}}$
1	$(1, 0)$	0	$b-1$	0	1	$= b$	$\sqrt{2(b+j)b}$
2	$(1, 0)$	0	$b-1$	0	1	$= b$	$\sqrt{(b+j)b}$
3	$(1, 1)$	0	$b-1$	1	0	$= \frac{2b+j}{3}$	$\sqrt{\frac{2(b+j)b}{3}}$
4	$(1, 1)$	0	$b-1$	1	-1	$= \frac{2b+j}{3}$	$\sqrt{\frac{(b+j)b}{2}}$
5	$(2, 1)$	0	$b-1$	1	0	$= \frac{3b+j}{5}$	$\sqrt{\frac{2(b+j)b}{5}}$
6	$(2, 2)$	0	$\frac{1}{3}b$	1	1	$\leq \frac{4b+2j}{7}$	$\sqrt{\frac{(b+j)b}{3}}$
	$(2, 1)$	$\frac{1}{3}b$	$b-1$	1	-1	$= \frac{3b+j}{5}$	
7	$(4, 4)$	0	$\frac{1}{7}b$	2	1	$\leq \frac{8b+4j}{15}$	$\sqrt{\frac{2(b+j)b}{7}}$
	$(4, 3)$	$\frac{1}{7}b$	$\frac{3}{8}b$	2	-1	$\leq \frac{7b+3j}{13}$	
	$(3, 1)$	$\frac{3}{8}b$	$b-1$	1	0	$= \frac{4b+j}{7}$	
8	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\sqrt{\frac{(b+j)b}{4}}$
	$(28, 21)$	$\frac{15}{49}b$	$\frac{13}{36}b$	12	1	$\leq \frac{49b+21j}{97}$	
	$(21, 15)$	$\frac{13}{36}b$	$\frac{11}{25}b$	9	-1	$\leq \frac{36b+15j}{71}$	
	$(15, 10)$	$\frac{11}{25}b$	$\frac{9}{16}b$	6	1	$\leq \frac{25b+10j}{49}$	
	$(10, 6)$	$\frac{9}{16}b$	$\frac{7}{9}b$	4	-1	$\leq \frac{16b+6j}{31}$	
	$(6, 3)$	$\frac{7}{9}b$	$b-1$	2	1	$\leq \frac{9b+3j}{17}$	

Table 3

r	Type of curve	The submaximality area $\ldots \leq j \leq \ldots$		m	δ	$\varepsilon(L;r)$	$\sqrt{\frac{L^2}{r}}$
1	$(1,0)$	0	$b-1$	0	1	$=b$	$\sqrt{2(kb+j)b}$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$2k$	$(1,0)$	0	$b-1$	0	1	$=b$	$\sqrt{\frac{(kb+j)b}{k}}$
$2k+1$	$(k,1)$	0	$b-1$	1	0	$=\frac{2kb+j}{2k+1}$	$\sqrt{\frac{2(kb+j)b}{2k+1}}$
$2k+2$	$(k,1)$	0	$b-1$	1	-1	$=\frac{2kb+j}{2k+1}$	$\sqrt{\frac{(kb+j)b}{k+1}}$
$2k+3$	$(k+1,1)$	0	$b-1$	1	0	$=\frac{(2k+1)b+j}{2k+3}$	$\sqrt{\frac{2(kb+j)b}{2k+3}}$
$2k+4$	$(k^2+k,k+1)$	0	$\frac{1}{k+2}b$	k	1	$\leq \frac{(k+1)(2kb+j)}{2k^2+4k+1}$	$\sqrt{\frac{(kb+j)b}{k+2}}$
	$(k+1,1)$	$\frac{1}{k+2}b$	$b-1$	1	-1	$=\frac{(2k+1)b+j}{2k+3}$	
$2k+5$	$(k+2,1)$	0	$b-1$	1	0	$=\frac{2(k+1)b+j}{2k+5}$	$\sqrt{\frac{2(kb+j)b}{2k+5}}$
$2k+6$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\leq \frac{(k+2)(2kb+j+b)}{2k^2+8k+7}$	$\sqrt{\frac{(kb+j)b}{k+3}}$
	$(k^2+3k+2,k+2)$	$\frac{k^2+3k+3}{k^2+4k+4}b$	$b-1$	$k+1$	1		
$2k+2n+7$	$(k+n+3,1)$	$(4+n-\sqrt{2k++2n+7})b$	$b-1$	1	0	$=\frac{(2k+3+n)b+j}{2k+2n+7}$	$\sqrt{\frac{2(kb+j)b}{2k+2n+7}}$

Table 4

3.1. Application to the problem of symplectic packing of $\mathbb{P}^1 \times \mathbb{P}^1$

As an application of Theorem 3.5 we prove the following:

THEOREM 3.18
Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ with the (a, b) polarization L. For every $N \geq R_0$ the polarized surface (X, L) admits full symplectic packing by N equal balls.

Proof. Fix r a number of points. Let $D \subset X$ of type (α, β) be a R-R expected submaximal curve. Let $M_D = (m, \ldots, m, m+\delta)$, where $\delta \in \{-1, 0, 1\}$ and $m \in \mathbb{Z}$, be its multiplicity vector. Since $h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\alpha, \beta)) = \alpha\beta + \alpha + \beta + 1$ and D is R-R expected, and by Proposition 1.3 there is no continuous family of submaximal curves, we must have

$$2\alpha\beta + 2\alpha + 2\beta = rm^2 + rm + 2m\delta + \delta^2 + \delta.$$

Rearranging terms on the right side we obtain that

$$rm^2 + rm + 2m\delta + \delta^2 + \delta = \sum_{i=1}^{r} m_i^2 + \sum_{i=1}^{r} m_i \tag{3.18.1}$$

(by m_i we mean the multiplicity D at x_i). By Theorem 3.5 we have that for $r \geq R_0$ points there are no R-R expected submaximal curves. In particular it means that there are no curves such that (3.18.1) becomes true. If this is the case, then the system of Diophantine equations in Theorem 3.1 does not have solutions and by the same theorem for $N \geq R_0$ we have $v_N = 1$.

3.2. Conjecture

As we remarked before, Seshadri constants are known only in few examples and in every such case, the computing curve was R-R expected. We observe also that on $\mathbb{P}^1 \times \mathbb{P}^1$ in that cases when there exists the full filling by N equal balls, there is no R-R expected submaximal curves at N points. This facts give us a reason to formulate the following conjecture.

CONJECTURE 3.19
In the case $\mathbb{P}^1 \times \mathbb{P}^1$ the number N_0 in the Nagata-Biran Conjecture can be replaced by R_0 .

REMARK 3.20
For the (pa, pb) polarization the number N_0, with respect to p, grows like a quadratic function. For the constant R_0 this is not the case. If we look at the Definition 3.2 then we can see, that R_0 is a rational function of a and b of degree 0 so the value of R_0 does not depend on p. In particular, it means that the Biran number N_0 can be optimally applied only for polarizations of type (a, b) with a and b coprime.

Acknowledgments

This article is a part of my PhD thesis. My stay at the University of Essen was made possible by the Leibnitz Preis of Esnault and Viehweg. I am very grateful to H. Esnault, T. Szemberg and E. Viehweg for invitation and helpful discussions. I would also thank H. Tutaj-Gasińska for discussions on symplectic packings.

References

[Bi1] P. Biran, *Symplectic packing in dimension* 4, Geom. Funct. Anal. **7** (1997), no. 3, 420-437.

[Bi2] P. Biran, *A stability property of symplectic packing*, Invent. Math. **136** (1999), no. 1, 123-155.

[CP] F. Campana, Th. Peternell, *Algebraicity of the ample cone of projective varieties*, J. Reine Angew. Math. **407** (1990), 160-166.

[De] J.-P. Demailly, *Singular Hermitian metrics on positive line bundles*, in: Complex algebraic varieties (Bayreuth, 1990), Lecture Notes in Math. **1507**, Springer, Berlin, 1992, 87-104.

[MP] D. McDuff, L. Polterovich, *Symplectic packings and algebraic geometry*, Invent. Math. **115** (1994), no. 3, 405-434.

[Og] K. Oguiso, *Seshadri constants in a family of surfaces*, Math. Ann. **323** (2002), no. 4, 625-631.

[S1] W. Syzdek, *Nagata submaximal curves on* $\mathbb{P}^1 \times \mathbb{P}^1$, Ann. Polon. Math. **80** (2003), 223-230.

[Sz] T. Szemberg, *Global and local positivity of line bundles*, Habilitationsschrift, Essen, 2001.

[Xu] G. Xu, *Ample line bundles on smooth surfaces*, J. Reine Angew. Math. **469** (1995), 199-209.

Institute of Mathematics
Pedagogical University
Podchorążych 2
PL-30-084 Kraków
E-mail: syzdek@ap.krakow.pl

Received: 6 January 2007; final version: 21 April 2007;
available online: 23 July 2007.

Annales Academiae Paedagogicae Cracoviensis

Folia 45 Studia Mathematica VI

Pavel Híc, Milan Pokorný

Randomly $C_n \cup C_m$ graphs

Abstract. A graph G is said to be a randomly H graph if and only if any subgraph of G without isolated vertices, which is isomorphic to a subgraph of H, can be extended to a subgraph F of G such that F is isomorphic to H. In this paper the problem of randomly H graphs, where $H = C_n \cup C_m$, $m \neq n$, is discussed.

1. Introduction

In 1951 Ore [12] studied arbitrarily traceable graphs, which were later referred to as randomly eulerian graphs. This concept was later extended by Chartrand and White [5], and Erickson [8]. In 1968 Chartrand and Kronk [2] introduced and characterized the concept of randomly hamiltonian graphs. Analogous questions were studied in [4], [6], [7], and [12].

In 1986 Chartrand, Oellermann, and Ruiz [3] generalized these concepts and introduced the term 'randomly H graph' as follows: Let G be a graph containing a subgraph H without isolated vertices. Then G is called a randomly H graph if whenever F is a subgraph of G without isolated vertices that is isomorphic to a subgraph of H, then F can be extended to a subgraph H_1 of G such that H_1 is isomorphic to H.

The graph G shown in Figure 1 is not randomly P_4 since the subgraph F of G cannot be extended to a subgraph of G isomorphic to P_4 , while the graph $K_{3,3}$ is randomly P_4 as well as randomly C_4 .

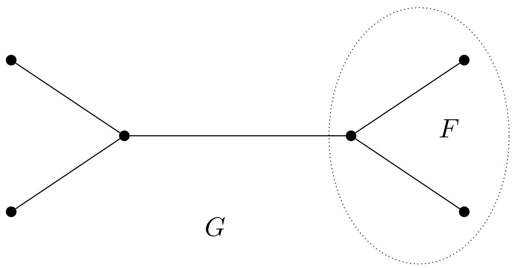


Figure 1

AMS (2000) Subject Classification: 05C75.

Every nonempty graph is randomly K_2, while every graph G without isolated vertices is a randomly G graph. K_n is randomly H for every $H \subseteq K_n$. The graph $K_{3,3}$ is randomly H for every subgraph H of $K_{3,3}$ (see [3], Theorem 1).

The requirement that both H and F are without isolated vertices follows from [3]. That is why we consider that both H and F are free of isolated vertices.

2. Preliminaries

The general question is 'For what classes of graphs H is it possible to characterize all those graphs G that are randomly H?'.

In [10] the characterization of randomly $K_{r,s}$ graphs was given, but in terms of H-closed graphs. In [1] Alavi, Lick, and Tian studied randomly complete n-partite graphs and characterized them.

The problem of characterization of randomly H graphs, where H is r-regular graph on p vertices, was given by Tomasta and Tomová (see [14]). In general, the characterization of such graphs seems to be difficult. However, there exist several results for some special values of r and p.

Theorem A (see Sumner [13])
Let H be a 1-regular graph on $2p$ vertices. A graph G on $2p$ vertices is randomly H (perfect matchable) if and only if

1. $G = K_{2p}$, *or*
2. $G = K_{p,p}$, *or*
3. $G = H$.

This is a list of results about randomly 2-regular connected graphs, which means randomly C_n graphs.

Theorem B (see Tomasta and Tomová [14])
Let G be a p-vertex graph which is randomly C_n, $n > 4$, $p > n$. Then $G = K_p$.

Theorem C (see Chartrand, Oellermann, and Ruiz [3])
A graph G is randomly C_3 if and only if each component of G is a complete graph of order at least 3.

Theorem D (see Chartrand, Oellermann, and Ruiz [3] and also Híc [10])
A graph G is randomly C_4 if and only if

1. $G = K_p$, *where* $p \geq 4$, *or*
2. $G = K_{r,s}$, *where* $2 \leq r \leq s$.

THEOREM E (see Chartrand, Oellermann, and Ruiz [3])
A graph G is randomly C_n , $n \geq 5$, if and only if

1. $G = K_p$, *where* $p \geq n$, *or*
2. $G = C_n$, *or*
3. $G = K_{\frac{n}{2},\frac{n}{2}}$ *and* n *is even.*

The following is a list of results about randomly 2-regular disconnected graphs, more specifically randomly $2C_n = C_n \cup C_n$ graphs.

THEOREM F (see Híc and Pokorný [11])
A graph G is randomly $2C_3$ if and only if

1. $G = K_p$, $p \geq 6$, *or*
2. $G = K_{p_1} \cup K_{p_2} \cup \ldots \cup K_{p_n}$, *where* $n \geq 2$, $p_i = 3$ *or* $p_i \geq 6$.

THEOREM G (see Híc and Pokorný [11])
A graph G is randomly $2C_{2n+1}$, where $n \geq 2$, if and only if

1. $G = 2C_{2n+1}$, *or*
2. $G = 2K_{2n+1}$, *or*
3. $G = C_{2n+1} \cup K_{2n+1}$, *or*
4. $G = K_p$, $p \geq 2(2n + 1)$.

THEOREM H (see Híc and Pokorný [11])
A graph G is randomly $2C_4$ if and only if

1. $G = K_{r,s}$, *where* $4 \leq r \leq s$, *or*
2. $G = 2C_4$, *or*
3. $G = 2K_4$, *or*
4. $G = C_4 \cup K_4$, *or*
5. $G = K_p$, *where* $p \geq 8$.

THEOREM I (see Híc and Pokorný [11])
A graph G is randomly $2C_{2n}$, where $n \geq 3$, if and only if

(i) $G = 2K_{2n}$, *or*
(ii) $G = 2C_{2n}$, *or*
(iii) $G = 2K_{n,n}$, *or*
(iv) $G = C_{2n} \cup K_{n,n}$, *or*
(v) $G = C_{2n} \cup K_{2n}$, *or*

(vi) $G = K_{n,n} \cup K_{2n}$, *or*

(vii) $G = K_{2n,2n}$, *or*

(viii) $G = K_p$, $p \geq 4n$.

This paper deals with randomly 2-regular graphs H, where $H = C_n \cup C_m$, $n \neq m$ (both components of H are circuits).

All the terms used in this paper can be found in [9]. Especially, if H is a subgraph of G, we will use $G - H = \langle V(G) - V(H) \rangle$ to denote the induced subgraph of the graph G with the vertex set $V(G) - V(H)$.

3. Results

LEMMA 1
Let G be a disconnected randomly $C_n \cup C_m$ graph, where $3 \leq n < m$. Then G has two components. Moreover, one of the components has n vertices and the other one has m vertices.

Proof. First, we will prove that G has two components.

a) Let G have k components, where $k > 2$. Let us construct a subgraph H of G which consists of three edges which belong to three different components of G. The subgraph H must be isomorphic to some subgraph of $C_n \cup C_m$. However, the subgraph H cannot be extended to $C_n \cup C_m$, a contradiction.

b) Let G have two components. Now we will prove that one of the components of G has n vertices and the other one has m vertices. We will discuss four different cases.

1. Obviously none of the components has less than n vertices. Moreover, one of the components has at least m vertices.

2. Let one of the components of G have k vertices, $k > m$. Let us construct a subgraph $H_1 = P_{m-2} \cup P_3$ of the component. Let H_2 be a subgraph of the other component of G which is isomorphic to P_2 . Then $H_1 \cup H_2$ should be isomorphic to a subgraph of $C_n \cup C_m$, but it cannot be extended to $C_n \cup C_m$, a contradiction. Thus none of the components of G has more then m vertices.

3. Let both components of G have m vertices. Let us construct a subgraph $H_1 = P_{n-\lfloor \frac{n}{2} \rfloor} \cup P_{\lfloor \frac{m}{2} \rfloor}$ of the first component of G and a subgraph $H_2 = P_{m-\lfloor \frac{m}{2} \rfloor} \cup P_{\lfloor \frac{n}{2} \rfloor}$ of the second component of G. Then $H_1 \cup H_2$ must be isomorphic to a subgraph of $C_n \cup C_m$, but it cannot be extended to $C_n \cup C_m$, a contradiction.

4. Let one of the components of G has k vertices, where $n < k < m$. According to parts 1 and 2 of this proof the other component of G has m vertices. Let us construct a subgraph $F = P_k \cup P_n$ of G, where P_k is a subgraph of the component of G with k vertices. Then F ought to be isomorphic to a subgraph of $C_n \cup C_m$, but it cannot be extended to $C_n \cup C_m$, a contradiction.

According to a) and b), G has two components. Moreover, one of them has n vertices and the other one has m vertices.

LEMMA 2
Let G be a disconnected randomly $C_n \cup C_m$ graph, where $3 \le n < m$. Then

(i) $G = C_n \cup C_m$ *, or*

(ii) $G = K_n \cup C_m$ *, or*

(iii) $G = K_{\frac{n}{2},\frac{n}{2}} \cup C_m$ *, where n is even.*

Proof. Let G be a disconnected randomly $C_n \cup C_m$ graph. According to Lemma 1, G has two components with n and m vertices. Obviously, one of the components is randomly C_n and the other one is randomly C_m. According to Theorem D and Theorem E, the first component can be C_n, K_n, or $K_{\frac{n}{2},\frac{n}{2}}$, where n is even, and the other component can be C_m, K_m, or $K_{\frac{m}{2},\frac{m}{2}}$, where m is even. We will prove that the second component can be neither K_m, nor $K_{\frac{m}{2},\frac{m}{2}}$. Let us construct a subgraph $F = C_n$ of this component. Then F is also a subgraph of G which is isomorphic to a subgraph of $C_n \cup C_m$, but it cannot be extended to $C_n \cup C_m$, a contradiction.

