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On sets associated to conditional equation of exponential function

Abstract. In the present paper we give a description and properties of the system of cones over $\mathbb{Q}$ which are one of parameters determining the solutions of the conditional equation of exponential function.

1. Introduction

F.S. Roberts, generalizing the mathematical description of choices introduced by himself in [10, 11], considers functions $f\colon \mathbb{R}(n) \longrightarrow \mathbb{R}(n)$, where $\mathbb{R}(n) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \ x_j \geq 0 \text{ for } j = 1, \ldots, n\}$, $0_n := (0, \ldots, 0) \in \mathbb{R}^n$, which satisfy, among others, the following conditional functional equation:

$$f(x) \cdot f(y) \neq 0_n \implies f(x+y) = f(x) \cdot f(y),$$

for $x, y \in \mathbb{R}(n)$. Here $x + y$ and $x \cdot y$ are defined in the following way:

$$x + y := (x_1 + y_1, \ldots, x_n + y_n), \qquad x \cdot y := (x_1 \cdot y_1, \ldots, x_n \cdot y_n).$$

Mathematical theory of this approach was developed by Z. S. Rosenbaum [12], Z. Moszner [3,7,8,9,], G.L. Forti and L. Paganoni [4,5] and A. Bahyrycz [1,2,3].

As a generalization, one may consider functions $f\colon \mathbb{R}(n) \longrightarrow \mathbb{R}(m)$ (where n, m are arbitrary natural numbers, independent of each other) satisfying the condition

$$\forall\, x, y \in \mathbb{R}(n) : \ f(x) \cdot f(y) \neq 0_m \implies f(x+y) = f(x) \cdot f(y). \tag{1}$$

It may be shown that in such a case the description of all the solutions $f = (f_1, \ldots, f_m)$ of equation (1) takes the form:

$$f_\nu(x) = \begin{cases} \exp a_\nu(x) & \text{for } x \in Z_\nu\,, \\ 0 & \text{for } x \in \mathbb{R}(n) \setminus Z_\nu\,, \end{cases} \tag{2}$$

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where $a_\nu\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ are additive functions for $\nu = 1, \ldots, m$, whereas the sets Z_ν satisfy the conditions

$$Z_1 \cup \ldots \cup Z_m = \mathbb{R}(n), \tag{3}$$

$$ij \neq 0_m \implies Z_1^{i_1} \cap \ldots \cap Z_m^{i_m} + Z_1^{j_1} \cap \ldots \cap Z_m^{j_m} \subset Z_1^{i_1 j_1} \cap \ldots \cap Z_m^{i_m j_m}, \tag{4}$$

where $i = (i_1, \ldots, i_m)$, $j = (j_1, \ldots, j_m) \in 0(m) := \{0,1\}^m \setminus \{0_m\}$,
$E_1 + E_2 := \{x + y : x \in E_1 \text{ and } y \in E_2\}$ for $E_1, E_2 \subset \mathbb{R}^n$,
$E^1 := E$, $E^0 := \mathbb{R}(n) \setminus E$ for $E \subset \mathbb{R}(n)$.
The proof of this fact is analogous to the proof for the case of $n = m$ in [7].

Because of the form of the solutions, equation (1), will be called the conditional equation of exponential function.

Let us notice that the parameters determining the solutions of equation (1) are systems of sets $Z_1, ..., Z_m$ satisfying conditions (3) and (4), as well as additive functions $a_\nu\colon \mathbb{R}^n \longrightarrow \mathbb{R}$. For this reason, it is interesting to find conditions equivalent to condition (4) under the assumption of condition (3).

2. Auxiliary lemma

Let us recall the following definition:

Definition 1

A set C is called a cone over an ordered field $\mathbb{K}$ ($\mathbb{K} = \mathbb{Q}$ or $\mathbb{K} = \mathbb{R}$), if $x + y \in C$ for all $x, y \in C$ and $\alpha x \in C$ for all $x \in C$ and $\alpha \in \mathbb{K} \cap (0, +\infty)$.