LEMMA 3
Let G be a connected randomly $C_n \cup C_m$ graph, where $3 \le n < m$. If $|V(G)| > m + n$, then G is a complete graph.

Proof. Let H be a subgraph of G isomorphic to C_n. Let $G' = G - H$. Obviously G' is randomly C_m. We will prove that G' is complete. Since $|V(G')| > m$, according to Theorem B, $G' = K_p$, $p > m$. Now we will prove that $G'' = \langle V(H) \rangle$ is complete, too. Let $H' = C_n$ be a subgraph of G'. If $G''' = G - H'$, then $G'' \subseteq G'''$. According to Theorem B, G''' is complete. Then G'' is complete, too. Finally, we will prove that for every $u \in V(G')$, $v \in V(G'')$ the graph G contains the edge $\{u, v\}$. Let us choose $u - v$ path on m vertices. Since both G' and G'' are complete and G is connected, the path always exists and can be extended to a graph which is isomorphic to $C_n \cup C_m$ only if we add the edge $\{u, v\}$ to the path. Since both u and v are arbitrary vertices, G is complete.

LEMMA 4
Let G be a connected randomly $C_n \cup C_m$ graph, where $4 \le n < m$, $|V(G)| = m + n$, and both m and n are even. If G contains a proper subgraph which is isomorphic to $K_{\frac{m+n}{2},\frac{m+n}{2}}$, then G is a complete graph.

Proof. Let $V(K_{\frac{m+n}{2},\frac{m+n}{2}}) = \{u_1, u_2, \ldots, u_{\frac{m+n}{2}}\} \cup \{v_1, v_2, \ldots, v_{\frac{m+n}{2}}\}$. Let $\{u_i, u_j\} \in E(G)$ and $\{u_i, u_j\} \notin E(K_{\frac{m+n}{2},\frac{m+n}{2}})$. Let v_k, v_t be arbitrary vertices that belong to the different partition set than u_i and u_j. Let us construct the path $v_k, u_i, u_j, v_s, u_s, \ldots, v_r, u_r, v_t$ of the length m. Since G is randomly $C_n \cup C_m$, the path can be extended to C_m only if we add the edge $\{v_k, v_t\}$. Since both v_k and v_t are arbitrary vertices, $\{v_k, v_t\} \in E(G)$ for every k, t. If we use a similar method with the edge $\{v_i, v_j\} \in E(G)$, we will prove that G is a complete graph.

Lemma 5

Let G be a connected randomly $C_n \cup C_m$ graph, where $3 \le n < m$, $|V(G)| = m+n$. Then

(i) $G = K_{\frac{m+n}{2},\frac{m+n}{2}}$ *if m and n are even, or*

(ii) $G = K_{m+n}$.

Proof. Let H be a subgraph of G isomorphic to C_n. Let $G' = G - H$. Obviously G' is randomly C_m. We will discuss three cases.

1. If m is odd, then according to Theorem E we have $G' = C_m$ or $G' = K_m$. We will prove that G' cannot be C_m. Assume the contrary. Let G' be isomorphic to C_m. Then $V(G') = \{v_1, v_2, \ldots, v_m\}$ and $E(G') = \{\{v_i, v_{i+1}\};\ i = 1, 2, \ldots, m-1\} \cup \{\{v_m, v_1\}\}$. Since G is connected, there exists an edge $\{u, v\}$, where $u \in V(H)$, $v \in V(G')$. Without loss of generality we may assume that $v = v_1$. Let us construct the path $u, v_1, v_2, \ldots, v_{m-1}$. This path can be extended to C_m only by adding the edge $\{v_{m-1}, u\}$. Now let us construct the path $v_m, v_{m-1}, u, v_1, v_2, \ldots, v_{m-3}$. This path can be extended to C_m only by adding $\{v_{m-3}, v_m\}$. So G' is not isomorphic to C_m, a contradiction. Then $G' = K_m$. If we choose a subgraph C_n of G' and we use similar ideas that we used in the proof of Lemma 3, we will prove that G is complete.

2. Similarly, if n is odd, then G is complete, too.

3. Let both m and n be even. According to Theorem E we have $G' = C_m$, $G' = K_m$, or $G' = K_{\frac{m}{2},\frac{m}{2}}$. It is easy to prove that G' cannot be C_m. In case $G' = K_m$ we can prove that G is complete. Let us consider that $G' = K_{\frac{m}{2},\frac{m}{2}}$. Let $G'' = \langle V(H) \rangle$. Note that G is randomly $C_n \cup C_m$. If we choose a subgraph $H' = C_m$ of G', then according to Theorem E it must be $G'' = C_n$, or $G'' = K_n$, or $G'' = K_{\frac{n}{2},\frac{n}{2}}$. Using similar ideas as in the part 1 of this proof we can prove that G'' cannot be C_n. If $G'' = K_n$, then G is complete. Now let us assume that $G'' = K_{\frac{n}{2},\frac{n}{2}}$. Let the vertex sets of G' and G'' be $V(G') = \{u_1, u_2, \ldots, u_{\frac{m}{2}}\} \cup \{v_1, v_2, \ldots, v_{\frac{m}{2}}\}$ and $V(G'') = \{w_1, w_2, \ldots, w_{\frac{n}{2}}\} \cup \{t_1, t_2, \ldots, t_{\frac{n}{2}}\}$. As G is a connected randomly $C_m \cup C_n$ graph, there exists at least one edge which connects a vertex of G' with a vertex of G''. Let us denote this edge $\{u_i, w_j\}$. We will prove that for every $r \in \{1, 2, \ldots, \frac{m}{2}\}$ and $s \in \{1, 2, \ldots, \frac{n}{2}\}$, $\{v_r, t_s\} \in E(G)$. Let us consider a path of the length m in G' and G'' that starts in v_r, ends in t_s, and contains the edge $\{u_i, w_j\}$. This path always exists. Since G is randomly $C_m \cup C_n$, the path

can be extended to C_m only by adding the edge $\{v_r, t_s\}$. Since r and s were arbitrary, we proved that every vertex from $\{v_1, v_2, \ldots, v_{\frac{m}{2}}\}$ is connected with every vertex from $\{t_1, t_2, \ldots, t_{\frac{n}{2}}\}$. If we repeat a similar procedure with the edge $\{v_r, t_s\}$ we can prove that every vertex from $\{u_1, u_2, \ldots, u_{\frac{m}{2}}\}$ is connected with every vertex from $\{w_1, w_2, \ldots, w_{\frac{n}{2}}\}$. It means that if G is randomly $C_n \cup C_m$ and both m and n are even, then $K_{\frac{m+n}{2}, \frac{m+n}{2}} \subseteq G \subseteq K_{m+n}$. According to Lemma 4, $G = K_{\frac{m+n}{2}, \frac{m+n}{2}}$ or $G = K_{m+n}$.

The following theorem summarizes the characterization of randomly $C_n \cup C_m$ graphs. It is easy to prove that each of the graphs that are mentioned in the theorem is randomly $C_n \cup C_m$. The rest of the theorem follows from Lemma 1-5.

Theorem 1
A graph G is randomly $C_n \cup C_m$, where $3 \leq n < m$ if and only if

(i) $G = C_n \cup C_m$, *or*

(ii) $G = K_n \cup C_m$, *or*

(iii) $G = K_{\frac{n}{2}, \frac{n}{2}} \cup C_m$ *where n is even, or*

(iv) $G = K_{\frac{m+n}{2}, \frac{m+n}{2}}$ *where both m and n are even, or*

(v) $G = K_p$, *where $p \geq m + n$.*

Conclusion

In the paper a characterization of randomly H graphs where $H = C_n \cup C_m$ is given. The case of 2-regular randomly H graphs, where H is a 2-regular graph which contains more than two components, remains open.

Acknowledgement

Research partially supported by VEGA grant No. 1/4001/07.

References

[1] Y. Alavi, D.R. Lick, S.L. Tian, *Randomly complete n-partite graphs*, Math. Slovaca **39** (1989), no. 3, 241-250.

[2] G. Chartrand, H.V. Kronk, *Randomly traceable graphs*, SIAM J. Appl. Math. **16** (1968), 696-700.

[3] G. Chartrand, O.R. Oellermann, S. Ruiz, *Randomly H graphs*, Math. Slovaca **36** (1986), no. 2, 129-136.

[4] G. Chartrand, H.V. Kronk, D.R. Lick, *Randomly hamiltonian digraphs*, Fund. Math. **65** (1969), 223-226.

[5] G. Chartrand, A.T. White, *Randomly traversable graphs*, Elem. Math. **25** (1970), 101-107.

[6] G. Chartrand, D.R. Lick, *Randomly Eulerian diagraphs*, Czechoslovak Math. J. **21**(96) (1971), 424-430.

[7] G.A. Dirac, C. Thomassen, *Graphs in which every finite path is contained in a circuit*, Math. Ann. **203** (1973), 65-75.

[8] D.B. Erickson, *Arbitrarily traceable graphs and digraphs*, J. Combinatorial Theory Ser. B **19** (1975), no. 1, 5-23.

[9] F. Harary, *Graph theory*, Addison–Wesley Publishing Co., Reading, Mass. – Menlo Park, Calif. – London. 1969.

[10] P. Híc, *A characterization of $K_{r,s}$-closed graphs*, Math. Slovaca **39** (1989), no. 4, 353-359.

[11] P. Híc, M. Pokorný, *Randomly $2C_n$ graphs*, in: Tatra Mountains Mathematical Publications, Bratislava (in print).

[12] O. Ore, *A problem regarding the tracing of graphs*, Elemente der Math. **6** (1951), 49-53.

[13] D.P. Sumner, *Randomly matchable graphs*, J. Graph Theory **3** (1979), no. 2, 183-186.

[14] P. Tomasta, E. Tomová, *On H-closed graphs*, Czechoslovak Math. J. **38**(113) (1988), no. 3, 404-419.

Department of Mathematics
and Computer Science
Faculty of Education
Trnava University
Priemyselná 4
SK–918 43 Trnava
Slovak Republic
E-mail: phic@truni.sk
E-mail: mpokorny@truni.sk

Received: 20 November 2006; final version: 16 May 2007;
available online: 23 July 2007.

Annales Academiae Paedagogicae Cracoviensis

Folia 45 Studia Mathematica VI

Vladimir Mityushev

Boundary value problem and functional equations for overlapping disks

Abstract. We compare applications of the method of functional equations to boundary value problem for circular multiply connected domains and to circular polygons generated by overlapping disks. The second part of the paper is devoted rather to the statement of a problem than its resolution.

1. Introduction

Boundary value problems for circular n-polygons can be solved using Christoffel–Schwarz integral. But it is necessary to find the corresponding accessor parameters. It is possible to do it for polygons with two or three vertexes [7]. In the general case, when $n > 3$, it is necessary to solve a nonlinear system of equations. In the present note, another approach based on functional equations is proposed. Only the question for 2-polygons is discussed. However, it seems that n-polygons can be treated by functional equations as well as discussed 2-polygons.

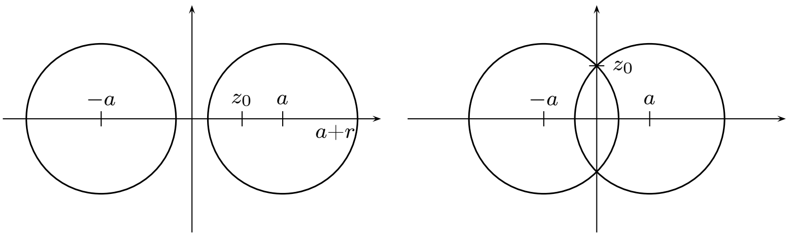
Boundary value problems for non-overlapping disks (circular multiply connected domains) have been solved in [6]. In the present note, it is proposed to develop the same method to overlapping disks. More questions than answers arise in this case.

Let z be a complex variable on the complex plane $\mathbb{C}$, $a > 0$. Consider two disks of radius r on the complex plane $D_1 = \{z \in \mathbb{C} : \ |z + a| < r\}$ and $D_2 = \{z \in \mathbb{C} : \ |z - a| < r\}$. Let D be the complement of the closures of D_1 and D_2 to the extended complex plane $\widehat{\mathbb{C}}$ and let $D^+ = D \cap \mathbb{C}^+$ be the upper half of D. We have the following two cases displayed in Fig. 1:

i) $r \leq a$,

ii) $r > a$.

AMS (2000) Subject Classification: 39B12, 30E25.



i) Non-overlapping disks ii) Overlapping disks

Figure 1

2. Functional equations [6]

Consider the case i). Following [6] we reduce a boundary value problem to a functional equation. For simplicity, the problem of conformal mapping of D^+ onto $\mathbb{C}^+$ is considered. The desired conformal mapping $\varphi(z) + z$ is determined up to purely imaginary constant by the condition

$$\mathrm{Im}(\varphi(t) + t) = 0, \qquad t \in \partial D^+. \tag{1}$$

Using the symmetry with respect to the real axis one can reduce (1) to the boundary value problem

$$\mathrm{Im}(\varphi(t) + t) = 0, \qquad t \in \partial D \tag{2}$$

with respect to $\varphi(z)$ analytic in doubly connected domain D and continuous in its closure.

Following [6] one can rewrite (2) as the $\mathbb{R}$-linear problem

$$\varphi(t) = \varphi_k(t) + \overline{\varphi_k(t)} - t, \qquad t \in \partial D_k \ (k = 1, 2), \tag{3}$$

where the auxiliary functions $\varphi_k(z)$ are analytic in D_k and continuous in the closures of the disks.

Consider inversions with respect to the circles $|z \pm a| = r$ and their compositions

$$z^*_{(1)} := \frac{r^2}{\overline{z} + a} - a, \quad z^*_{(2)} := \frac{r^2}{\overline{z} - a} + a, \tag{4}$$

$$\alpha(z) = z^*_{(12)} := \left(z^*_{(2)}\right)^*_{(1)}, \ \ \alpha^{-1}(z) = z^*_{(21)} := \left(z^*_{(1)}\right)^*_{(2)}, \ \ z^*_{(121)} := \left(z^*_{(21)}\right)^*_{(1)} \tag{5}$$

and so forth. The functions (4), (5) generate a Schottky type group $\mathcal{S} = \{\gamma_s, \ s \in \mathbb{Z}\}$ each element of which is presented in the form of the composition of inversions (4)

$$\begin{aligned} \gamma_0(z) := z, \quad & \gamma_1(\overline{z}) := z^*_{(1)}, & \gamma_{-1}(\overline{z}) &:= z^*_{(2)} \\ & \gamma_2(z) := \alpha(z), & \gamma_{-2}(z) &:= \alpha^{-1}(z), \\ & \gamma_3(\overline{z}) := \alpha(z^*_{(1)}), & \gamma_{-3}(\overline{z}) &:= \alpha^{-1}(z^*_{(2)}), \ldots \end{aligned} \tag{6}$$

When s is even, γ_s is a Möbius transformation in z. If s is odd, it is a transformation in $\overline{z}$. The number $|s|$ is called the level of the mapping γ_s.

Let us introduce the function

$$\Phi(z) := \begin{cases} \varphi_1(z) - \overline{\varphi_2\left(z^*_{(2)}\right)} - z, & |z+a| \le r, \\ \varphi_2(z) - \overline{\varphi_1\left(z^*_{(1)}\right)} - z, & |z-a| \le r, \\ \varphi(z) - \overline{\varphi_1\left(z^*_{(1)}\right)} - \overline{\varphi_2\left(z^*_{(2)}\right)}, & z \in D. \end{cases}$$

It follows from (3) that the jump of $\Phi(z)$ across the circles $|z \pm a| = r$ is equal to zero. Then the Analytic Continuation Principle implies that $\Phi(z)$ is analytic in the extended complex plane. By the Liouville theorem, $\Phi(z)$ must be constant: $\Phi(z) \equiv c$. The definition of $\Phi(z)$ in $|z \pm a| \le r$ yields the following system of functional equations

$$\varphi_1(z) = \overline{\varphi_2\left(z^*_{(2)}\right)} + z + c, \qquad |z+a| \le r, \tag{7}$$

$$\varphi_2(z) = \overline{\varphi_1\left(z^*_{(1)}\right)} + z + c, \qquad |z-a| \le r. \tag{8}$$

Elimination of φ_2 from (7)-(8) yields the classical iterative functional equation with the shift into domain [3], [5]

$$\varphi_1(z) = \varphi_1[\alpha(z)] + g(z), \qquad |z+a| \le r, \tag{9}$$

where

$$g(z) = z + \overline{z^*_{(2)}} + 2\mathrm{Re}\, c, \qquad |z+a| \le r. \tag{10}$$

Investigate the function

$$\alpha(z) = \frac{(r^2 - 2a^2)z - 2a(r^2 - a^2)}{2az + r^2 - 2a^2}. \tag{11}$$

One can see that $\alpha(z)$ has two fixed points $\pm z_0$, where $z_0 = \sqrt{a^2 - r^2}$ (see Fig. 1. i). Substitution of the attractive fixed point $-z_0 \in D_1$ into (10) yields $\mathrm{Re}\, c = 0$. General solution of (10) has the form

$$\varphi_1(z) = \sum_{k=0}^{\infty} g[\alpha^k(z)] + c_1\,, \tag{12}$$

where c_1 is an arbitrary constant. Using the definition of Φ in D we obtain up to an additive purely imaginary constant

$$z + \varphi(z) = \sum_{s \in 2\mathbb{Z}} \gamma_s(z) + \sum_{s \in 2\mathbb{Z}+1} \gamma_s(\overline{z}). \tag{13}$$

The series (13) converges absolutely and uniformly in each compact subset of D. One can check directly that (13) satisfies (2).

The series (13) can be directly obtained from (2) by Grave's method of symmetry [6, 1, 4] or from the alternating Schwarz method [6]. The idea of Grave's method consists in the using of all symmetries generating a group and the analytical continuation by symmetry. The derivative of the series (13) is a θ-Poincaré series [6].

The briefly presented method of functional equations is applied to arbitrary number of non-overlapping disks and gives solution to boundary value problems in the form of the general θ-Poincaré series [6].

3. Overlapping disks

The main question of the present note can be stated as follows. Is it possible to extend the method of functional equations to overlapping disks?

We proceed to study two overlapping disks.

3.1. Functional equations

The function $\alpha(z)$ has two neutral fixed points $\pm z_0$, where $z_0 = i\sqrt{r^2 - a^2}$ (see Fig. 1. ii). Consider the conformal mapping

$$\zeta = \frac{z - z_0}{z + z_0}. \tag{14}$$

On the plane ζ the circles $|z \pm a| = r$ becomes the straight lines $\arg\zeta = \pm\theta$, where 2θ is the angle between the circles $|z \pm a| = r$. The function $\alpha(z)$ becomes the composition of two symmetries with respect to the lines $\arg\zeta = \pm\theta$, hence it becomes the transformation $e^{4i\theta}\zeta$. Therefore, the functional equation (9) becomes

$$\psi(\zeta) = \psi[e^{4i\theta}\zeta] + h(\zeta), \tag{15}$$

where $\psi(\zeta) = \varphi_1(z)$. However, the functional equation (15) is not reduced now from the boundary value problem

$$\operatorname{Im}(\varphi(t) + t) = 0, \qquad t \in \partial D, \tag{16}$$

where $D = \{z \in \mathbb{C} : \ |z \pm a| > r\}$. (16) is obtained formally from (9) without construction of $\Phi(z)$.

Formally, (13) satisfies (16). But does it converge? Consider the case when the circles meet at the angle $2\theta = \frac{\pi}{2}$. Then Grave's method yields the group

$$\left\{z, z^*_{(1)}, z^*_{(2)}, z^*_{(21)}\right\} \equiv \{\zeta, e^{\frac{\pi i}{2}}\overline{\zeta}, e^{-\frac{\pi i}{2}}\overline{\zeta}, e^{\pi i}\zeta\}$$

consisting of four elements. Hence (13) have to be replaced by

$$z + \varphi(z) = z + \overline{z^*_{(1)}} + \overline{z^*_{(2)}} + z^*_{(21)}. \tag{17}$$

In this case the functional equation (15) implies the same result (17).

3.2. Schwarz alternating method

Another interesting question related to the Schwarz alternating method arises [2, 6]. It is known that the method converges for overlapping domains. But does it yield the series (13)? In the case $2\theta = \frac{\pi}{2}$ the series (13) diverges. Then, what is the difference between (13) and a series obtained by the Schwarz alternating method?

3.3. Conformal mapping

I think, it is possible to apply the method of conformal mapping to the problem (16) and to compare the result with (13). The domain D is mapped onto the upper half plane $\operatorname{Im} w > 0$ by the function $w = i\zeta^{\frac{\pi}{2\theta}}$, where ζ is given by (14). Then the problem (16) becomes

$$\operatorname{Im}\Phi(w) = G(w), \qquad w \in \mathbb{R}, \tag{18}$$

where

$$G(w) = -\operatorname{Im}\left(z_0 \frac{1 + e^{-i\theta} w^{\frac{2\theta}{\pi}}}{1 - e^{-i\theta} w^{\frac{2\theta}{\pi}}} \right)$$

$\Phi(w)$ is analytic in $\operatorname{Im} w > 0$, continuous in $\operatorname{Im} w \geq 0$ including infinity. Applying the Poisson formula we obtain

$$\Phi(w) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} G(t) \frac{\eta\, dt}{(t-\xi)^2 + \eta^2}, \tag{19}$$

where $w = \xi + i\eta$. When does (19) coincide with (13)? We have to substitute in (13)

$$z = z_0 \frac{1 + e^{-i\theta} w^{\frac{2\theta}{\pi}}}{1 - e^{-i\theta} w^{\frac{2\theta}{\pi}}}.$$

References

[1] F.A. Apel'tsin, *A generalization of D.A. Grave's method for plane boundary-value problems in harmonic potential theory*, Applied mathematics and information science, No. 1. Comput. Math. Model. **11** (2000), no. 1, 1-14.

[2] L.V. Kantorovich, V.I. Krylov, *Approximate methods of higher analysis*, Interscience Publishers, Inc., New York; P. Noordhoff Ltd., Groningen, 1958.

[3] M. Kuczma, B. Choczewski, R. Ger, *Iterative functional equations*, Encyclopedia of Mathematics and its Applications, **32**, Cambridge University Press, Cambridge, 1990.

[4] R.F. Millar, *Application of the Schwarz function to boundary problems for Laplace's equation*, Math. Methods Appl. Sci. **10** (1988), no. 1, 67-86.

[5] V. Mityushev, *Solution to a linear functional equation with a shift into domain*, Vesti Akad. navuk BSSR. Ser. Phys.-Math. **5** (1983), 117.

[6] V.V. Mityushev, S.V. Rogosin, *Constructive methods for linear and nonlinear boundary value problems for analytic functions. Theory and applications*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, **108**, Chapman & Hall/CRC, Boca Raton, 2000.

[7] Z. Nehari, *Conformal mapping*, Dover Publications Inc., New York, 1975.

Institute of Mathematics
Pedagogical University
Podchorążych 2
PL-30-084 Kraków
E-mail: vmityu@yahoo.com

Received: 8 May 2007; available online: 23 July 2007.

Annales Academiae Paedagogicae Cracoviensis
Folia 45 Studia Mathematica VI

Report of Meeting

11th International Conference on Functional Equations and Inequalities, Będlewo, September 17 - 23, 2006

Contents

Abstracts of Talks **138**
Problems and Remarks **165**
List of Participants **175**

The *Eleventh International Conference on Functional Equations and Inequalities* was held from September 17 to September 23, 2006 in Będlewo, Poland. The series of ICFEI meetings has been organized by the *Institute of Mathematics of the Pedagogical University of Cracow* since 1984. For the second time, the conference was organized jointly with the *Stefan Banach International Mathematical Center* and hosted by the *Mathematical Research and Conference Center* in Będlewo.

The Organizing Committee consisted of Professor Janusz Brzdęk (Chairman), Dr. Paweł Solarz, Miss Janina Wiercioch and Mr. Władyław Wilk.

The Scientific Committee consisted of Professors Nicole Brillouët-Belluot, Dobiesław Brydak (Honorary Chairman), Janusz Brzdęk (Chairman), Bogdan Choczewski, Roman Ger, Hans-Heinrich Kairies, László Losonczi, Marek Cezary Zdun and Dr. Jacek Chmieliński (Scientific Secretary).

The 73 participants came from 11 countries: Austria, Canada, China, Croatia, France, Germany, Hungary, India, Israel, Poland and Russia.

Professor J. Brzdęk welcomed participants in the name of the Organizing Committee and then an opening address was given by Professor Eugeniusz Wachnicki, the Vice-Rector of the Pedagogical University of Cracow. The opening ceremony was followed by the first scientific session chaired by Professor János Aczél with the first lecture given by Professor Walter Benz.

During 20 regular sessions 65 talk were delivered. They focused on functional equations in single and several variables, functional inequalities, stability theory, convexity, multifunctions, theory of iteration, means, differential and difference equations, dynamical systems, applications of functional equations in physics and other topics. Several contributions have been made during special *Problems and Remarks*' sessions. Additional session devoted to stability problems was organized by Professor Boris Paneah on Friday evening, September 22.

On Tuesday, September 19, a picnic was organized in the park surrounding the Center. On the next day afternoon participants visited the castle and arboretum in Kórnik, as well as the park with ancient oaks in Rogalin. In the evening the piano recital was performed by Professor Hans-Heinrich Kairies.

On Thursday, September 21, a banquet was held in the Palace in Będlewo, and another piano recital was performed by Dr. Marek Czerni. On the following day a guitar concert *An Introduction to Flamenco* was given by Dr. Grzegorz Guzik.

The final session on Saturday, September 23 was chaired by Professor Bogdan Choczewski who also closed the conference. In the closing address, he gave some concluding information about the meeting and conveyed best regards for the participants from the Honorary Chairman of the ICFEI, Professor Dobiesław Brydak. It was announced that the 12th ICFEI will be organized in 2008.

The following part of the report contains abstracts of talks (in alphabetical order of the authors), problems and remarks (in chronological order of presentation) and the list of participants (with addresses). It has been compiled by Dr. Jacek Chmieliński.

Abstracts of Talks

Mirosław Adamek *On generalized affine functions*

Let $\mathcal{F}$ be a family of real functions defined on a nonempty interval $I \subset \mathbb{R}$. We say that $\mathcal{F}$ is a *two-parameter family* on I if, for any two different points $x_1, x_2 \in I$ and for any $t_1, t_2 \in \mathbb{R}$, there exists exactly one $\varphi \in \mathcal{F}$ such that

$$\varphi(x_i) = t_i \qquad \text{for } i = 1, 2.$$

The unique function $\varphi \in \mathcal{F}$ determined by the points $x_1, x_2 \in I$ and values $t_1, t_2 \in \mathbb{R}$ will be denoted by $\varphi_{(x_1,t_1)(x_2,t_2)}$.

A function $f\colon I \longrightarrow \mathbb{R}$ is called $\mathcal{F}$-*affine* if for any different points $x_1, x_2 \in I$ and $t \in [0,1]$ the following equality holds

$$f(tx_1 + (1-t)x_2) = \varphi_{(x_1,f(x_1))(x_2,f(x_2))}(tx_1 + (1-t)x_2).$$

If a function f satisfies the above equality only with a fixed $t \in (0,1)$, then such function we call $(\mathcal{F}, t)$-*affine*.

In the talk we present some general properties about them and also Kuchn-type theorem for $(\mathcal{F}, t)$-affine functions.

Roman Badora *On the stability of the nonlinear functional equation*

In the talk we present an elementary proof of the following generalization of Baker's theorem (J.A. Baker, *The stability of certain functional equations*, Proc. Amer. Math. Soc. **112** (1991), 729-732) on the stability of the nonlinear functional equation.

Theorem

Let S be a nonempty set and let (X, d) be a complete metric space. Assume that $f\colon S \longrightarrow S$ and the function $F\colon S \times X \longrightarrow X$ satisfies

$$d(F(t,x), F(t,y)) \leq \lambda(t) d(x,y), \qquad t \in S,\ x, y \in X,$$

where $\lambda\colon S \longrightarrow \mathbb{R}$. Suppose that $\phi\colon S \longrightarrow X$ satisfies

$$d(\phi(t), F(t, \phi(f(t)))) \leq \varepsilon(t), \qquad t \in S,$$

where $\varepsilon\colon S \longrightarrow \mathbb{R}$ and

$$\sum_{n=2}^{\infty} \varepsilon(f^{n-1}(t)) \prod_{i=0}^{n-2} \lambda(f^i(t)) < +\infty, \qquad t \in S.$$

Then there exists a unique function $\Phi\colon S \longrightarrow X$ such that

$$\Phi(t) = F(t, \Phi(f(t))), \qquad t \in S$$

and

$$d(\Phi(t), \phi(t)) \leq \varepsilon(t) + \sum_{n=2}^{\infty} \varepsilon(f^{n-1}(t)) \prod_{i=0}^{n-2} \lambda(f^i(t)), \qquad t \in S.$$

Anna Bahyrycz *Forti's example of an unstable homomorphism equation*

We present a proof of some property of the function introduced by G.L. Forti, which is used to show that the homomorphism equation for some group is not stable.

[1] G.L. Forti, *Remark*, Aequationes Math. **29** (1985), 90-91.

[2] G.L. Forti, *The stability of homomorphisms and amenability, with applications to functional equations*, Abh. Math. Sem. Univ. Hamburg **57** (1987), 215-226.

Karol Baron *On linear iterative equations for distribution functions*

Given a probability space $(\Omega, \mathcal{A}, P)$ and a function $\tau: \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ we consider the equation

$$F(x) = \int_{\Omega} F(\tau(x, \omega)) P(d\omega)$$

in some classes of distribution functions. We get results on existence, uniqueness and convergence of successive approximations.

Bogdan Batko *Superstability of some alternative Cauchy functional equations*

We are going to discuss the superstability of some alternative functional equations. Consider, for instance, Mikusinski's functional equation

$$f(x+y) \cdot (f(x+y) - f(x) - f(y)) = 0, \qquad x, y \in G \tag{1}$$

in the class of complex functions defined on an abelian group G. We ask if every unbounded solution of the inequality

$$|f(x+y) \cdot (f(x+y) - f(x) - f(y))| \leq \varepsilon, \qquad x, y \in G,$$

with given $\varepsilon > 0$, must be an exact solution of Mikusinski's equation (1).

Walter Benz *A conditional functional inequality*

Let X be a real inner product space of infinite or finite dimension ≥ 3, and t be a fixed element of X with $t^2 = 1$. Put $H = \{h \in X | \;\; ht = 0\}$ and observe that to every $x \in X$ there exists uniquely determined $\bar{x} \in H$ and $x_0 \in \mathbb{R}$ such that $x = \bar{x} + x_0 t$. If $x, y \in X$, $x \leq y$ is defined by $\|\bar{y} - \bar{x}\| \leq y_0 - x_0$. We are interested in special solutions $f: X \longrightarrow X$, so-called casual automorphisms, of the conditional functional inequality

$$\forall\, x, y \in X \;\; x \leq y \implies f(x) \leq f(y),$$

generalizing a theorem of A.D. Alexandrov, V.V. Ovchinnikova and E.C. Zeeman. (See my book "Classical Geometries in Modern Contexts", Birkhäuser, Basel – Boston – Berlin, 2005.)

Elena V. Blinova *On Ω-explosions in smooth skew products of interval maps*

Joint work with L.S. Efremova.

This work is continuation of [1]. it is devoted to the research of C^0- and C^1-Ω-explosions in C^1-smooth skew products of interval maps, i.e., maps of the type

$$F(x, y) = (f(x), g_x(y)), \tag{1}$$

where $(x, y) \in I$, I is the closed rectangle in the plane, in the additional assumption on the closure of the set $\mathrm{Per}(F)$ of F-periodic points.

Definition
Let $z_1 = (x, y_1)$, $z_2 = (x, y_2)$ be periodic points of F. The point z_1 is called the accessible from z_2 ($z_2 \to^a z_1$) if for every $\varepsilon > 0$ there is an ε-chain from z_2 to z_1 by the map F restricted to $\bigcup_{y \in \mathrm{Orb}(x)} \{y\} \times I$, where $\mathrm{Orb}(x)$ is f-periodic orbit of x.

The criterion of C^0-Ω-explosion for a map (1) is given in the terms of the properties of the set $\mathrm{Per}(F)$.

Theorem A
C^1-smooth map F with the closed set of periodic points permits a C^0-Ω-explosion if and only if F satisfies one of the following conditions:

1. *the set $\mathrm{Per}(F)$ is connected, and there exists at least one point $x \in \mathrm{Per}(f)$ such that the set $\mathrm{Per}(\tilde{g_x})$ is not connected, here $\tilde{g_x} = g_{f^{n-1}(x)} \circ \ldots \circ g_{f(x)} \circ g_x$, n is the least period of x;*

2. *the set $\mathrm{Per}(F)$ is not connected (let K_i be the connected components of $\mathrm{Per}(F)$), and either one of the connected components satisfies the condition (1) or there exists a finite number of connected components K_i, $i = 1, 2, \ldots, m$ of $\mathrm{Per}(F)$ such that for all $i = 1, 2, \ldots, m$, $K_i \to^a K_{i+1}$, where $K_{m+1} = K_1$.*

Theorem B
If the set $\mathrm{Per}(F)$ of C^1-smooth map (1) is closed, then there is $\varepsilon > 0$ such that for every map $F^* \in B^1_\varepsilon(F)$ $\mathrm{Per}(F^*)$ is a closed set, here $B^1_\varepsilon(F)$ is ε-neighborhood of F in the space of C^1-smooth skew products of interval maps with C^1-norm.

The next theorem is the main result in the research of C^1-Ω-explosions in C^1-smooth maps (1).

Theorem C
C^1-smooth map (1) with a closed set of periodic points doesn't permit C^1-Ω-explosions.

Finally, we present the example of the one-parameter family of C^1-smooth maps of type (1) with the closed set of periodic points, which depend continuously (but not smoothly) on the parameter, where one can observe the appearing of the periodic orbits with periods $2, 4, \ldots, 2^n$ ($n \geq 1$) at once from the fixed point.

This research is partially supported by RFBR, grant No 04-01-00457.

[1] L.S. Efremova, *On the nonwandering set and the center of triangular maps with closed set of periodic points in the base* (in Russian), Dynamical Syst. and Nonlinear Phenomena. Kiev., 1990, 15-25.

[2] J. Kupka, *Triangular maps with the chain recurrent points periodic*, Acta Math. Univ. Comenianae, **72.2** (2003), 245-251.

Nicole Brillouët-Belluot *On applications of functional equations in physics*

Nowadays, problems in Physics are generally modelled by partial differential equations. Before the development of the differential calculus, the physical processes were often described by functional equations.

Functional equations represent an alternative way of modelling problems in Physics.

In this talk we present some applications of functional equations in Physics. Through these examples, I will explain how the functional equations appear in the physical process and what can be the interest of modelling physical problems by functional equations.

Bruno Brive *Differential equations of infinite order*

Joint work with Prof. Atzmon (Tel Aviv University, Israel).

We consider the following functional equation

$$\sum_{n\geq 0} a_n \frac{d^n}{dz^n} f(z) = g(z) \tag{$*$}$$

where f and g are functions of a complex variable z and a_n are complex numbers. This is an inhomogeneous linear differential equation of (possibly) infinite order with constant coefficients. Examples of such equations are given by linear difference-differential equations with constant coefficients. Many authors have studied equations $(*)$ in various contexts: Nörlund, Valiron, Malgrange, Martineau, Guelfond, … Many results are given in Berenstein and Gay, *Complex Analysis and Special Topics in Harmonic Analysis*, Springer, 1995.

We look at $(*)$ from a functional analysis viewpoint. We consider the equation $(*)$ when f and g belong to a weighted L^p space of entire functions. Under general assumptions, the differentiation operator $D = \frac{d}{dz}$ is bounded on this space. We define the left-hand side of $(*)$ by Riesz holomorphic functional calculus.

We give a necessary and sufficient condition for the operator $\sum_{n\geq 0} a_n D^n$ to be surjective. In this case, it admits moreover a bounded linear right-inverse.

We also give an application to the particular equation

$$f(z+1) - f(z) = g(z)$$

which was first investigated by Guichard and Hurwitz.

Janusz Brzdęk *On stability of the linear functional equation*

Joint work with Dorian Popa and Bing Xu.