Let us observe that if the sets $Z_1, \ldots, Z_m$ satisfy condition (3), then for every $a \in \mathbb{R}(n)$ and every $k \in \{1, \ldots, m\}$ there exists a unique $i_k \in \{0,1\}$ such that $a \in Z_k^{i_k}$ (further on it will be denoted by $i_k(a)$) and at least one j such that $i_j(a) = 1$.

Lemma 1

If a system of sets $Z_1, \ldots, Z_m$ satisfies conditions (3) and (4), then for every non-empty subset $\{l_1, \ldots, l_p\}$ of the set $\{1, \ldots, m\}$ and for every $(i_{l_1}, \ldots, i_{l_p}) \in 0(p)$ the set $Z_{l_1}^{i_{l_1}} \cap \ldots \cap Z_{l_p}^{i_{l_p}}$ is a cone over $\mathbb{Q}$.

Proof.

Step 1°

Consider arbitrary $\emptyset \neq \{l_1, \ldots, l_p\} \subset \{1, \ldots, m\}$ and $(i_{l_1}, \ldots, i_{l_p}) \in 0(p)$. Take $x \in Z_{l_1}^{i_{l_1}} \cap \ldots \cap Z_{l_p}^{i_{l_p}}$ and $y \in Z_{l_1}^{i_{l_1}} \cap \ldots \cap Z_{l_p}^{i_{l_p}}$. Since $(i_{l_1}, \ldots, i_{l_p}) \neq 0_p$, there exists $\nu \in \{l_1, \ldots, l_p\}$ such that $i_\nu = 1$. Obviously, $x \in Z_1^{i_1(x)} \cap \ldots \cap Z_m^{i_m(x)}$, $y \in Z_1^{i_1(y)} \cap \ldots \cap Z_m^{i_m(y)}$ and $i_{l_k}(x) = i_{l_k}(y) = i_{l_k}$ for every $k \in \{1, \ldots, p\}$. Since $i_\nu(x) = i_\nu(y) = 1$, from condition (4) it follows that

$$\begin{aligned} x+y &\in Z_1^{i_1(x)} \cap \ldots \cap Z_m^{i_m(x)} + Z_1^{i_1(y)} \cap \ldots \cap Z_m^{i_m(y)} \\ &\subset Z_1^{i_1(x) i_1(y)} \cap \ldots \cap Z_m^{i_m(x) i_m(y)} \\ &\subset Z_{l_1}^{(i_{l_1})^2} \cap \ldots \cap Z_{l_p}^{(i_{l_p})^2} \\ &= Z_{l_1}^{i_{l_1}} \cap \ldots \cap Z_{l_p}^{i_{l_p}}. \end{aligned}$$

Therefore, for every $\emptyset \neq \{l_1, \ldots, l_p\} \subset \{1, \ldots, m\}$ and every $(i_{l_1}, \ldots, i_{l_p}) \in 0(p)$ the set $Z_{l_1}^{i_{l_1}} \cap \ldots \cap Z_{l_p}^{i_{l_p}}$ is closed under addition.

Step 2°

Take arbitrary $\emptyset \neq \{l_1, \ldots, l_p\} \subset \{1, \ldots, m\}$ and $(i_{l_1}, \ldots, i_{l_p}) \in 0(p)$. Consider the set $Z_{l_1}^{i_{l_1}} \cap \ldots \cap Z_{l_p}^{i_{l_p}}$. Take $x \in Z_{l_1}^{i_{l_1}} \cap \ldots \cap Z_{l_p}^{i_{l_p}}$ and $k \in \mathbb{N}$. The set $Z_{l_1}^{i_{l_1}} \cap \ldots \cap Z_{l_p}^{i_{l_p}}$ is closed under addition, so $kx \in Z_{l_1}^{i_{l_1}} \cap \ldots \cap Z_{l_p}^{i_{l_p}}$. Consider $\frac{1}{k}x$. Obviously,

$$\frac{1}{k}x \in Z_1^{i_1\left(\frac{1}{k}x\right)} \cap \ldots \cap Z_m^{i_m\left(\frac{1}{k}x\right)}$$

and since $Z_1 \cup \ldots \cup Z_m = \mathbb{R}(n)$, there exists $\nu \in \{1, \ldots, m\}$ such that $i_\nu(\frac{1}{k}x) = 1$.