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, X be a Banach space over $\mathbb{K}$, S be a nonempty set, $f\colon S \longrightarrow S$, $F\colon S \longrightarrow X$, m be a positive integer, and $a_j\colon S \longrightarrow \mathbb{K}$ for $j =$

$1, \dots, m$. We present some results concerning stability of the general linear functional equation (in single variable)

$$\varphi(f^m(x)) = a_1(x)\varphi(f^{m-1}(x)) + \ldots + a_{m-1}(x)\varphi(f(x)) + a_m(x)\varphi(x) + F(x), \quad (1)$$

where $\varphi\colon S \longrightarrow X$ is the unknown function and f^p denotes the p-th iterate of f, i.e., $f^0(x) = x$ and $f^{p+1}(x) = f(f^p(x))$ for $p = 0, 1, 2, \ldots$.

For instance, in the special case where all the functions $a_1, \dots, a_p$ are constant we have the following result.

Theorem
Suppose that $r_1, \dots, r_m \in \mathbb{K}$ are the roots of the characteristic equation

$$r^m - a_1 r^{m-1} - \ldots - a_{m-1} r - a_m = 0,$$

$\delta > 0$, and one of the following two conditions holds.

(i) $|r_j| > 1$ *for every* $j = 1, \dots, m$.

(ii) f *is bijective and* $|r_j| \neq 1$ *for every* $j = 1, \dots, m$.

If a function $\varphi_s\colon S \longrightarrow X$ satisfies

$$\left\| \varphi_s(f^m(x)) - \sum_{i=1}^{m} a_i \varphi_s(f^{m-i}(x)) - F(x) \right\| \leq \delta, \qquad \forall x \in S,$$

then the equation

$$\varphi(f^m(x)) = a_1\varphi(f^{m-1}(x)) + \ldots + a_{m-1}\varphi(f(x)) + a_m\varphi(x) + F(x)$$

has a unique solution $\varphi\colon S \longrightarrow X$ such that

$$\|\varphi_s(x) - \varphi(x)\| \leq \frac{\delta}{||r_1| - 1| \cdot \ldots \cdot ||r_m| - 1|}, \qquad \forall x \in S.$$

Pál Burai *On the equivalence of equations involving means and the solution to a problem of Daróczy*

In this work we prove the equivalence of the following two functional equations:

$$f(\mathcal{A}(x, y; p)) + f(\mathcal{H}(x, y; 1 - p)) = f(x) + f(y) \qquad x, y \in I,$$

and

$$2f(\mathcal{G}(x, y)) = f(x) + f(y) \qquad x, y \in I.$$

Here I is a nonempty open interval on the positive real line, and $\mathcal{A}(x, y; p)$, $\mathcal{H}(x, y; 1 - p)$, $\mathcal{G}(x, y)$ the weighted arithmetic mean with the weight p, the weighted harmonic mean with the weight $1-p$, and the geometric mean respectively.

[1] Z. Daróczy, B. Ebanks, Zs. Páles, *Jelentés a 2001. évi Schweitzer Miklós Matematikai emlékversenyröl*, Matematikai Lapok, 2001-2002/2, 58-60.

[2] Z. Daróczy, Zs. Páles, *Gauss-composition of means and the solution of the Matkowski–Sutô problem*, Publ. Math. Debrecen, **61** (2002), 157-218.

[3] Z. Daróczy, K. Lajkó, R. Lovas, Gy. Maksa, Zs. Páles, *Functional equations involving means*, submitted.

[4] Z. Daróczy, Gy. Maksa, Zs. Páles, *Functional equations involving means and their Gauss composition*, Proc. Amer. Math. Soc. **134** (2006), 521-530.

[5] B. Ebanks, *Solution of some functional equations involving symmetric means*, Publ. Math. Debrecen **61** (2002), 579-588.

[6] A. Járai, *Regularity Properties of Functional Equations in Several Variables*, Springer, Advances in Mathematics, Vol. **8**, 2005.

Jacek Chmieliński *Orthogonality preserving property, Wigner equation and stability*

We deal with the stability of the orthogonality preserving property in the class of mappings phase-equivalent to linear or conjugate-linear ones. We give a characterization of approximately orthogonality preserving mappings in this class and we show some connections between the considered stability and the stability of the Wigner equation.

Jacek Chudziak *Stability of the Gołąb–Schinzel functional equation*

Let X be a linear space over a field K of real or complex numbers. At the 38th International Symposium on Functional Equations (2000, Noszvaj, Hungary) Professor Roman Ger raised, among others, the problem of Hyers–Ulam stability of the Gołąb–Schinzel functional equation

$$f(x+f(x)y)=f(x)f(y) \tag{1}$$

for $x,y\in X$. In [1] it has been proved that in the class of functions $f\colon X\longrightarrow K$ satisfying some weak regularity assumptions, the equation (1) is superstable, i.e., every solution of the inequality

$$|f(x+f(x)y)-f(x)f(y)|\le\varepsilon \tag{2}$$

for $x,y\in X$, where ε is a fixed nonnegative real number, either is bounded or satisfies (1). However, it is known (cf. [2, 3]) that the phenomenon of superstability is caused by the fact that we mix two operations. Namely, on the right-hand side of (1) we have the product, but in (2) we measure the distance between the two sides of (1) using the difference. Therefore, it is more natural, to measure the difference between 1 and the quotients of the sides of the equation (1). In the present talk we deal with this problem.

[1] J. Chudziak, J. Tabor, *On the stability of the Gołąb–Schinzel functional equation*, J. Math. Anal. Appl. **302** (2005), 196-200.

[2] R. Ger, *Superstability is not natural*, Wyż. Szkoła Ped. Kraków Rocznik Nauk.-Dydakt. Prace Matematyczne **13** (1993), 109-123.

[3] R. Ger, P. Šemrl, *The stability of the exponential equation*, Proc. Amer. Math. Soc. **124** (1996), 779-787.

Marek Czerni *Comparison theorems for nonlinear functional inequalities*

We present some comparison theorems for the solutions ψ of nonlinear functional inequalities

$$\begin{cases} \psi[f(x)] \le G(x,\psi(x)), \\ (-1)^p\psi[f^{2M}(x)] \le (-1)^p g_{2M}(x,\psi(x)) \end{cases}$$

where $p \in \{0,1\}$, M is a fixed positive integer and functional sequence g_n is defined by the recurrent formula

$$\begin{cases} g_0(x,y) = y, \\ g_{n+1}(x,y) = G(f^n(x), g_n(x,y)), \qquad n = 0,1,2,\dots\ . \end{cases}$$

In the talk we shall consider the case when the given function G is strictly decreasing with respect to the second variable. Moreover, we shall obtain a characterization of this results in terms of lattice theory.

Thomas M.K. Davison *On the functional equation $g(xy) + g(x\tau(y)) = 2g(x)g(y)$*

A function g on G is *basic* if $\{u \in G:\ g(xuy) = g(xy),\ \text{for all } x, y \text{ in } G\} = \{e\}$. Using Stetkær's results [1], we prove that if g is basic, satisfies our equation, and the domain G of g is non-abelian then the centre of G is $\mathrm{Fix}(\tau) := \{z \in G:\ \tau(z) = z\}$.

[1] H. Stetkær, *D'Alembert's functional equations on metabelian groups*, Aequationes Math. **59** (2000), 306-320.

Joachim Domsta *Regular iteration of homeomorphisms of intervals*

According to the known results, the regular iteration of a self mapping f of $\mathbb{R}_+ := (0,\infty)$ without fixed points is closely related to regularly varying solutions Ψ of the corresponding Schröder equation

$$g(\Psi(x)) = \Psi(f(x)) \qquad \text{for } x \in \mathbb{R}_+\,,$$

where g is a multiplication by a positive constant, not equal to 1. In more general cases, g is also a selfmapping of $\mathbb{R}_+$ and the equation states a (weak) conjugacy of g to f. For regular Ψ joining f and g the relation between the regular iteration groups of f and g is analysed.

Lyudmila S. Efremova *On homoclinic points of C^1-smooth skew products of interval maps*

The structure of a neighborhood of the transverse homoclinic trajectory to the saddle periodic orbit of a C^1-smooth skew product of interval maps is investigated.

In the set of C^1-smooth Ω-stable skew products of interval maps the criterion of the existence of a homoclinic trajectory is proved.

In the space of C^1-smooth skew products of interval maps the subset is distinguished in which the maps with transverse homoclinic trajectories to saddle periodic orbits are everywhere dense.

The examples of C^1-smooth skew products belonging to the boundary of the Ω-stability domain and having the "exotic" properties are given.

The author is partially supported by RFBR, grant No 04-01-00457.

[1] L.S. Efremova, *On the fundamental property of quasiminimal sets of skew products of interval maps*, Intern. Conference "Tikhonov and Contemporary Mathematics", Moscow, Russia, June 19-25, 2006; Abstracts of session "Functional Analysis and Differential Equations", 2006, 64-65.

Danièle Fournier-Prunaret *Attractors bifurcations and basins in two-dimensional and three-dimensional biological models based on logistic maps*

Joint work with Ricardo Lopez-Ruiz.

We consider 2-D and 3-D biological models given by coupling between logistic maps. The considered maps are noninvertible. We study the evolution of attractors (periodic orbits and chaotic attractors) and their basins when parameters change. An important tool is that of critical manifolds, which are specific to noninvertible maps and separate the state space in areas where points have a different number of preimages.

Roman Ger *On a functional congruence related to Gołąb–Schinzel equation*

Anna Mureńko, in her doctoral dissertation [1] devoted to the functional equation

$$f\left(x + M(f(x))y\right) = f(x) \circ f(y),$$

has come across a functional congruence

$$F(x \circ y) - M(x)F(y) - F(x) \in T.$$

We are looking for a readable description of the solutions of that congruence in the case where given a linear space X over a field K, a subgroup $(T, +)$ of the additive group $(X,+)$ and a groupoid $W \subset K \setminus \{0\}$, F maps W into X and M stands for a selfmapping of K.

[1] Anna Mureńko, *O rozwiązaniach pewnego równania funkcyjnego typu Gołąba–Schinzla*, Doctoral dissertation, Kraków 2006.

Attila Gilányi *Bernstein–Doetsch and Sierpiński theorems for (M,N)-convex functions*

Joint work with Zsolt Páles.

One of the classical results of the theory of convex functions is the theorem of F. Bernstein and G. Doetsch [1] which states that if a real valued Jensen-convex function defined on an open interval I is locally bounded above at one point in I then it is continuous. According to a related result by W. Sierpiński [4], the Lebesgue measurability of a Jensen-convex function implies its continuity, too.

In this talk we generalize the theorems above for (M,N)-convex functions, calling a function $f\colon I \to J$, (M,N)-convex (c.f., e.g.: [4]) if it satisfies the inequality

$$f(M(x,y)) \leq N(f(x), f(y))$$

for all $x, y \in I$, where I and J are open intervals, M and N are suitable means on I and J, respectively. Our statements also generalize T. Zgraja's recent results on (M,M)-convex functions (c.f.: [5]). A special case of our theorems was presented in [2].

[1] F. Bernstein, G. Doetsch, *Zür Theorie der konvexen Funktionen*, Math. Ann. **76** (1915), 514-526.

[2] A. Gilányi, Zs. Páles, *Bernstein–Doetsch theorem for (M,N)-convex functions*, Talk, 42nd International Symposium on Functional Equations, Hradec nad Moravici, Czech Republic, June 20-27, 2004.

[3] C. Niculescu, L.E. Persson, Convex Functions and Their Applications, CMS Books in Mathematics, Springer, 2006.

[4] W. Sierpiński, *Sur les fonctions convexes mesurables*, Fund. Math. **1** (1920), 125-128.

[5] T. Zgraja, *Continuity of functions which are convex with respect to means*, Publ. Math. Debrecen **63** (2003), 401-411.

Alina Gleska *Oscillatory properties of solutions of nonhomogeneous difference equations*

Joint work with Jarosław Werbowski.

Our goal in this talk is to investigate the monotonic and oscillatory properties of solutions of the nonhomogeneous difference equation

$$(-1)^z \Delta^m y(n) = f(n, y(r_1(n)), y(r_2(n)), \ldots, y(r_k(n))), \qquad (\mathrm{E_z})$$

where $z, k \in \mathbb{N}$, $m \geq 2$, $r_i: \mathbb{N}_{n_0} \longrightarrow \mathbb{N}_{n_0}$, $\lim_{n\to\infty} r_i(n) = \infty$ and the function $f: \mathbb{N}_{n_0} \times \mathbb{R}^k \longrightarrow \mathbb{R}$ satisfies the condition

$$f(n, x_1, x_2, \ldots, x_k) \operatorname{sgn} x_1 \geq \sum_{i=1}^{k} p_i(n)|x_i|, \tag{C}$$

where $p_i: \mathbb{N}_{n_0} \longrightarrow \mathbb{R}_+ \cup \{0\}$ $(i = 1, 2, \ldots, k)$.

[1] I. Györi, G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1991.

[2] G. Ladas, Ch.G. Philos, Y.G. Sficas, *Sharp conditions for the oscillation of delay difference equations*, J. Appl. Math. Simulation **2** (1989), 101-111.

Dorota Głazowska *An invariance of geometric mean with respect to Lagrangean means*

Joint work with Janusz Matkowski.

The invariance of the geometric mean G with respect to the Lagrangean mean-type mapping (L^f, L^g), i.e., the equation $G \circ (L^f, L^g) = G$, is considered. We show that the functions f and g must be of high class regularity. This fact allows to reduce the problem to a differential equation and determine the second derivatives of the generators f and g.

Dijana Ilišević *Quadratic functionals and sesquilinear forms*

The problem of the representability of quadratic functionals by sesquilinear forms arises from the well-known Jordan–von Neumann characterization of inner product spaces among normed spaces via the parallelogram identity. The aim of this talk is to present some recent algebraic Jordan–von Neumann type theorems in the setting of modules over involutive rings and algebras.

Eliza Jabłońska *Solutions of a generalized Gołąb–Schinzel functional equation*

Let X be a linear space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We consider solutions $f: X \longrightarrow \mathbb{K}$ and $M: \mathbb{K} \longrightarrow \mathbb{K}$ of the functional equation

$$f(x + M(f(x))y) = f(x)f(y) \qquad \text{for } x, y \in X \tag{1}$$

such that f is bounded on a set "big" in some sense. As a consequence we obtain measurable in Lebesgue and Baire sense solutions of (1). Our results refer to results of C.G. Popa and J. Brzdęk.

Justyna Jarczyk *Invariance in the class of quasi-arithmetic means with function weights*

Let $I \subset \mathbb{R}$ be an open interval. Given a function $\mu: I \times I \longrightarrow (0, 1)$ and a strictly monotonic function $\varphi: I \longrightarrow \mathbb{R}$ we consider the mean $M_\mu^\varphi: I \times I \longrightarrow \mathbb{R}$ defined by

$$M_\mu^\varphi(x,y) = \varphi^{-1}\left(\mu(x,y)\varphi(x) + (1-\mu(x,y))\varphi(y)\right).$$

We study the invariance of M_λ^{id} in such a class of means, that is the functional equation

$$\lambda(x,y)M_\mu^\varphi(x,y) + (1-\lambda(x,y))M_\nu^\psi(x,y) = \lambda(x,y)x + (1-\lambda(x,y))y.$$

In particular, we are interested in the case when

$$\lambda(x,y) = \tfrac{r(x)}{r(x)+r(y)}, \quad \mu(x,y) = \tfrac{s(x)}{s(x)+s(y)} \quad \text{and} \quad \nu(x,y) = \tfrac{t(x)}{t(x)+t(y)}$$

for every $x, y \in I$, where r, s, t are given positive functions on I. As a special case we obtain a recent result of J. Domsta and J. Matkowski [Aequationes Math. **71** (2006), 70-85; Theorem 2].

In particular, we come also to the Bajraktarević means. They satisfy

$$M_\mu^\varphi(x,y) + M_{1-\mu}^\psi(x,y) = x+y, \qquad x,y \in I,$$

i.e., the arithmetic mean is invariant with respect to $(M_\mu^\varphi, M_{1-\mu}^\psi)$.

Witold Jarczyk *Almost convex functions on Abelian groups*

Joint work with Miklós Laczkovich.

A σ-ideal $\mathcal{I}$ in $\mathbb{R}^n$ is called linearly invariant if

$$A \in \mathcal{I} \text{ and } x \in \mathbb{R}^n \text{ imply } x - A \in \mathcal{I}.$$

We say that σ-ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ in $\mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}^n$, respectively, are conjugate if they fulfil the following Fubini condition:

> for every $A \in \mathcal{I}_2$ the sections $\{y \in \mathbb{R}^n : (x,y) \in A\}$ are in $\mathcal{I}_1$ for $\mathcal{I}_1$-a.a. $x \in \mathbb{R}^n$.

In 1970 Marek Kuczma published the following result [Colloq. Math. **21** (1970), 279-284]. Here ϕ stands for the transformation of $\mathbb{R}^n \times \mathbb{R}^n$ defined by $\phi(x,y) = (x+y, x-y)$.

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be conjugate proper linearly invariant σ-ideals in $\mathbb{R}^n$ and $\mathbb{R}^{2n}$, respectively, fulfilling the conditions

> *if $A \in \mathcal{I}_1$ then $aA \in \mathcal{I}_1$ for every $a \in \mathbb{R}$,*
>
> *if $A \in \mathcal{I}_2$ then $\phi^{-1}(A) \in \mathcal{I}_2$.*

If $D \subset \mathbb{R}^n$ is an open convex set and $f\colon D \longrightarrow \mathbb{R}$ is an $\mathcal{I}_2$-almost convex function, i.e.,

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$$

for $\mathcal{I}_2$-a.a. $(x, y) \in D$, then there exists a unique Jensen convex function $g\colon D \longrightarrow \mathbb{R}$ such that $g(x) = f(x)$ for $\mathcal{I}_1$-a.a. $x \in D$.

We present a generalization of Kuczma's result for functions defined on a subset of an Abelian group G. Convexity [almost convexity] of $f\colon A \longrightarrow \mathbb{R}$ means here that the inequality

$$2f(x) \le f(x+h) + f(x-h)$$

holds for all [$\mathcal{I}_2$-a.a.] $(x, y) \in G \times G$ such that $x, x+h, x-h \in A$.

Hans-Heinrich Kairies *On some problems concerning a sum type operator*

The sum type operator F, given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} 2^{-k} \varphi(2^k x),$$

has been thoroughly discussed in the last years. Nevertheless, there remained some open problems. We state some of them which are connected with

1. Images and pre- images of F,
2. Spectral properties of F,
3. The maximal domain of F,
4. Characterizations of $F[\varphi]$,
5. A two parameter extension of F.

Zoltán Kaiser *On stability of the Fréchet equation*

Let X be a linear space and Y be a normed space over the field of rational numbers. The stability problem concerning the Fréchet equation is the following:

Let the n-th differences of the function $f\colon X \longrightarrow Y$ are bounded, i.e.,

$$\|\Delta_{y_1,\dots,y_n} f(x)\| \le \varepsilon \qquad (y_1, \dots, y_n, x \in X) \tag{1}$$

for some $\varepsilon > 0$. Is there any generalized polynomial g of degree at most $n-1$, for which $f - g$ is bounded?

Without any regularity condition of f, the first positive answer of this problem was given by D.H. Hyers [3]. M. Albert and J.A. Baker [1] gave a result in a more general form, with a shorter proof.

C. Borelli and C. Invernizzi [2] dealt with the stability of the Fréchet equation in the case that the right hand-side of (1) is an α-homogeneous function, but there was a mistake in the proof of the main theorem. Motivated by these results, we prove a stability theorem of the Fréchet equation in Banach spaces over fields with valuation.

[1] M. Albert, J.A. Baker, *Functions with bounded nth differences*, Ann. Polon. Math. **43** (1983), 93-103.

[2] C. Borelli, C. Invernizzi, *Sulla stabilitá dell'equazione funzionale dei polinomi*, Rend. Sem. Mat. Univ. Politec. Torino **57** (1999), 197-208.

[3] D.H. Hyers, *Transformations with bounded mth differences*, Pacific J. Math. **11** (1961), 591-602.

Barbara Koclęga-Kulpa *On some equation connected with Hadamard inequalities*

Joint work with Tomasz Szostok.

We consider some equations connected with Hadamard inequalities. Namely, we observe that the function $f(x) = x^2$ satisfies the condition

$$\int_x^y f(t)\,dt = (y-x)\left[\frac{2}{3}f\left(\frac{x+y}{2}\right) + \frac{1}{6}f(x) + \frac{1}{6}f(y)\right]. \tag{1}$$

We ask about functions having properties of this kind. Moreover, we present some generalization of the equation (1), i.e.,

$$f(y) - g(y) = (y-x)[h(x+y) + \phi(x) + \psi(y)],$$

which was considered in [1] for functions acting on $\mathbb{R}$. We determine all solutions of this equation in more general case — for integral domains.

[1] T. Riedel, P.K. Sahoo, *Mean value theorems and functional equations*, World Scientific, Singapore – New Jersey – London – Hong Kong, 1998.

Imre Kocsis *A bisymmetry equation on restricted domain*

Let $X \subset \mathbb{R}$ be an interval of positive length and define the set $\Delta = \{(x,y) \in X \times X \mid x \geq y\}$. In this note we give the solution of the equation

$$F(G_1(x,y), G_2(u,v)) = G(F(x,u), F(y,v)), \qquad (x,y) \in \Delta,\ (u,v) \in \Delta,$$

where the functions $F\colon X \times X \longrightarrow X$, $G_1\colon \Delta \longrightarrow X$, $G_2\colon \Delta \longrightarrow X$, and $G\colon F(X,X) \times F(X,X) \longrightarrow X$ are continuous and strictly monotonic (in the same sense) in each variable. The result is a generalization of a previous one investigated by the author (under publication in Aequationes Mathematicae). The original problem was published by R. Duncan Luce and J.A. Marley in The Journal of Risk and Uncertainty (**30:1** (2005), 21-62).

Dorota Krassowska *On a nonlinear simultaneous system of functional inequalities*

Under some conditions on given real functions f, F, g, G we determine all the continuous at least at one point solutions φ of the simultaneous system of functional inequalities

$$\begin{cases} \varphi\left(f(x)\right) \le F\left(\varphi(x)\right) \\ \varphi\left(g(x)\right) \le G\left(\varphi(x)\right) \end{cases}, \qquad x \in I,$$

where $I \subset \mathbb{R}$ is an arbitrary interval.

Xiaopei Li *An iterative equation on the unit circle*

Joint work with Shengfu Deng.

A functional equation of nonlinear iterates is discussed on the circle S^1 for its continuous solutions and differentiable solutions. By lifting to $\mathbb{R}$, the existence, uniqueness and stability of those solutions are obtained. Techniques of continuation are used to guarantee the preservation of continuity and differentiability in lifting.

Arkadiusz Lisak *A characterization of some operators by functional equations*

Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ and let r_n and q_n be two sequences of real numbers for every $n \in \mathbb{N}$. We define for fixed $t \neq 0$, $y, z \in \mathbb{R}$, $y \neq z$ sequence of operators ${}_t\phi_{y,z}^{(n)}$ in the following way

$$\begin{aligned} {}_t\phi_{y,z}^{(1)} f(x) &= t\frac{f(\frac{x+y}{t}) - f(\frac{x+z}{t})}{y-z}, \\ {}_t\phi_{y,z}^{(n+1)} f &= t[r_n \; {}_t\phi_{y,z}^{(n)} f + q_n \; {}_t\phi_{2y,2z}^{(n)} f] \end{aligned}$$

for $x \in \mathbb{R}$. For every $n \in \mathbb{N}$ we consider functional equations

$${}_t\phi_{y,z}^{(n)} f(x) = g(x),$$

where $f, g\colon \mathbb{R} \longrightarrow \mathbb{R}$ are unknown functions, and we solve them in special cases. One of the special cases has been dealt with in [1].

Next we consider and solve the equations on the Abelian groups. We show that the equations characterize polynomials (or generalized polynomials) and their derivatives (or homomorphisms of a special form).

[1] T. Riedel, M. Sablik, A. Sklar, *Polynomials and divided differences*, Publ. Math. Debrecen **66** (2005), 313-326.

Lászlo Losonczi *Polynomials with all zeros on the unit circle*

Joint work with P. Lakatos.

We summarize recent results on polynomials all of whose zeros are on the unit circle. We give sufficient conditions for this and also necessary conditions (in terms of the coefficients), describe the methods used. Finally we mention a very general new sufficient condition for self-inversive polynomials.

Grażyna Łydzińska *On some set-valued iteration semigroups*

Let X be an arbitrary set. We present the necessary and sufficient conditions for a set-valued function $A\colon X \longrightarrow 2^{\mathbb{R}}$ under which a family of multifunctions of the form

$$A^{-1}\left(A(x) + \min\{t, q - \inf A(x)\}\right)$$

where $q := \sup A(X)$, naturally occurring in the iteration theory, is an iteration semigroup.

Andrzej Mach *On some functional equations involving involutions*

Joint work with Zenon Moszner.

We present some theorems characterizing solutions of the equation

$$f(x) = f(\varphi(x)) + g(x),$$

where φ is a given involution, and particularly differentiable solutions of the equation $f(x) = f(1-x) + 2x - 1$. The stability of this equation and non-existence of the extremal points of the set of solutions are proved.

[1] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA **27** (1941), 222-224.

[2] A. Mach, Z. Moszner, *On some functional equations involving involutions*, Sitzungsberichte of the Austrian Academy of Sciences (ÖAW), in print.

[3] P. Volkmann, *Caractérisation de la fonction $f(x) = x$ par un système de deux équations fonctionnelles*, C. R. Math. Rep. Acad. Sci. Canada, **5** (1983), 27-28.

Elena Makhrova *Dendrites with the periodic points property*

Dendrite is a locally connected continuum without subsets homeomorphic to a circle.

A dendrite X is said to have the periodic points property provided that for any continuous map $f\colon X \longrightarrow X$ and for an arbitrary subcontinuum $Y \subset X$ such that $Y \subseteq f(Y)$ the last inclusion implies $Y \cap \mathrm{Per}(f) \neq \emptyset$, $\mathrm{Per}(f)$ is the periodic points set of f.

In [1] it is shown that a finite tree (a dendrite with a finite ramification points set and points of finite order) has the periodic points property, the example of a dendrite which has no the periodic points property is constructed.

In the report the structure of dendrites having the periodic points property is investigated. The next result is true.

Theorem A
Let X be a dendrite such that the derivative of the ramification points set of X is at most countable. Then X has the periodic points property.

Theorem A does not occur if the derivative of the ramification points set of a dendrite X is uncountable. The example of such dendrite is constructed here.

Necessary conditions of structure of dendrites having the periodic points property are presented.

This research is partially supported by RFBR, grant No 04-01-00457.

[1] E.N. Makhrova, *On the existence of periodic points of continuous maps of dendrites*, Some Problems of Fundamental and Applied Mathematics, Moscow 2006 (to appear).

Janusz Matkowski *On extension of solutions of simultaneous systems of functional equations*

Some sufficient conditions which allow to extend every local solution of a simultaneous system of equations to a global one are presented.

Fruzsina Mészáros *Functional equations on group*

Joint work with Zs. Ádám, K. Lajkó and Gy. Maksa.

Let G be an arbitrary group written additively. We give the general solution of the functional equation

$$f(x)\,f(x+y) = f(y)^2\,f(x-y)^2\,g(y) \qquad (x, y \in G)$$

and all the solutions of

$$f(x)\,f(x+y) = f(y)^2\,f(x-y)^2\,g(x) \qquad (x, y \in G)$$

with the additional supposition $g(x) \neq 0$ for all $x \in G$. In both cases $f, g\colon G \longrightarrow \mathbb{R}$ are unknown functions.

Janusz Morawiec *On a refinement type equation*

Joint work with Rafał Kapica.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. We show that the trivial function is the unique L^1-solution of the following refinement type equation

$$f(x) = \int_\Omega \left|\frac{\partial A}{\partial x}(x,\omega)\right| f(A(x,\omega))\,dP(\omega)$$

in a wide class of the given functions A. This class contains functions of the form $A(x,\omega) = \alpha(\omega)x - \beta(\omega)$ with $-\infty < \int_\Omega \log|\alpha(\omega)|\,dP(\omega) < 0$.

Jacek Mrowiec *Generalized convex functions in linear spaces*

The notion of generalized convex functions has been introduced by E.F. Beckenbach in the following way:
Let $\mathcal{F}$ be a two-parameter family of continuous real-valued functions defined on an open interval (a,b) such that for any two distinct points $x_1, x_2 \in (a,b)$ and any $t_1, t_2 \in \mathbb{R}$ there exists exactly one $\varphi = \varphi_{(x_1,t_1)\,(x_2,t_2)} \in \mathcal{F}$ satisfying

$$\varphi(x_i) = t_i, \qquad i = 1,2.$$

We say that a function $f\colon (a,b) \longrightarrow \mathbb{R}$ is $\mathcal{F}$-convex if for any distinct $x_1, x_2 \in (a,b)$

$$f(x) \le \varphi_{(x_1,f(x_1))\,(x_2,f(x_2))}(x) \qquad \text{for every } x \in [x_1,x_2].$$

We present a method for generating two-parameter families not only on the real line. This method allows us to extend the notion of generalized convex functions to linear spaces. Such functions have many properties of functions, which are convex in the usual sense, and proofs of their properties are easy.

Anna Mureńko *On solutions of some conditional generalizations of the Gołąb-Schinzel equation*

We deal with the conditional functional equations

$$\text{if } x, y, x + M(g(x))y > 0, \text{ then } g(x + M(g(x))y) = g(x) \circ g(y),$$

$$\text{if } x, y, x + M(g(x))y > 0, \text{ then } g(x + M(g(x))y) = g(x)g(y),$$

where $M\colon \mathbb{R} \longrightarrow \mathbb{R}$, $\circ\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $g\colon (0,\infty) \longrightarrow \mathbb{R}$ is Lebesgue measurable or Baire measurable. We consider the above equations under some additional (different for each equation) assumptions.

Veerapazham Murugan *Smooth solutions for a functional equation involving series of iterates*

Joint work with P.V. Subrahmanyam.

In this talk we give sufficient conditions for the existence and uniqueness of C^2 solution for the functional equation

$$\sum_{i=1}^{\infty} \lambda_i f^i(x) = F(x), \qquad x \in [a,b] \subset \mathbb{R},$$

where λ_i's are nonnegative real numbers with $\sum_{i=1}^{\infty} \lambda_i = 1$ and F is a given C^2 function on $[a,b]$ satisfying some additional conditions. Such functional equations have been studied earlier by Kulczycki, Shengfu, Tabor, Xiaopei, Żołdak and the present authors.

Adam Najdecki *On the stability of some generalization of Cauchy, d'Alembert and quadratic functional equations*

Let $X \neq \emptyset$ be a set, $(Y,+)$ be a commutative semigroup with a complete invariant metric, $k \in \mathbb{N}$ and let $A, B\colon Y \longrightarrow Y$, $G_i\colon X \times X \longrightarrow X$ for $i \in \{1, 2, \ldots, k\}$. We consider stability of the functional equation

$$\sum_{i=1}^{k} f(G_i(x,y)) = A(f(x)) + B(f(y))$$

in the class of functions $f\colon X \longrightarrow Y$, as well as of the equation

$$\sum_{i=1}^{k} f(G_i(x,y)) = kf(x)f(y)$$

in the class of functions mapping X into a real or complex normed algebra with a multiplicative norm.

Wiesława Nowakowska *Sufficient conditions for the oscillation of solutions of iterative functional equations*

Joint work with Jarosław Werbowski.

Sufficient conditions for the oscillation of all solutions of iterative functional equations will be presented. Oscillation criteria for difference equations will be obtained.

Andrzej Olbryś *A characterization of $(t_1, \ldots, t_n)$-Wright affine functions*

Let $t_1, \ldots, t_n$, $n \geq 2$, be fixed positive numbers, let X be a linear space over the field $L(t_1, \ldots, t_n)$ generated by $t_1, \ldots, t_n$ (i.e., the smallest field containing the set $\{t_1, \ldots, t_n\}$) and let Y be a commutative group.

Following [1] where the definition of $(t_1, \ldots, t_n)$-Wright convex function was given we introduce the definition of $(t_1, \ldots, t_n)$-Wright affine function as a function $f\colon D \longrightarrow Y$ satisfying the following functional equation:

$$\Delta_{t_1 z, \ldots, t_n z} f(x) = 0, \qquad (x,z) \in D \times X : x + (t_1 z + \ldots + t_n z) \in D,$$

where $D \subset X$ is a $L(t_1, \ldots, t_n)$-convex set.

In the paper [2] K. Lajkó has given a characterization of $(t, 1-t)$-Wright affine functions. We extend this result to $(t_1, \ldots, t_n)$-Wright affine functions of an arbitrary order.

[1] A. Gilányi, Zs. Páles, *On Dinghas-type derivatives and convex functions of higher order*, Real Anal. Exchange **27** (2001/2002), 485-494.

[2] K. Lajkó, *On a functional equation of Alsina and Garcia-Roig*, Publ. Math. Debrecen **52** (1998), 507-515.

[3] A.Olbryś, *A characterization of* $(t_1, \ldots, t_n)$*-Wright affine functions*, submitted.

Ágota Orosz *Difference equations on discrete polynomial hypergroups*

In the classical theory of difference equations the translate of a function by n and the translation of the function n-times by 1 give the same result for all n in $\mathbb{N}$. But in the hypergroup case there are two different ways to define difference equations along these two interpretations. In this talk we give the solutions of a homogenous linear difference equation of order N on a polynomial hypergroup (which is actually a difference equation with nonconstant coefficients in the classical sense) in both cases.

Boris Paneah *Strong stability of functional equations in several variables*

We deal with compact supported Banach-valued functions F satisfying

$$(\mathcal{P}F)(x,y) := F(a(x,y)) - \sum_{j=1}^{n} \alpha_j(x,y)F(a_j(x,y)) = H(x,y)$$

for all (x,y) in a bounded domain $D \subset \mathbb{R}^2$. The prototype is the Jensen equation with $a = \alpha_1 x + \alpha_2 y$, $a_1 = x$, $a_2 = y$, and real positive numbers α_1, α_2 satisfying $\alpha_1 + \alpha_2 = 1$. The problem of its stability goes back to Ulam (1940). Our approach is novel in two essential ways.

The first is that if $a = \sum \alpha_j a_j$ on a one-dimensional submanifold $\Gamma \subset \overline{D}$ (weak Jensen operator $\mathcal{P}$), then under quite general conditions the stability problem for $\mathcal{P}$ is overdetermined: the smallness of H **only** on Γ implies the nearness of F to a linear function (strong stability).

The second is a functional analytic point of view. We consider the linear operator $\mathcal{P}_\Gamma$ – the restriction of $\mathcal{P}$ to Γ – between appropriate function spaces and give conditions of its surjectivity. The stability then follows from functional analytic considerations.

Iwona Pawlikowska *Flett-type means II*

We continue the investigation of properties of means obtained from Flett's mean value theorem. We take into account two generalizations of Flett mean value theorem and we show that means obtained from these theorems are unique. There is also discussed problem of equivalence Flett-type means to the well known means.

Zsolt Páles *A regularity problem concerning the equality of generalized quasi-arithmetic means*

Let $I \subset \mathbb{R}$ be a nonvoid open interval. Given a continuous strictly monotone function $\varphi: I \longrightarrow \mathbb{R}$ and a Borel probability measure μ on $[0,1]$, the mean $M_{\varphi,\mu}: I^2 \longrightarrow I$ is defined by

$$M_{\varphi,\mu}(x,y) := \varphi^{-1}\Bigg(\int_0^1 \varphi(tx+(1-t)y)\,d\mu(t)\Bigg) \qquad (x,y\in I).$$

The equality problem of these means is to describe all pairs (φ,μ) and (ψ,ν) such that

$$M_{\varphi,\mu}(x,y) = M_{\psi,\nu}(x,y) \qquad (x,y\in I).$$

By a recent result obtained jointly with Z. Makó, if there exists a point $p\in I$ such that φ and ψ are differentiable at p and $\varphi'(p)\psi'(p)\neq 0$, then a necessary condition for the above equality problem is that the first moments of the measures μ and ν be equal, i.e.,

$$\mu_1 := \int_0^1 t\,d\mu(t) = \int_0^1 t\,d\nu(t) =: \nu_1\,.$$

Introducing the notion of quasi-differentiability, we deduce this (and more general conditions) under much weaker assumptions.