It follows from step 1° that the set $Z_1^{i_1\left(\frac{1}{k}x\right)} \cap \ldots \cap Z_m^{i_m\left(\frac{1}{k}x\right)}$ is closed under addition, so consequently

$$x = k \cdot \Big(\frac{1}{k}x\Big) \in Z_1^{i_1\left(\frac{1}{k}x\right)} \cap \ldots \cap Z_m^{i_m\left(\frac{1}{k}x\right)}.$$

As a result, for every $l \in \{1, \ldots, m\}$ we have

$$i_l(x) = i_l\Big(\frac{1}{k}x\Big),$$

which means that

$$\frac{1}{k}x \in Z_{l_1}^{i_{l_1}} \cap \ldots \cap Z_{l_p}^{i_{l_p}}.$$

We have shown that the set $Z_{l_1}^{i_{l_1}} \cap \ldots \cap Z_{l_p}^{i_{l_p}}$ is a cone over $\mathbb{Q}$.

Remark 1

Notice that for $m \neq 1$ the converse of Lemma 1 is false, and here is an example for $m = n = 2$.

Define

$$\begin{aligned} Z_1 &= \{(x,y) \in \mathbb{R}(2) : y \leq 2x\}, \\ Z_2 &= \{(x,y) \in \mathbb{R}(2) : y \geq \tfrac{1}{2}x\}. \end{aligned}$$

It is obvious that the sets Z_1^1, Z_2^1, $Z_1^1 \cap Z_2^1$, $Z_1^1 \cap Z_2^0$ and $Z_1^0 \cap Z_2^1$ are cones over $\mathbb{R}$ (because the sets Z_1^1, Z_2^1, Z_1^0, Z_2^0 are cones over $\mathbb{R}$). Condition (4) is not satisfied, since $(1,0) \in Z_1^1 \cap Z_2^0$ and $(2,2) \in Z_1^1 \cap Z_2^1$ whereas $(3,2) \notin Z_1^1 \cap Z_2^0$.

Remark 2

Let us observe that if the sets $Z_1, \dots, Z_m$ satisfy conditions (3) and (4), then for $m \in \{1, 2\}$ the set Z_1^0 in the case of $m = 1$ and the sets Z_1^0, Z_2^0, $Z_1^0 \cap Z_2^0$ for $m = 2$ are also cones over $\mathbb{Q}$. If $m = 1$, then $Z_1 = \mathbb{R}(n)$, so $Z_1^0 = \emptyset$. If $m = 2$, then $Z_1 \cup Z_2 = \mathbb{R}(n)$, so $\mathbb{R}(n) \setminus Z_1^1 \subset Z_2^1$ and $\mathbb{R}(n) \setminus Z_2^1 \subset Z_1^1$, and consequently $Z_1^0 \subset Z_2^1$ and $Z_2^0 \subset Z_1^1$. Therefore

$$Z_1^0 \cap Z_2^1 = Z_1^0 \qquad \text{and} \qquad Z_2^0 \cap Z_1^1 = Z_2^0\,,$$

and, by Lemma 1, Z_1^0 and Z_2^0 are cones over $\mathbb{Q}$ and the set $Z_1^0 \cap Z_2^0$ is empty.

If the sets $Z_1, \dots, Z_m$ satisfy conditions (3) and (4), then for $m > 2$ not every set

$$Z_{l_1}^0 \cap \ldots \cap Z_{l_p}^0\,,$$

where $\emptyset \neq \{l_1, \dots, l_p\}$ is necessarily a cone over $\mathbb{Q}$. Here is a suitable example for $n = 2$ and $m = 3$.