Magdalena Piszczek *On a multivalued second order differential problem*

Let K be a closed convex cone with the nonempty interior in a real Banach space and let $\mathrm{cc}(K)$ denote the family of all nonempty convex compact subsets of K. Assume that continuous linear multifunctions $H,\Psi\colon K\longrightarrow \mathrm{cc}(K)$ are given. We consider the following problem

$$\begin{aligned} D^2\,\Phi(t,x) &= \Phi(t,H(x)),\\ D\,\Phi(t,x)|_{t=0} &= \{0\},\\ \Phi(0,x) &= \Psi(x) \end{aligned}$$

for $t\geq 0$ and $x\in K$, where $D\,\Phi(t,x)$ denotes the Hukuhara derivative of $\Phi(t,x)$ with respect to t and $D^2\,\Phi(t,x) = D(D\,\Phi(t,x))$.

Barbara Przebieracz *Near iterability*

Inspired by Problem (3.1.12) posed by E. Jen in [T] we present various approaches to the concept of near-iterability. We deal with selfmappings of a real compact interval, characterize and compare a few classes of near-iterable functions in a sense. That includes

- almost iterable functions, that is continuous $f\colon X\longrightarrow X$, for which there exists an iterable $g\colon X\longrightarrow X$ such that

  $$f^n - g^n \text{ converges to 0 everywhere in } X \tag{1}$$

  and the convergence is uniform on every interval with endpoints being two consecutive fixed points of f (cf. [J]);

- functions satisfying (1);

and some weaker condition than (1), that is

- functions f for which there exists an iterable g such that

$$f^n(x) - g^n(x) \text{ converges to } 0 \text{ for every } x \in X \setminus M, \tag{2}$$

 where the set M has empty interior;

- approximately iterable functions, that is continuous $f\colon X \longrightarrow X$, such that for every $\varepsilon > 0$ there exists an iterable function $g\colon X \longrightarrow X$ and a positive integer n_0 satisfying the inequality $|f^n(x) - g^n(x)| < \varepsilon$, $n \geq n_0$, $x \in X$;

- closure of the set of all iterable functions.

(This is the continuation of my talk at 10th ICFEI).

[J] W. Jarczyk, *Almost iterable functions*, Aequationes Math. **42** (1991), 202-219.

[T] Gy. Targonski, *New directions and open problems in iteration theory*, Ber. Math.-Statist. Sekt. Forschungsgesellsch. Joanneum, No. **229**, Forschungszentrum, Graz, 1984.

Maciej Sablik *A generalization of generalized bisymmetry*

Many authors (cf. e.g. the references below) have been concerned with functional equations of generalized bisymmetry. In particular, under some assumptions the form of solutions has been determined in the case where unknown functions map finite products of intervals into reals. In the present talk we determine (again, under some regularity assumptions) the form of operators M defined in some function spaces of measurable functions and satisfying a "Fubini type" equality

$$M_{[s]}M_{[t]}x(s,t) = M_{[t]}M_{[s]}x(s,t)$$

for every real function x such that $x(s,\cdot)$, $x(\cdot,t)$ belong to a given function space. The proofs use the results obtained earlier for the generalized bisymmetry.

[1] J. Aczél, *Lectures on Functional Equations and their Applications*, Academic Press, New York – London, 1966.

[2] J. Aczél, Gy. Maksa, M. Taylor, *Equations of generalized bisymmetry and of consistent aggregation: Weakly surjective solutions which may be discontinuous at places*, J. Math. Anal. Appl. **214** (1997), 22-35.

[3] Gy. Maksa, *Solution of generalized bisymmetry type equations without surjectivity assumptions*, Aequationes Math. **57** (1999), 50-74.

[4] Gy. Maksa, *Quasisums and generalized associativity*, Aequationes Math. **69** (2005), 6-27.

[5] A. Münnich, Gy. Maksa, R.J. Mokken, *n-variable bisection*, J. Math. Psych. **44** (2000), 569-581.

Vsevolod Sakbaev *On the spaces of functions integrable with respect to finite additive measure and the generalized convergence*

The investigation of qualitative properties of the dynamical systems and the ill-posed boundary-value problems by regularization methods lead to considering of divergent sequences. To describe the behavior of the divergent sequence by the regular methods of generalized summation and to obtain the distributions on the set of its limit points we consider the measures on the set of natural numbers $\mathbb{N}$ which concentrated on the infinity. Then any nonnegative normalized measure which concentrated on the infinity defines the regular method of generalized summation of sequences such that any bounded sequence is summable. We investigate the procedure of weak integration of vector-valued functions on the set with bounded additive measure and the properties of the spaces of integrable functions. The Hilbert space of square integrable function is applied to description of the regularization of ill-posed Cauchy problem ([1]).

This work is partially supported by RFBR, grant No 04-01-00457.

[1] V.Zh. Sakbaev, *Set-valued mappings specified by regularization of the Schrödinger equation with degeneration*, Comp. Math. Math. Phys. **46** (2006), 651-665.

Adolf Schleiermacher *On real places*

Let K be a formally real field and $\varphi\colon K \longrightarrow \mathbb{R} \cup \{\infty\}$ a mapping which satisfies

$$\varphi(a+b) = \varphi(a) + \varphi(b) \quad \text{and} \quad \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b) \qquad (*)$$

whenever the righthand sides in these equations are defined. Here the operations are extended as usual to the symbol ∞ so that $x+\infty = \infty$, $y \cdot \infty = \infty$ for $x, y \in \mathbb{R}$, $y \neq 0$ and $\infty \cdot \infty = \infty$ while $\infty + \infty$ and $0 \cdot \infty$ remain undefined. A mapping φ satisfying $(*)$ is called a real place if in addition $\varphi(1) = 1$. This last requirement serves to exclude trivial cases for which $\varphi(K) \subseteq \{0, \infty\}$. In K we introduce a partial ordering $\leq_P$ defined as usual by its set P of non-negative elements. A real place φ will be called order preserving if for $a \leq_P b$ and $\varphi(a), \varphi(b) \in \mathbb{R}$ we have $\varphi(a) \leq \varphi(b)$. If $\varphi(K) \subseteq \mathbb{R}$ then a real place φ is simply an isomorphic embedding of K in $\mathbb{R}$. The object of this talk is to investigate the fields K with partial ordering $\leq_P$ for which all order preserving real places are embeddings.

It is known that this occurs when all orderings of K compatible with $\leq_P$ are Archimedean. Other necessary and sufficient conditions will be obtained by studying various properties of the partial ordering $\leq_P$. For instance, we consider the topology $\mathcal{T}$ defined by $\leq_P$ or we study the unit interval $T = \{x : -1 \leq_P x \leq_P 1\}$ as a convex set in the vector space $K_{\mathbb{Q}}$ over $\mathbb{Q}$.

For a subset S of $K_{\mathbb{Q}}$ the core of S consists of all $x \in S$ such that each line through x contains an open segment (a,b) with $x \in (a,b)$ and $(a,b) \subseteq S$. Here of course, $(a,b) = \{\rho a + (1-\rho)b : \rho \in \mathbb{Q}$ and $0 < \rho < 1\}$. Note that in infinite dimensional vector spaces there exist convex sets whose affine hull is the whole space but whose core is nevertheless empty. Conditions characterizing the fields for which all order preserving real places are embeddings are for instance:

1. *The sequence* $\frac{1}{2^n}$ *converges to zero with respect to topology* $\mathcal{T}$.
2. *In the vector space* $K_{\mathbb{Q}}$ *the (convex) set* T *has non-empty core.*
3. *The topology* $\mathcal{T}$ *can be defined by a spectral norm* μ *satisfying* $\mu(1) = 1$, $\mu(\lambda x) = |\lambda|\mu(x)$ *for all* $x \in K$, $\lambda \in \mathbb{Q}$, *and* $\mu(x) \leq 1$ *if and only if* $x \in T$.

Stanisław Siudut *Stability of the Cauchy equation for convolutions*

Let S be a real or complex normed algebra with multiplication $*$ and let F be a S-valued function defined on S.

The equation

$$F(x*y) - F(x)*F(y) = 0 \qquad (x,y \in S)$$

will be called superstable if for each F satisfying

$$\|F(x*y) - F(x)*F(y)\| \leq \delta, \qquad x,y \in S,$$

where $\delta > 0$, either F is a bounded function or $F(x*y) = F(x)*F(y)$ for all $x,y \in S$.

The above equation is not superstable for some functions algebras with convolution multiplication $*$. However, under some additional assumptions on the range $F(S)$ of F, if the set $\{F(x*y) - F(x)*F(y) :\ x,y \in S\}$ is bounded in S then F is a bounded function or $\|F(x*y) - F(x)*F(y)\| = 0$ for all $x,y \in S$.

Dariusz Sokołowski *Solutions with exponential character to a linear functional equation and roots of its characteristic equation*

We deal with the functional equation

$$\varphi(x) = \int_S \varphi\left(x + M(s)\right)\sigma(ds) \tag{1}$$

and its characteristic equation

$$\int_S e^{\lambda M(s)}\sigma(ds) = 1 \tag{2}$$

assuming that (S,Σ,σ) is a finite measure space, $M\colon S \longrightarrow \mathbb{R}$ is a Σ-measurable bounded function and $\sigma(M \neq 0) > 0$. By a *solution* of (1) we mean a real function φ defined on a set of the form $(a, +\infty) \cap W$ with $W + \langle M(S) \rangle \subset W$, and such that for every

$$x \in (a + \sup\{|M(s)| : \ s \in S\}, +\infty) \cap W$$

the integral $\int_S \varphi(x + M(s))\sigma(ds)$ exists and (1) holds. Our main result reads.

Theorem

Assume λ is a real number. If for some solution φ of (1) either a finite and nonzero limit

$$\lim_{x \to +\infty} \frac{\varphi(x)}{xe^{\lambda x}}$$

exists, or

$$0 < \liminf_{x \to +\infty} \frac{\varphi(x)}{e^{\lambda x}} \leqslant \limsup_{x \to +\infty} \frac{\varphi(x)}{e^{\lambda x}} < +\infty,$$

then λ is a root of (2).

Paweł Solarz *Iterative roots for homeomorphisms with a rational rotation number*

Let $F\colon S^1 \longrightarrow S^1$ be an orientation-preserving homeomorphism such that its rotation number is rational, i.e., $\alpha(F) = \frac{q}{n}$, where $q, n \in \mathbb{N}$, $\gcd(q, n) = 1$ and $q < n$. Denote by $\operatorname{Per} F$ the set of all periodic points of F and let

$$\mathcal{M}_F^+ := \Big\{ u \in \operatorname{Per} F : \exists\, w \in \operatorname{Per} F \setminus \{u\} : \\ \big(\overrightarrow{(u, w)} \cap \operatorname{Per} F = \emptyset \wedge \forall\, z \in \overrightarrow{(u, w)} \ z \in \overrightarrow{(u, F^n(z))} \big) \Big\}.$$

The continuous and orientation-preserving solutions of the equation

$$G^m(z) = F(z), \qquad z \in S^1,$$

where $m \geq 2$ is an integer, exist if and only if some orientation-preserving continuous solution $\Phi\colon \operatorname{Per} F \to S^1$ of the equation

$$\Phi(F(z)) = e^{2\pi \mathrm{i} \frac{q}{n}} \Phi(z), \qquad z \in \operatorname{Per} F$$

satisfies

$$e^{2\pi \mathrm{i} \frac{q + jn}{mn}} \Phi[\operatorname{Per} F] = \Phi[\operatorname{Per} F] \quad \text{and} \quad e^{2\pi \mathrm{i} \frac{q + jn}{mn}} \Phi[\mathcal{M}_F^+] = \Phi[\mathcal{M}_F^+]$$

for some $j \in \{0, \ldots, m - 1\}$.

Tomasz Szostok *On some conditional Cauchy equation*

Let X be a real inner product space. The Cauchy equation with the right-hand side multiplied by some constant is considered. This equation is assumed for all $x, y \in X$ satisfying the equality $\frac{\|x-y\|}{\|x+y\|} = \alpha$ where $\alpha \in (0, \infty)$ is given. Solutions of this conditional equation under some assumptions are determined.

Jacek Tabor *Shadowing with multidimensional time in Banach spaces*

As it is well-known, many stability problems can be reduced to the following one concerning difference equations:

Problem
Let $(A_k)_{k\in\mathbb{Z}}$ be a sequence of bounded linear operators in a Banach space X and let $(y_k)_{k\in\mathbb{Z}} \subset X$ be such that

$$\|y_{k+1} - A_k y_k\| \le \delta \qquad \text{for } k \in \mathbb{Z}.$$

Does there exist a solution to $x_{k+1} = Ax_k$ such that

$$\|x_k - y_k\| \le \varepsilon\,?$$

The aim of the talk is to deal with the generalization of the above problem to the case of multidimensional time. Applying the Taylor functional calculus [3] we generalize the results from [2] to the case of Banach spaces [1].

[1] Z. Mączyńska, Jacek Tabor, *Shadowing with multidimensional time in Banach spaces,* J. Math. Anal. Appl., to appear.

[2] S.Yu. Pilyugin, Sergei B. Tikhomirov, *Shadowing in actions of some Abelian groups*, Fund. Math. **179** (2003), 83-96.

[3] J.L. Taylor, *The analytic-functional calculus for several commuting operators*, Acta Math. **125** (1970), 1-38.

Józef Tabor *Restricted stability of the Cauchy equation in metric semigroups*

Joint work with Jacek Tabor.

Let $f\colon [0,\infty) \longrightarrow [1,\infty)$ be defined by

$$f(x) := \frac{1}{3}x + 1.$$

Then

$$f(x+y) - f(x) - f(y)| \le 1 \qquad \text{for } x, y \in [0,\infty),$$

but there is no additive function $a\colon [0,\infty) \longrightarrow [1,\infty)$ satisfying the condition

$$\sup_{x\in[0,\infty)} |f(x) - a(x)| < \infty.$$

However, function f can be approximated by an additive one on $[3,\infty)$. Namely, we have

$$\left|f(x) - \frac{1}{3}x\right| \le 1 \qquad \text{for } x \in [3,\infty).$$

Such an effect is caused by the fact that the division by 2 is not globally performable in the set $[1,\infty)$.

Our aim is to deal with the stability of the Cauchy equation in such a frame.

Peter Volkmann *Characterization of the absolute value of complex linear functionals by functional equations*

Joint work with Karol Baron.

Let V be a complex vector space. The functions $f(x) = |\varphi(x)|$ $(x \in V)$ with some linear $\varphi\colon V \longrightarrow \mathbb{C}$ are characterized by each of the two equations

$$\sup_{\alpha\in\mathbb{R}} f(x + e^{\alpha i}y) = f(x) + f(y) \qquad \text{and} \qquad \inf_{\alpha\in\mathbb{R}} f(x + e^{\alpha i}y) = |f(x) - f(y)|.$$

The paper will appear in Sem. LV (http://www.mathematik.uni-karlsruhe.de/~semlv).

Janusz Walorski *On some solutions of the Schröder equation in Banach spaces*

Let X be a Banach space. We consider the problem of existence and uniqueness of solutions of the Schröder equation

$$\varphi(f(x)) = A\varphi(x),$$

where the function $f\colon X \longrightarrow X$ and the bounded linear operator $A\colon X \longrightarrow X$ are given.

Szymon Wąsowicz *On error bounds for Gauss–Legendre and Lobatto quadrature rules*

For a function $f\colon [-1,1] \longrightarrow \mathbb{R}$ let

$$\begin{aligned}
\mathcal{G}_2(f) &:= \frac{1}{2}\left(f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)\right),\\
\mathcal{G}_3(f) &:= \frac{5}{18}f\left(-\frac{\sqrt{15}}{5}\right) + \frac{4}{9}f(0) + \frac{5}{18}f\left(\frac{\sqrt{15}}{5}\right),\\
\mathcal{L}(f) &:= \frac{1}{12}f(-1) + \frac{5}{12}f\left(-\frac{\sqrt{5}}{5}\right) + \frac{5}{12}f\left(\frac{\sqrt{5}}{5}\right) + \frac{1}{12}f(1),\\
\mathcal{S}(f) &:= \frac{1}{6}\left(f(-1) + 4f(0) + f(1)\right).
\end{aligned}$$

The operators $\mathcal{G}_2$ and $\mathcal{G}_3$ are connected with Gauss–Legendre quadrature rules. The operators $\mathcal{L}$ and $\mathcal{S}$ are connected with Lobatto and Simpson's quadrature rules, respectively. We establish the following inequalities:

PROPOSITION
If $f\colon [-1,1] \longrightarrow \mathbb{R}$ is 3-convex, then $\mathcal{G}_2(f) \le \mathcal{G}_3(f) \le \mathcal{S}(f)$ and $\mathcal{L}(f) \le \mathcal{S}(f)$.

We apply this result to give the error bounds for quadrature rules $\mathcal{G}_3$ and $\mathcal{L}$ for four times differentiable functions (instead of six times differentiable functions as in the classical results known from numerical analysis).

[1] Sz. Wąsowicz, *On error bounds for Gauss–Legendre and Lobatto quadrature rules*, J. Inequal. Pure Appl. Math. **7**(3) (2006), Art. 84, 1-7.

Marek Żołdak *Asymptotic stability of isometries in p-homogeneous F-spaces*

Joint work with Józef Tabor.

The equation of isometry in Banach spaces is stable in Hyers–Ulam sense. It happens that in complete Frechet spaces this equation is not stable. We prove that if $p \in (0,1]$, $r > 0$, $\varepsilon > 0$; X, Y are complete p-homogeneous spaces, $f\colon X \longrightarrow Y$ is a surjective mapping such that $f(0) = 0$ and

$$|\, \|f(x) - f(y)\|^r - \|x - y\|^r \,| \le \varepsilon \qquad \text{for } x, y \in X,$$

then there exist a linear surjective isometry U and a constant $L > 0$ such that

$$\|f(x) - U(x)\| \le L(\varepsilon^p \|x\|^{1-pr} + \varepsilon^{\frac{1}{r}}) \qquad \text{for } x \in X,$$

when $pr < 1$, and

$$\|f(x) - U(x)\| \le L\varepsilon^{\frac{1}{r}} \qquad \text{for } x \in X,$$

when $pr > 1$.