Define

$$\begin{aligned} Z_1 &= \{(x, y) \in \mathbb{R}(2) :\ y \leq \tfrac{1}{2}x\}, \\ Z_2 &= \{(x, y) \in \mathbb{R}(2) :\ \tfrac{1}{2}x < y \leq 2x\}, \\ Z_3 &= \{(x, y) \in \mathbb{R}(2) :\ y > 2x\}. \end{aligned}$$

It can be easily checked that the sets Z_1, Z_2, Z_3 satisfy conditions (3) and (4) and that the set

$$Z_2^0 = \{(x, y) \in \mathbb{R}(2) :\ y \leq \tfrac{1}{2}x\} \cup \{(x, y) \in \mathbb{R}(2) :\ y > 2x\}$$

is not a cone over $\mathbb{Q}$.

3. Main result

The following theorem gives the conditions equivalent to condition (4) under the assumption of condition (3).

Theorem 1

Assume that sets $Z_1, \dots, Z_m$ satisfy condition (3). The following conditions are equivalent:

(i) *condition (4);*

(ii) *the sets $Z_1, \dots, Z_m$ are cones over $\mathbb{Q}$ for which*

$$Z_l^1 + Z_l^1 \cap Z_k^0 \subset Z_l^1 \cap Z_k^0 \tag{5}$$

for all $k, l \in \{1, \dots, m\}$ (it is enough to consider $k \neq l$);

(iii) *the sets $Z_1, \dots, Z_m$ satisfy the conditions*

$$Z_k^1 + Z_k^1 \subset Z_k^1\,, \tag{6}$$

for every $k \in \{1,\dots,m\}$ *and condition* (5) *for all* $k,l \in \{1,\dots,m\}$*;*

(iv) *for all* $x,y \in \mathbb{R}(n)$ *if there exists* $\nu \in \{1,\dots,m\}$ *such that* $x \in Z_\nu$ *and* $y \in Z_\nu$ *, then*

$$\forall\, k \in \{1,\dots,m\}: \ x+y \in Z_k \iff x \in Z_k \text{ and } y \in Z_k\,, \tag{7}$$

(v) *for all* $k,l \in \{1,\dots,m\}$ *the following implication holds*

$$ij \neq 0_2 \implies Z_k^{i_k} \cap Z_l^{i_l} + Z_k^{j_k} \cap Z_l^{j_l} \subset Z_k^{i_k j_k} \cap Z_l^{i_l j_l}, \tag{8}$$

where $i = (i_k, i_l)$*,* $j = (j_k, j_l) \in 0(2)$*.*

Proof. (i)⇒(ii) By Lemma 1, the sets $Z_1,\dots,Z_m$ are cones over $\mathbb{Q}$. Assume that $z \in Z_l^1 + Z_l^1 \cap Z_k^0$ for $k,l \in \{1,\dots,m\}$ such that $k \neq l$. It implies that there exist such $x \in Z_l^1$ and $y \in Z_l^1 \cap Z_k^0$ that $x + y = z$. Clearly, $x \in Z_1^{i_1(x)} \cap \ldots \cap Z_m^{i_m(x)}$, $y \in Z_1^{i_1(y)} \cap \ldots \cap Z_m^{i_m(y)}$ and $i_l(x) = i_l(y) = 1$. By applying condition (4) we get

$$\begin{aligned} x + y &\in Z_1^{i_1(x)} \cap \ldots \cap Z_m^{i_m(x)} + Z_1^{i_1(y)} \cap \ldots \cap Z_m^{i_m(y)} \\ &\subset Z_1^{i_1(x) i_1(y)} \cap \ldots \cap Z_m^{i_m(x) i_m(y)} \subset Z_l^{i_l(x) i_l(y)} \cap Z_k^{i_k(x) i_k(y)} \\ &= Z_l^1 \cap Z_k^0\,. \end{aligned}$$

(ii)⇒(iii) The sets $Z_1,\dots,Z_m$ are cones over $\mathbb{Q}$, so each of them is closed under addition, condition (6) is therefore satisfied for every $k \in \{1,\dots,m\}$, which completes the proof.