Problems and Remarks

1. Remark. *Dynamical systems generated by two maps*

Let $\delta_j\colon I \longrightarrow I$, $j = 1, 2$, be continuous maps of the interval $I = [-1, 1]$ into itself satisfying:

1° all δ_j do not decrease; 2° $\mathcal{R}(\delta_1) \cap \mathcal{R}(\delta_2) = \{0\}$; 3° $\mathcal{R}(\delta_1) \cup \mathcal{R}(\delta_2) = I$

with $\mathcal{R}(\delta_j)$ a range of δ_j. The semigroup Φ_δ generated by δ_1, δ_2 consists of al maps $\delta_J\colon I \longrightarrow I$ of the form $\delta_J = \delta_{j_n} \circ \ldots \circ \delta_{j_1}$, where $J = (j_1, \ldots, j_n)$ is an arbitrary multi-index with $j_k = 1$ or $j_k = 2$. A sequence $(t_1, \ldots, t_n, \ldots)$ of points $t_k \in I$ is called *orbit* if for all $k = 1, 2, \ldots$ we have

$$t_{k+1} = \delta_{j_k}(t_k), \qquad j_k \in \{1, 2\}. \tag{$*$}$$

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be arbitrary disjoint closed subsets in I and $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$. An orbit $(t_1, t_2, \ldots)$ is called $\mathcal{T}$-*guiding* if in $(*)$ $j_k = 1$ as $t_k \in \mathcal{T}_2$ and $j_k = 2$ as $t_k \in \mathcal{T}_1$. This notion plays a crucial role when studying various forms of the solvability of general linear functional equations. For example, when describing the kernel of the Cauchy type operator $CF := F(\delta_1 + \delta_2) - F(\delta_1) - F(\delta_2)$ with the above

δ_1, δ_2 the result follows immediately if we note that the maximal value of any element $F \in \ker C$ spreads along $\mathcal{T}$-guiding orbits, where $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ and $\mathcal{T}_j = \{t \mid \delta_j'(t) = 0\}$, see [B. Paneah, Funct. Anal. Appl., **37** (2003), 46-60]. Finally, *attractor* $\mathcal{A}$ in Φ_δ is a collection of points $x \in I$ such that for any point $t \in I$ there is a $\mathcal{T}$-guiding orbit $(t, \delta_{j_1}(t), \ldots)$ converging to x. The main problem (solution of which finds immediately many applications) is as follows: given maps δ_1, δ_2 and sets $\mathcal{T}_1$, $\mathcal{T}_2$ to describe all attractors of the dynamical system Φ_δ. A particular solution of the problem is given in the above mentioned paper.

Boris Paneah

2. Remark.

During the 44th International Symposium on Functional Equations held in Louisville in May, 2006, Janusz Brzdęk asked on all self-mappings of a given semigroup satisfying the equation

$$f(x) + f(y + f(y)) = f(y) + f(x + f(y)). \tag{1}$$

Recently, Marcin Balcerowski from Katowice proved some results on (1) as well as on the more general equation

$$f(x) + f(y + g(y)) = f(y) + f(x + g(y)). \tag{2}$$

Among them the following can be proved.

THEOREM

Let G be a group and let $g\colon G \longrightarrow G$. Assume that the group $\langle g(G)\rangle$ generated by $g(G)$ is G. Let H be an Abelian group. Then $f\colon G \longrightarrow H$ satisfies (2) if and only if it is affine, that is

$$f(x) = a(x) + b, \qquad x \in G$$

with an additive $a\colon G \longrightarrow H$ and a $b \in H$.

COROLLARIES

1. *Let G be an abelian group and let $f\colon G \longrightarrow G$. Assume that $\langle f(G)\rangle = G$. Then f is a solution of (1) if and only if it is affine.*
2. *A function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous solution of (1) if and only if*

$$f(x) = ax + b, \qquad x \in \mathbb{R},$$

with some $a, b \in \mathbb{R}$.

3. *A function $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ is an analytic solution of (1) if and only if*

$$f(z) = az + b, \qquad z \in \mathbb{C},$$

with some $a, b \in \mathbb{C}$.

Witold Jarczyk

3. Problem.

Let $D \subset \mathbb{R}^2$ be an open region. *Determine the general solution of*

$$k(x+y) = f(x)g(y) + h(y) \qquad ((x,y) \in D). \tag{1}$$

More exactly, determine all $f\colon D_1 \longrightarrow \mathbb{R}$, $g, h\colon D_2 \longrightarrow \mathbb{R}$, $k\colon D_+ \longrightarrow \mathbb{R}$ satisfying (1), where

$$\begin{aligned} D_1 &:= \{x \mid \exists y : (x,y) \in D\}, \\ D_2 &:= \{y \mid \exists x : (x,y) \in D\}, \\ D_+ &:= \{x+y \mid (x,y) \in D\}. \end{aligned} \tag{2}$$

BACKGROUND

I solved equation (1) (Proc. Amer. Math. Soc. **133** (2005), 3227-3233) when k is *locally nonconstant* (not constant on neighbourhood of any point in D_+; called *philandering* by Lundberg, Sablik et al.)

No other assumption. The *problem is to eliminate this one assumption.*

Why is the equation (1) *interesting?*

$$f(x+y) = f(x)g(y) + h(y) \qquad (k = f) \tag{3}$$

is fundamental to characterising power means among quasiarithmetic means.

For $k = h$ the equation

$$k(x+y) = f(x)g(y) + h(y) \qquad ((x,y) \in D) \tag{1}$$

i.e.

$$h(x+y) = f(x)g(y) + h(y) \qquad ((x,y) \in D) \tag{4}$$

played an important role in comparison of utility representations (Gilányi-Ng-Aczél, J. Math. Anal. Appl. **304** (2005), 572-583).

Of course, also *the Pexider equations*

$$k(x+y) = f(x)g(y) \qquad ((x,y) \in D) \tag{5}$$

and

$$k(x+y) = f(x) + h(y) \qquad ((x,y) \in D) \tag{6}$$

are particular cases of (1).

As is known, (6) *can be solved by extension*, that is there exist $F, H, K\colon \mathbb{R} \longrightarrow \mathbb{R}$ satisfying

$$F = f \text{ on } D_1\,, \qquad H = h \text{ on } D_2\,, \qquad K = k \text{ on} D_+$$

and

$$K(u+v) = F(u) + H(v) \qquad \text{for } (u,v) \in \mathbb{R}^2.$$

Surprisingly, *for* (5) *such an extension is in general, not possible* (possible only if k is nowhere zero on D_+), as Fulvia Skoff showed by *counterexample.*

By a constructive method, Baker, Aczél and Skoff *found the general solutions of* (5). Similarly, *if k is not locally nonconstant, extension would not work* in general for (1), *another (constructive?) method* would be needed to find the general solution of (1).

János Aczél

4. Remark and Problem. *On the stability of the Hermite–Hadamard inequality*

The convexity of a continuous real function $f: I \longrightarrow \mathbb{R}$ defined on an open interval $I \subseteq \mathbb{R}$ is characterized by both sides of the well-known Hermite–Hadamard inequality, i.e., we have the following

Fact 1

The following three assertions are equivalent:

(i) *f is convex;*

(ii) $$\frac{1}{y-x}\int_x^y f(t)\,dt \le \frac{f(x)+f(y)}{2} \qquad (x, y \in I,\ x<y);$$

(iii) $$f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(t)\,dt \qquad (x, y \in I,\ x<y).$$

For a proof and further generalizations see the book of Niculescu and Persson [4] and the paper [1].

Related to ε-convexity, we have the next (easy to verify)

Fact 2

Assume that f is ε-convex in the following sense

(i)* $$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+\varepsilon \qquad (x, y \in I,\ t \in [0,1]).$$

Then

(ii)* $$\frac{1}{y-x}\int_x^y f(t)\,dt \le \frac{f(x)+f(y)}{2}+\varepsilon \qquad (x, y \in I,\ x<y);$$

(iii)* $$f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(t)\,dt+\varepsilon \qquad (x, y \in I,\ x<y).$$

Conversely, if (ii)* *and* (iii)* *hold then f is 4ε-convex.*

Proof. Assume that f is ε-convex. Then, integrating (i)* with respect to t over $[0,1]$, one obtains (ii)*. To deduce (iii)*, observe that (i)* implies

$$f\left(\frac{x+y}{2}\right) \le \frac{f(tx+(1-t)y)+f(ty+(1-t)x)}{2}+\varepsilon \qquad (x,y\in I,\ t\in[0,1]).$$

Integrating this inequality with respect to t over $[0,1]$, one arrives at (iii)*.

If (ii)* and (iii)* hold then

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}+2\varepsilon \qquad (x,y\in I).$$

Now, using the result of Ng and Nikodem [3], the 4ε-convexity of f follows.

A problem presented at the 5th Katowice–Debrecen Winter Seminar in Będlewo was if any of the inequalities (ii)* or (iii)* implies the $c\varepsilon$-convexity of f for some positive constant c. By a recent paper of Nikodem, Riedel and Sahoo [5], the answers to both of these questions are negative, i.e., neither (ii)* nor (iii)* imply the $c\varepsilon$-convexity of f for any $c>0$.

Briefly, in [5] the following result was proved:

1. The function $fx) := \ln x$, $(x>0)$ satisfies (ii)* with $\varepsilon=1$ but it is not c-convex for any $c>0$.
2. For all $n\in\mathbb{N}$ there exists a function f_n which satisfies (iii)* with $\varepsilon=1$ but not c-convex for any $0<c<n$.

Related to another version of approximate convexity that was studied in [2] we have

Fact 3

Assume that f is $(\varepsilon,1)$-Jensen-convex in the following sense

(i)** $$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}+\varepsilon|x-y| \qquad (x,y\in I,\ t\in[0,1]).$$

Then

(ii)** $$\frac{1}{y-x}\int_x^y f(t)\,dt \le \frac{f(x)+f(y)}{2}+\varepsilon|x-y| \qquad (x,y\in I,\ x<y);$$

(iii)** $$f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(t)\,dt+\frac{\varepsilon}{2}|x-y| \qquad (x,y\in I,\ x<y).$$

Conversely, if (ii)** *and* (iii)** *hold then f is $\left(\frac{3}{2}\varepsilon,1\right)$-Jensen-convex.*

Proof. Assume that f is $(\varepsilon,1)$-convex. Then, by the main result of [2],

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+2\varepsilon T(t)|x-y| \qquad (x,y\in I,\ t\in[0,1]),$$

where $T\colon\mathbb{R}\longrightarrow\mathbb{R}$ denotes the Takagi-function defined by

$$T(t) := \sum_{n=0}^{\infty} \frac{\operatorname{dist}(2^n t, \mathbb{Z})}{2^n} \qquad (t \in \mathbb{R}).$$

Now, integrating this inequality with respect to t over $[0,1]$, using that $\int_0^1 T(t)\,dt = \frac{1}{2}$, one obtains (ii)**. To deduce (iii)**, observe that (i)** implies, for $x, y \in I$, $t \in [0,1]$,

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(tx+(1-t)y)+f(ty+(1-t)x)}{2} + \varepsilon\,|1-2t|\,|x-y|.$$

Integrating this inequality with respect to t over $[0,1]$, one gets (iii)*.

If (ii)** and (iii)** hold then

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \frac{3}{2}\varepsilon|x-y| \qquad (x, y \in I),$$

which means the $(\frac{3}{2}\varepsilon, 1)$-Jensen-convexity of f.

Motivated by the above fact, we can raise the following

Problem

Does either (ii)** or (iii)** imply the $(c\varepsilon, 1)$-convexity of f for some positive constant c?

[1] M. Bessenyei, Zs. Páles, *Characterizations of convexity via Hadamard's inequality*, Math. Inequal. Appl. **9** (2006), 53-62.

[2] A. Házy, Zs. Páles, *On approximately midconvex functions*, Bull. London Math. Soc. **36** (2004), 339-350.

[3] C.T. Ng, K. Nikodem, *On approximately convex functions*, Proc. Amer. Math. Soc. **118** (1993), 103-108.

[4] C.P. Niculescu, L.E. Persson, *Convex Functions and Their Applications. A Contemporary Approach*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, **23**, Springer, New York, 2006.

[5] K. Nikodem, T. Riedel, P. Sahoo, *The stability problem of the Hermite–Hadamard inequality*, submitted.

Zsolt Páles

5. Remark. *Functional equations involving weighted quasi-arithmetic means and their Gauss composition* (presented by Zs. Páles)

Let $I \subset \mathbb{R}$ be a nonvoid open interval. Let $M_i\colon I^2 \longrightarrow I$ $(i = 1, 2, 3)$ be weighted quasi-arithmetic means with the property

$$M_3 = M_1 \otimes M_2\,,$$

where $\otimes$ denote the Gauss composition of M_1 and M_2. We consider the following two functional equations for the unknown $f\colon I \longrightarrow \mathbb{R}$:

(1) $f(M_1(x,y)) + f(M_2(x,y)) = f(x) + f(y) \qquad (x, y \in I)$,

(2) $2f(M_3(x,y)) = f(x) + f(y) \quad (x, y \in I)$.

It is known, that all solutions of (2) are solutions of (1), too. We give a complete characterization for the means M_i $(i = 1, 2, 3)$ so that arbitrary solution of (1) also satisfy (2).

Zoltán Daróczy

6. Remark.

In 1960 the following system of equalities was solved by Aczél and Gołąb (see [1], also [2])

$$H(s,t,x) = H(u,t,H(s,u,x)), \tag{1}$$

$$H(s,s,x) = x. \tag{2}$$

One can observe that equations (1) and (2) themselves do not need any algebraic structures in the domain of H so we could assume that the function H acts as follows $H: S \times S \times X \longrightarrow X$ where S and X are sets.

Moreover, it is known that if $(S, +)$ is a group and $F: S \times X \longrightarrow X$ satisfies the translation equation

$$F(s+t,x) = F(t, F(s,x))$$

with natural initial condition

$$F(0,x) = x, \tag{3}$$

then the function $H: S \times S \times X \longrightarrow X$ defined by

$$H(s,t,x) := F(s-t,x)$$

satisfies the system of (1) and (2).

Nevertheless, condition (3) is common but in some situations is not fulfilled by solution of the translation equation. It leads to the idea of solving equation (1) without equality (2). In this direction we have proved the following proposition.

Proposition

Let S, X be sets and let $H: S \times S \times X \longrightarrow X$ be a solution of equation (1). *Therefore there are functions $\Phi, \Psi: S \times X \longrightarrow X$ such that*

$$H(s,t,x) := \Psi(t, \Phi(s,x)) \qquad \text{for every } s,t \in S,\ x \in X. \tag{4}$$

Moreover, if $\Phi, \Psi: S \times X \longrightarrow X$ are functions such that for every $u \in S$: $\Psi(u,\cdot)^{-1} = \Phi(u,\cdot)$ on the set $\Phi(S \times X)$, then the function $H: S \times S \times X \longrightarrow X$ given by the formula (4) *is a solution of equation* (1).

[1] J. Aczél, S. Gołąb, *Funktionalgleichungen der Theorie der Geometrischen Objekte*, PWN, Warszawa, 1960.

[2] Z. Moszner, *Les equations et les inégalités liées à l'équation de translation*, Opuscula Math. **19** (1999), 19-43.

Grzegorz Guzik

7. Problem.

Is the following conjecture true?

CONJECTURE

Let the diffeomorphism $\Psi\colon (0,\infty) \longrightarrow (0,\infty)$ have no fixed point. If for every increasing self-diffeomorphism g of the closed interval $[0,\infty)$ the function

$$g_\Psi(x) := \Psi^{-1}\left(g(\Psi(x))\right), \qquad x > 0,$$

(with value 0 at zero) is again a self-diffeomorphism of $[0,\infty)$, then the derivative $D\Psi$ of Ψ is slowly varying at zero.

For making the problem more readable, let us sketch a proof of the inverse claim. For, let the diffeomorphism Ψ have slowly varying derivative, i.e., let

$$\lim_{x\to 0} \frac{D\Psi(\lambda\cdot x)}{D\Psi(x)} = 1 \qquad \text{for all } \lambda > 0.$$

Then both, Ψ and Ψ^{-1} are regularly varying with exponent 1 (we are omitting the details). Moreover for the derivative of g_Ψ we have

$$Dg_\Psi(x) = \frac{D\Psi(x)}{D\Psi(\Psi^{-1}\circ g\circ\Psi(x))}\cdot Dg(\Psi(x)).$$

With the use of $Dg(0) > 0$, by the regular variability of Ψ^{-1} we obtain that the ratio of the arguments of Ψ has a finite and positive limit as follows,

$$\lim_{x\to 0}\frac{x}{\Psi^{-1}\circ g\circ\Psi(x)} = \lim_{x\to 0}\frac{\Psi^{-1}\circ\Psi(x)}{\Psi^{-1}\circ g\circ\Psi(x)} = \lim_{x\to 0}\frac{\Psi(x)}{g\circ\Psi(x)}$$
$$= \lim_{y\to 0}\frac{y}{g(y)} = (Dg(0))^{-1} \in (0,\infty).$$

By the slow variability of $D\Psi$ and by continuity of Dg, the limit $Dg_\Psi(0^+)$ equals $1\cdot Dg(0)$. By similar arguments from $Dg_\Psi(0) = \lim_{x\to 0}\frac{g_\Psi(x)}{x}$ one can get that $Dg_\Psi(0) = Dg(0)$, too. Thus, Dg_Ψ is continuous at zero, which closes the most important step for the inverse claim.

Joachim Domsta

8. Remark.

The Theorem formulated on p. 159 of the report on the 10th ICFEI (Ann. Acad. Paed. Cracov. Studia Math., **5** (2006)) is not true. A counterexample: $F(x,y) = x$, $(x,y) \in \mathbb{R}^2$, was communicated to the speaker by Professor Karol Baron.