(iii)⇒(iv) Let $x,y \in \mathbb{R}(n)$, $\nu \in \{1,\dots,m\}$ be arbitrary and such that $x \in Z_\nu$ and $y \in Z_\nu$.

(⇐ in (7)) Lets us take an arbitrary $k \in \{1,\dots,m\}$ such that $x \in Z_k^1$ and $y \in Z_k^1$. Condition (6) yields

$$x + y \in Z_k^1 + Z_k^1 \subset Z_k^1\,.$$

(⇒ in (7)) Fix an arbitrary $k \in \{1,\dots,m\}$ such that $x + y \in Z_k^1$. If $k = \nu$, then, by assumption, $x \in Z_k^1$ and $y \in Z_k^1$. If $k \neq \nu$, then suppose that $x \in Z_k^0$ or $y \in Z_k^0$. Without loss of generality we may assume that $y \in Z_k^0$. Then, by (5), we obtain

$$x + y \in Z_\nu^1 + Z_\nu^1 \cap Z_k^0 \subset Z_\nu^1 \cap Z_k^0 \subset Z_k^0\,,$$

which is a contradiction, since the sets Z_k^1 and Z_k^0 are mutually disjoint. It means that

$$x \in Z_k^1 \qquad \text{and} \qquad y \in Z_k^1\,,$$

which finishes the proof.

(iv)⇒(v) Let $k, l \in \{1, \ldots, m\}$ and $i = (i_k, i_l)$, $j = (j_k, j_l) \in 0(2)$ be such that $ij \neq 0_2$. Consider $z \in Z_k^{i_k} \cap Z_l^{i_l} + Z_k^{j_k} \cap Z_l^{j_l}$. Then there exist $x \in Z_k^{i_k} \cap Z_l^{i_l}$ and $y \in Z_k^{j_k} \cap Z_l^{j_l}$ such that $z = x + y$. Since $ij \neq 0_2$, there exist $\nu \in \{k, l\}$ such that $i_\nu = j_\nu = 1$, and condition (7) gives

$$x + y \in Z_\nu^1 = Z_\nu^{i_\nu j_\nu}.$$

For $t \in \{k, l\} \setminus \{\nu\}$ the following cases are possible:

a) $i_t = j_t = 1$,

b) $i_t = 0$ or $j_t = 0$.

Case a). If $i_t = j_t = 1$, then $x \in Z_t^1$ and $y \in Z_t^1$, so, by condition (7), we obtain

$$x + y \in Z_t^1 = Z_t^{i_t j_t}.$$

Case b). If $i_t = 0$ or $j_t = 0$, then it is not true that $x \in Z_t^1$ and $y \in Z_t^1$. Since $x \in Z_\nu^1$ and $y \in Z_\nu^1$, from condition (iv) it follows that it is not true that $x + y \in Z_t^1$, therefore

$$x + y \in Z_t^0 = Z_t^{i_t j_t}.$$

Thus

$$x + y \in Z_k^{i_k j_k} \cap Z_l^{i_l j_l}.$$

(v)⇒(i) Let $i, j \in 0(m)$ be such that $ij \neq 0_m$. Consider $z \in Z_1^{i_1} \cap \ldots \cap Z_m^{i_m} + Z_1^{j_1} \cap \ldots \cap Z_m^{j_m}$. Then there exist $x \in Z_1^{i_1} \cap \ldots \cap Z_m^{i_m}$ and $y \in Z_1^{j_1} \cap \ldots \cap Z_m^{j_m}$ such that $z = x + y$. Since $ij \neq 0_m$, there exists $k \in \{1, \ldots, m\}$ such that $i_k = j_k = 1$.

Let $l \in \{1, \ldots, m\}$. In such a case $(i_k, i_l) \cdot (j_k, j_l) \neq 0_2$ and, by condition (v), we obtain

$$x + y \in Z_k^{i_k j_k} \cap Z_l^{i_l j_l} \subset Z_l^{i_l j_l},$$

so

$$x + y \in Z_1^{i_1 j_1} \cap \ldots \cap Z_m^{i_m j_m},$$

which completes the proof of Theorem 1.

Theorem 1 leads to the following

Corollary 1
If sets $Z_1, \ldots, Z_m$ are pairwise disjoint and satisfy condition (3)*, then condition* (4) *is equivalent to the following condition: $Z_1, \ldots, Z_m$ are cones over $\mathbb{Q}$.*

Proof. Assume that sets $Z_1, \ldots, Z_m$ are pairwise disjoint and satisfy condition (3). Then, by Theorem 1, condition (4) is equivalent to condition (ii), namely, the sets $Z_1, \ldots, Z_m$ are cones over $\mathbb{Q}$ which satisfy condition (5) for all $k, l \in \{1, \ldots, m\}$ such that $k \neq l$. Because of the fact that for all $k, l \in \{1, \ldots, m\}$ such that $k \neq l$ the sets Z_k, Z_l are disjoint, the condition

$$Z_l^1 + Z_l^1 \cap Z_k^0 \subset Z_l^1 \cap Z_k^0$$

is reduced to the condition

$$Z_l^1 + Z_l^1 \subset Z_l^1,$$

because $Z_l^1 \subset \mathbb{R}(n) \setminus Z_k^1 = Z_k^0$ (since $Z_k^1 \cap Z_l^1 = \emptyset$), and therefore $Z_l^1 \cap Z_k^0 = Z_l^1$. Similarly, the condition

$$Z_k^1 + Z_k^1 \cap Z_l^0 \subset Z_k^1 \cap Z_l^0$$

is reduced to the condition

$$Z_k^1 + Z_k^1 \subset Z_k^1 .$$

Thus, in order to verify condition (5) for two disjoint sets contained in $\mathbb{R}(n)$ it suffices to check whether these sets are closed under addition, which completes the proof.

Remark 3
Notice that if sets $Z_1, \ldots, Z_m$ satisfy condition (3), then the fact that they fulfil condition (4) implies that they satisfy condition (5). The converse implication is not true, the assumption that the sets $Z_1, \ldots, Z_m$ are cones over $\mathbb{Q}$ cannot be omitted in condition (ii). Here is an example for $m = n = 2$.

Define

$$Z_1 = \{(x, y) \in \mathbb{R}(2) : \ x \leq 1\},$$
$$Z_2 = \mathbb{R}(2).$$

The sets Z_1, Z_2 satisfy condition (3), as well as the conditions

$$Z_1^1 + Z_1^1 \cap Z_2^0 = \emptyset = Z_1^1 \cap Z_2^0 , \quad Z_2^1 + Z_2^1 \cap Z_1^0 = \{(x, y) \in \mathbb{R}(2) : x > 1\} = Z_2^1 \cap Z_1^0.$$

Therefore, condition (5) is satisfied for $k, l \in \{1, 2\}$. Obviously, the sets Z_1, Z_2 do not satisfy condition (4), since the set Z_1 is not a cone over $\mathbb{Q}$.

Remark 4
Let us have a closer look at condition (4). Notice that the number of pairs $(i, j) \in \{0, 1\}^m \times \{0, 1\}^m$ equals 4^m. The product $ij = 0_m$ if and only if $i_k = 0$ and $j_k = 0$ or $i_k = 1$ and $j_k = 0$ or $i_k = 0$ and $j_k = 1$ for every $k \in \{1, \ldots, m\}$. Therefore the number of pairs $(i, j) \in 0(m) \times 0(m)$ satisfying the condition $ij \neq 0_m$ is equal to $4^m - 3^m$.