Bogdan Choczewski

9. Remark. *Regular variability in functional equations*

This remark is related to the lecture presented by professor Zsolt Páles (see Abstracts of Talks, page 157). Some of the results use the regular variability almost everywhere for obtaining uniqueness of the generating function from the mean, dependent additionally on some generating measure (mean of a mixed type). We want to point at the fact that the regular variability has been used already in the following (obviously much simpler) problem of restoring f from the mean defined as follows

$$M_f(x,y) := f^{-1}\left(\frac{xf(x)+yf(y)}{x+y}\right), \qquad x,y \in I \tag{1}$$

where f is a continuous and strictly monotonic function defined on an interval I of positive reals. For a point $x_0 \in I$ let the auxiliary function

$$\delta_0(u) := f(x_0+u) - f(x_0) \qquad \text{whenever } x_0+u \in I \tag{2}$$

be regularly varying at 0 with non-zero exponent, i.e., let

$$\lim_{u\to 0} \frac{\delta_0(\lambda \cdot u)}{\delta_0(u)} = \lambda^\rho \qquad \text{for } \lambda > 0, \text{ where } \rho \in (0,\infty). \tag{3}$$

(For measurable functions the definition is equivalent to the notion introduced by J. Karamata in [5]; for review of the regular variability see [1], [4] or [6], and for the facts suitable for the functional equations, see [1].) In terms of

$$\mu_0(u) := M_f(x_0, x_0+u) - x_0, \qquad w_0(u) := \frac{x_0+u}{2x_0+u}, \tag{4}$$

definition (1) implies the following homogeneous equation

$$\delta_0(\mu_0(u)) = w_0(u) \cdot \delta_0(u) \qquad \text{whenever } u \in I - x_0\,, \tag{5}$$

where $I - x_0 := \{x - x_0 : \ x \in I\}$. It is shown in [3] that

$$f(x) = f(x_0) + (f(x_1) - f(x_0)) \cdot \lim_{n\to\infty} \left(\frac{{\mu_0}^n(x-x_0)}{{\mu_0}^n(x_1-x_0)}\right)^\rho \cdot \frac{W_{0;n}(x_1-x_0)}{W_{0;n}(x-x_0)}$$

where μ_0 is given by the x_0-cut of M_f according to (4), and

$$W_{0;n}(u) := \prod_{j=0}^{n-1} w_0({\mu_0}^j(u)) \qquad \text{for } u \in I - x_0\,. \tag{6}$$

whenever $(x - x_0) \cdot (x_1 - x_0) > 0$, $x_1, x \in I$.

[1] N.H. Bingham, C.M. Goldie, J.L. Teugels, *Regular Variation*, Encyclopedia of Mathematics and Its Applications **27**, Cambridge University Press, Cambridge – New York – New Rochelle – Melbourne – Sydney, 1987.

[2] J. Domsta, *Regularly Varying Solutions of Functional Equations in a Single Variable – Applications to the Regular Iteration*, Uniwersytet Gdański, Gdańsk, 2002.

[3] J. Domsta, J. Matkowski, *Invariance of the arithmetic mean with respect to special mean-type mappings*, Aequationes Math. **71** (2006), 70-85.

[4] W. Feller, *An Introduction to Probability Theory and its Applications*, vol. **2**, John Wiley and Sons, Inc., New York, 1966.

[5] J. Karamata, *Sur un mode de croissance régulière des fonctions*, Mathematica (Cluj), **4** (1930), 38-53.

[6] E. Seneta, *Regularly varying functions*, Lecture Notes in Math. **508**, Springer-Verlag, Berlin – Heidelberg – New York, 1976.

Joachim Domsta

10. Remark. *Embedding commuting functions into a regular iteration group*

An increasing continuous self-mapping $f\colon (0,\infty) \to (0,\infty)$ is said to be Szekeresian if $f(x) < x$, for all $x > 0$, possesses the derivative at zero $D_0 f := \lim_{x\to 0} \frac{f(x)}{x}$ in $(0,1)$ and if the Szekeres principal function

$$\varphi_f(x\|y) := \lim_{n\to\infty} \frac{f^n(x)}{f^n(y)}, \qquad x > 0,$$

is continuous for some $y > 0$. If additionally f is homeomorphic onto $(0,\infty)$ then $\varphi_f(\cdot\,\|y)$ is the unique regularly varying at zero solution of the canonical Schröder equation

$$\varphi_f(f(x)\|y) = d \cdot \varphi_f(x\|y), \qquad x > 0, \text{ where } d = D_0 f$$

equal 1 at y, for arbitrary positive y (for details, see [1]). The following are considerations which were suggested to me by prof. J. Matkowski. Let f and g be commuting Szekeresian homeomorphisms. Then $\varphi_f := \varphi_f(x\|y)$, with fixed y, satisfies

$$d \cdot (\varphi_f \circ g) = \varphi_f \circ f \circ g = (\varphi_f \circ g) \circ f,$$

which means that $\varphi_f \circ g$ is again a regularly varying solution of the canonical Schröder equation for f. By a suitable uniqueness theorem, for some positive constant C

$$\varphi_f \circ g = C \cdot \varphi_f\,.$$

All the facts together show, that φ_f is the Szekeres principal function for g and that $C = D_0 g$. Let us introduce $\rho := \frac{\log D_0 g}{\log D_0 f}$.

Corollary

If φ_f is homeomorphic, then there is exactly one regular iteration group containing f and g. Moreover the iterates are given by the formula:

$$f_t = (\varphi_f)^{-1} \circ (d^t \cdot \varphi_f), \qquad t \in (-\infty, \infty),$$

and $f = f_1$ and $g = f_\rho$.

Conjecture

If $\rho \notin \mathbb{Q}$, then there is a regular iteration group $(f_t;\ t \in T)$ indexed by the (dense) additive subgroup T generated by 1 and ρ and such that $f = f_1$ and $g = f_\rho$.

[1] J. Domsta, *Regularly Varying Solutions of Functional Equations in a Single Variable – Applications to the Regular Iteration*, Uniwersytet Gdański, Gdańsk, 2002.

Joachim Domsta

List of Participants

ACZÉL János, Deptartment of Pure Mathematics, University of Waterloo, WATERLOO, Ontario, Canada N2L 3G1, e-mail: jdaczel@math.uwaterloo.ca

ADAMEK Mirosław, Katedra Matematyki, Akademia Techniczno-Humanistyczna, ul. Willowa 2, 43-309 BIELSKO-BIAŁA, Poland, e-mail: madamek@ath.bielsko.pl

BADORA Roman, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: robadora@ux2.math.us.edu.pl

BAHYRYCZ Anna, Instytut Matematyki, Akademia Pedagogiczna, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: bah@ap.krakow.pl

BARON Karol, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: baron@us.edu.pl

BARTŁOMIEJCZYK Lech, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: lech@gate.math.us.edu.pl

BATKO Bogdan, Instytut Matematyki, Akademia Pedagogiczna, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: bbatko@ap.krakow.pl

BENZ Walter, Department of Mathematics, University of Hamburg, Bundesstr. 55, 20146 Hamburg, Germany, e-mail: wbenz@mac.com

BLINOVA Elena, Mechanical and Mathematical Faculty, Nizhni Novgorod State University, Gagarin ave., 23, build. 6, NIZHNI NOVGOROD 603950, Russia, e-mail: vmb@infonet.nnov.ru

BRILLOUËT-BELLUOT Nicole, Départ. d'Inform. et de Mathématiques, Ecole Centrale de Nantes, 1 rue de la Noë, B.P. 92101, 44321 NANTES-Cedex 03, France, e-mail: Nicole.Belluot@ec-nantes.fr

BRIVE Bruno, Université des Sciences et Technologies de Lille, Mathématiques – Bâtiment M2 – Cité Scientifique, 59 655 VILLENEUVE d'ASCQ Cedex, France (p/o Prof. Anne DUVAL), e-mail: brive@math.univ-lille1.fr

BRZDĘK Janusz, Instytut Matematyki, Akademia Pedagogiczna, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: jbrzdek@ap.krakow.pl

BURAI Pál, Institute of Mathematics, University of Debrecen, Pf. 12, H-4010 DEBRECEN, Hungary, e-mail: buraip@math.klte.hu

CHMIELIŃSKI Jacek, Instytut Matematyki, Akademia Pedagogiczna, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: jacek@ap.krakow.pl

CHOCZEWSKI Bogdan, Wydział Matematyki Stosowanej, Akademia Górniczo-Hutnicza, al. Mickiewicza 30, 30-059 KRAKÓW, Poland, e-mail: smchocze@cyf-kr.edu.pl

CHUDZIAK Jacek, Instytut Matematyki, Uniwersytet Rzeszowski, ul. Rejtana 16 A, 35-959 RZESZÓW, Poland, e-mail: chudziak@univ.rzeszow.pl

CZERNI Marek, Instytut Matematyki, Akademia Pedagogiczna, ul. Podchorążych 2, 30-084 KRAKÓW, Poland

DAVISON Thomas M.K., Department of Mathematics and Statistics, McMaster University, 1280 Main Street West, HAMILTON, Ontario, Canada L8S 4K1, e-mail: davison@univmail.cis.mcmaster.ca

DOMSTA Joachim, Wydział Fizyki Techniki i Matematyki Stosowanej, Politechnika Gdańska, ul. Narutowicza 11/12, 80-952 GDAŃSK-WRZESZCZ, Poland, e-mail: jdomsta@mif.pg.gda.pl

EFREMOVA Lyudmila, Mechanical and Mathematical Faculty, University of Nizhni Novgorod, Gagarin ave., 23, building 2, 603600 NIZHNI NOVGOROD, Russia, e-mail: lef@uic.nnov.ru

FOURNIER-PRUNARET Danièle, Laboratoire d'Etude des Systèmes Automatiques et Institut National des Sciences Appliquées (INSA) 135 av. de Rangueil, 31077 TOULOUSE Cedex 4, France, e-mail: Daniele.Fournier@insa-toulouse.fr

FÖRG-ROB Wolfgang, Institut für Mathematik, Universität Innsbruck, Technikerstr. 25, 6020 INNSBRUCK, Austria, e-mail: wolfgang.foerg-rob@uibk.ac.at

GER Roman, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: romanger@us.edu.pl

GILÁNYI Attila, Institute of Mathematics, University of Debrecen, Pf. 12, H-4010 DEBRECEN, Hungary, e-mail: gilanyi@math.klte.hu

GLESKA Alina, Instytut Matematyki, Politechnika Poznańska, ul. Piotrowo 3a, 60-965 POZNAŃ, Poland, e-mail: agleska@math.put.poznan.pl

GŁAZOWSKA Dorota, Wydział Matematyki, Informatyki i Ekonometrii, Uniwersytet Zielonogórski, ul. Szafrana 4a, 65-516 ZIELONA GÓRA, Poland, e-mail: D.Glazowska@wmie.uz.zgora.pl

GUZIK Grzegorz, Wydział Matematyki Stosowanej, Akademia Górniczo-Hutnicza, al. Mickiewicza 30, 30-059 KRAKÓW, Poland, e-mail: guzikgrz@wms.mat.agh.edu.pl

ILIŠEVIĆ Dijana, Department of Mathematics, University of Zagreb, Bijenička 30, P.O. Box 335, 10 002 ZAGREB, Croatia, e-mail: ilisevic@math.hr

JABŁOŃSKA Eliza, Katedra Matematyki, Politechnika Rzeszowska, W. Pola 2, 35-959 RZESZÓW, Poland, e-mail: elizapie@prz.rzeszow.pl

JARCZYK Justyna, Wydział Matematyki, Informatyki i Ekonometrii, Uniwersytet Zielonogórski, ul. Szafrana 4a, 65-516 ZIELONA GÓRA, Poland,
e-mail: j.jarczyk@wmie.uz.zgora.pl

JARCZYK Witold, Wydział Matematyki, Informatyki i Ekonometrii, Uniwersytet Zielonogórski, ul. Szafrana 4 a, 65-516 ZIELONA GÓRA, Poland,
e-mail: wjarczyk@uz.zgora.pl

KAIRIES Hans-Heinrich, Institut für Mathematik, Technische Universität Clausthal, Erzstrasse 1, 38678 CLAUSTHAL–ZELLERFELD 1, Germany,
e-mail: kairies@math.tu-clausthal.de

KAISER Zoltán, Institute of Mathematics, University of Debrecen, Pf. 12, H-4010 DEBRECEN, Hungary, e-mail: kaiserz@math.klte.hu

KOCLĘGA-KULPA Barbara, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: koclega@ux2.math.us.edu.pl

KOCSIS Imre, Faculty of Engineering, University of Debrecen, Ótemetö u. 2-4, H-4028 DEBRECEN, Hungary, e-mail: kocsisi@mfk.unideb.hu

KRASSOWSKA Dorota, Wydział Matematyki Informatyki i Ekonometrii, Uniwersytet Zielonogórski, ul. Szafrana 4a, 65-516 ZIELONA GÓRA, Poland,
e-mail: D.Krassowska@wmie.uz.zgora.pl

LI Xiaopei, Department of Mathematics, Zhanjiang Normal University, Guangdong province, 524048 ZHANJIANG, China, e-mail: lixp27333@sina.com

LISAK Arkadiusz, Katedra Matematyki, Wyższa Szkoła Informatyki i Zarządzania, ul. Sucharskiego 2, 35-225 RZESZÓW, Poland, e-mail: alisak@wsiz.rzeszow.pl

LOSONCZI László, Department of Economic Analysis and Information, Technical University of Debrecen, Kassai üt 26, H-4028 DEBRECEN, Hungary,
e-mail: laszlo.losonczi@econ.unideb.hu

ŁYDZIŃSKA Grażyna, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: lydzinska@ux2.math.us.edu.pl

MACH Andrzej, Instytut Matematyki, Akademia Świętokrzyska, ul. Świętokrzyska 15, 25-406 KIELCE, Poland, e-mail: amach@pu.kielce.pl

MAKHROVA Elena, Mechanical and Mathematical Faculty, University of Nizhni Novgorod, Gagarin ave., 23, building 6, 603600 NIZHNI NOVGOROD, Russia,
e-mail: elena_makhrova@inbox.ru

MATKOWSKI Janusz, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: J.Matkowski@im.uz.zgora.pl

MÉSZÁROS Fruzsina, Institute of Mathematics, University of Debrecen, Pf. 12, H-4010 DEBRECEN, Hungary, e-mail: mefru@math.klte.hu

MORAWIEC Janusz, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: morawiec@ux2.math.us.edu.pl

MROWIEC Jacek, Katedra Matematyki, Akademia Techniczno-Humanistyczna, ul. Willowa 2, 43-309 BIELSKO-BIAŁA, Poland, e-mail: jmrowiec@ath.bielsko.pl

MUREŃKO Anna, Instytut Matematyki, Uniwersytet Rzeszowski, ul. Rejtana 16 A, 35-959 RZESZÓW, Poland, e-mail: aniam@univ.rzeszow.pl

MURUGAN Veerapazham, Department of Mathematics, Indian Institute of Technology Madras, CHENNAI 600 036, India, e-mail: murugan@iitm.ac.in; mrgniitm@gmail.com

NAJDECKI Adam, Instytut Matematyki, Uniwersytet Rzeszowski, ul. Rejtana 16 A, 35-959 RZESZÓW, Poland, e-mail: najdecki@univ.rzeszow.pl

NOWAKOWSKA Wiesława, Instytut Matematyki, Wydział Elektryczny, Politechnika Poznańska, ul.Piotrowo 3 a, 60-965 POZNAŃ, Poland,
e-mail: wnowakow@math.put.poznan.pl

OLBRYŚ Andrzej, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: andrzej.olbrys@wp.pl

OROSZ Ágota, Department of Mathematics, Corvinus University of Budapest, P.O. Box 489, H-1828 BUDAPEST, Hungary, e-mail: agota.orosz@uni-corvinus.hu

PANEAH Boris, Department of Mathematics, Technion, 32000 HAIFA, Israel, e-mail: peter@techunix.technion.ac.il

PAWLIKOWSKA Iwona, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: pawlikow@us.edu.pl

PÁLES Zsolt, Institute of Mathematics, University of Debrecen, Pf. 12, H-4010 DEBRECEN, Hungary, e-mail: pales@math.klte.hu

PISZCZEK Magdalena, Instytut Matematyki, Akademia Pedagogiczna, ul. Podchorążych 2, 30-084 KRAKÓW, e-mail: magdap@ap.krakow.pl

PRZEBIERACZ Barbara, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: przebieraczb@ux2.math.us.edu.pl

SABLIK Maciej, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: mssablik@us.edu.pl

SAKBAEV Vsevolod, Moscow Institute of Physics and Technology, KVM, Institutskiy per. 9, 141700 DOLGOPRUDNY, Moscow Reg., Russia, e-mail: fumi2003@mail.ru

SCHLEIERMACHER Adolf, Rablstr. 18/V, 81669 MÜNCHEN, Germany, e-mail: adsle@aol.com

SIKORSKA Justyna, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: sikorska@ux2.math.us.edu.pl

SIUDUT Stanisław, Instytut Matematyki, Akademia Pedagogiczna, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: siudut@ap.krakow.pl

SOKOŁOWSKI Dariusz, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: sokolowski@ux2.math.us.edu.pl

SOLARZ Paweł, Instytut Matematyki, Akademia Pedagogiczna, Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: psolarz@ap.krakow.pl

SZCZAWIŃSKA Joanna, Instytut Matematyki, Akademia Pedagogiczna, ul. Podchorążych 2, 30-084 KRAKÓW, Poland, e-mail: jszczaw@ap.krakow.pl

SZOSTOK Tomasz, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: szostok@ux2.math.us.edu.pl

TABOR Jacek, Instytut Matematyki, Uniwersytet Jagielloński, ul. Reymonta 4, 30-059 KRAKÓW, Poland, e-mail: tabor@im.uj.edu.pl; Jacek.Tabor@im.uj.edu.pl

TABOR Józef, Instytut Matematyki, Uniwersytet Rzeszowski, ul. Rejtana 16 A, 35-959 RZESZÓW, Poland, e-mail: tabor@univ.rzeszow.pl

VOLKMANN Peter, Institut für Analysis, Universität Karlsruhe, 76128 KARLSRUHE, Germany, no e-mail

WALORSKI Janusz, Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, 40-007 KATOWICE, Poland, e-mail: walorski@ux2.math.us.edu.pl

WĄSOWICZ Szymon, Katedra Matematyki i Informatyki, Akademia Techniczno-Humanistyczna, ul. Willowa 2, 43-309 BIELSKO-BIAŁA, Poland, e-mail: swasowicz@ath.bielsko.pl

WYRWIŃSKA Aleksandra, Instytut Matematyki, Politechnika Poznańska, ul. Piotrowo 3A, PL-60-965 POZNAŃ, Poland, e-mail: awyrwin@math.put.poznan.pl

ŻOŁDAK Marek, Instytut Matematyki, Uniwersytet Rzeszowski, ul. Rejtana 16 A, 35-959 RZESZÓW, Poland, e-mail: marek_z2@op.pl

projekt graficzny Jadwiga Burek

ISSN 1643-6555

Wydawnictwo Naukowe AP
30-084 Kraków, ul. Podchorążych 2
tel./fax (012) 662-63-83
tel. (012) 662-67-56
e-mail: wydawnictwo@ap.krakow.pl

Zapraszamy na stronę internetową
http://www.wydawnictwoap.pl

druk i oprawa
Wydawnictwo Naukowe AP, zam. 34/07

ISSN 1643-6555