Observe that condition (4) is symmetrical with respect to i and j, so instead of verifying $4^m - 3^m$ conditions in order to verify condition (4) we will show that it suffices to verify $\frac{4^m-3^m+2^m-1}{2}$ conditions. The number of pairs $(i,i) \in 0(m)\times 0(m)$ is equal to 2^m-1 and for all remaining pairs $(i,j)\in 0(m)\times 0(m)$ such that $ij \neq 0_m$ and $i\neq j$ it suffices to verify condition (4) for half of them; that is to say, if it is verified for the pair (i,j), then there is no need to verify it for the pair (j,i) so we have

$$2^m-1+\frac{4^m-3^m-(2^m-1)}{2}=\frac{4^m-3^m+2^m-1}{2}.$$

Notice that in order to verify condition (iii) of Theorem 1 (which is equivalent to condition (4) if condition (3) is assumed) it suffices to verify m^2 conditions (m in order to verify condition (6) and $m(m-1)$ to verify condition (5)).

m	1	2	3	4	5
$\frac{4^m-3^m+2^m-1}{2}$	1	5	22	95	406
m^2	1	4	9	16	25

Table 1

Hence, verification of condition (iii) of Theorem 1 for $m\geq 2$ requires examining less conditions then it is the case for condition (4). Let us additionally observe that the conditions obtained from (iii) of Theorem 1 are of a simpler form than the ones obtained from (4).

REMARK 5
We will show that the system of conditions obtained from condition (iii) of Theorem 1 for $n>1$ and $m>1$ is complete.

I. Consider the following sets:

$$Z_1=\{(x,y,0,\dots,0)\in\mathbb{R}(n):\ y\geq x\},$$
$$Z_2=\mathbb{R}(n)$$

and if $m\geq 3$, then

$$Z_3=\ldots=Z_m=\emptyset.$$

Notice that condition (6) is satisfied for every $k\in\{1,\dots,m\}$, since the sets $Z_1,\dots,Z_m$ are cones over $\mathbb{R}$. Condition (5) is satisfied for all $(k,l)\in\{1,\dots,m\}^2\setminus\{(1,2)\}$, and for $k=1$ and $l=2$ we have

$$\begin{aligned}\mathbb{R}(n)+\mathbb{R}(n)\cap Z_1^0 &= \mathbb{R}(n)\setminus\{(0,x_2,0,\dots,0)\in\mathbb{R}(n)\}\\ &\not\subset \mathbb{R}(n)\cap Z_1^0\\ &= Z_1^0.\end{aligned}$$

II. Consider the following sets:

$$Z_1 = \{(x, x, 0, \dots, 0) \in \mathbb{R}(n)\},$$
$$Z_2 = \mathbb{R}(n) \setminus Z_1$$

and if $m \geq 3$, then

$$Z_3 = \ldots = Z_m = \emptyset.$$

Condition (5) is satisfied for all $k, l \in \{1, \dots, m\}$, since the sets $Z_1, \dots, Z_m$ are pairwise disjoint. Condition (6) is satisfied for every $k \in \{1, \dots, m\} \setminus \{2\}$, since the sets Z_k for $k \in \{1, \dots, m\} \setminus \{2\}$ are cones over $\mathbb{R}$. Condition (6) is not satisfied for $k = 2$, because $(1, 0, \dots, 0) \in Z_2$ and $(0, 1, 0, \dots, 0) \in Z_2$ whereas $(1, 1, 0, \dots, 0) \notin Z_2$.

The independence of the conditions obtained from condition (iii) of Theorem 1, which occurs even under additional assumption that the sets $Z_1, \dots, Z_m$ are cones over $\mathbb{R}$, means that when verifying condition (iii) it is necessary to consider m^2 conditions (none of them may be omitted).

Remark 6

We are going to show that if sets $Z_1, \dots, Z_m$ fulfilling condition (3) satisfy condition (iii) of Theorem 1, then, as a consequence, they satisfy the conditions

$$Z_k^1 + Z_k^1 = Z_k^1,$$
$$Z_l^1 + Z_l^1 \cap Z_k^0 = Z_l^1 \cap Z_k^0$$

for all $k, l \in \{1, \dots, m\}$.

It suffices to prove that $Z_k^1 \subset Z_k^1 + Z_k^1$ and $Z_l^1 \cap Z_k^0 \subset Z_l^1 + Z_l^1 \cap Z_k^0$.

Fix $x \in Z_k^1$ and $y \in Z_l^1 \cap Z_k^0$. On account of Theorem 1, the system $Z_1, \dots, Z_m$ satisfies condition (3), which fact, combined with Lemma 1, implies that the sets Z_k^1 and $Z_l^1 \cap Z_k^0$ are cones over $\mathbb{Q}$, so

$$\frac{1}{2}x \in Z_k^1 \quad \text{and} \quad x = \frac{1}{2}x + \frac{1}{2}x \in Z_k^1 + Z_k^1,$$

$$\frac{1}{2}y \in Z_l^1 \cap Z_k^0 \quad \text{and} \quad y = \frac{1}{2}y + \frac{1}{2}y \in Z_l^1 \cap Z_k^0 + Z_l^1 \cap Z_k^0 \subset Z_l^1 + Z_l^1 \cap Z_k^0.$$

Hence, in conditions (5) and (6) of condition (iii) of Theorem 1 the inclusion may by replaced by equality. Analogous reasoning proves that in condition (5) of condition (ii) of Theorem 1 the inclusion may by replaced by equality.

We will show that the fact that sets $Z_1, \dots, Z_m$ satisfy condition (3) and condition (i) of Theorem 1 does not necessarily imply that the condition

$$ij \neq 0_m \implies Z_1^{i_1} \cap \ldots \cap Z_m^{i_m} + Z_1^{j_1} \cap \ldots \cap Z_m^{j_m} = Z_1^{i_1 j_1} \cap \ldots \cap Z_m^{i_m j_m},$$

is satisfied (that is to say, that the inclusion in condition (4) cannot be replaced with equality). Here is an example for $m > 1$.

Put

$$Z_1 = \mathbb{R}(n),$$
$$Z_2 = \ldots = Z_m = \emptyset.$$

Notice that

$$Z_1^1 \cap Z_2^0 \cap \ldots \cap Z_m^0 + Z_1^1 \cap Z_2^1 \cap Z_3^0 \cap \ldots \cap Z_m^0$$
$$= \emptyset \subsetneq Z_1^1 \cap Z_2^0 \cap \ldots \cap Z_m^0 = \mathbb{R}(n).$$

Similarly, in condition (v) of Theorem 1 the inclusion cannot be replaced with equality. It suffices to consider the same sets as above and put $k = 1$, $l = 2$, $i = (1, 0)$, $j = (1, 1)$ and we obtain

$$Z_1^1 \cap Z_2^0 + Z_1^1 \cap Z_2^1 = \emptyset \subsetneq Z_1^1 \cap Z_2^0 = \mathbb{R}(n).$$

Remark 7

Characterizing a function $f\colon \mathbb{R}(2) \longrightarrow \mathbb{R}(2)$ satisfying equation (1), Z. Moszner in [9] replaces condition (4) with four conditions which are equivalent to (4) under the assumption of condition (3). These are the following conditions:

(a) $Z_1 + Z_1 \subset Z_1$,

(b) $Z_2 + Z_2 \subset Z_2$,

(c) $Z_1^0 + Z_2 \subset Z_1^0$,

(d) $Z_2^0 + Z_1 \subset Z_2^0$.

Let us compare the above conditions with the ones obtained by expanding condition (iii) of Theorem 1 for the case of $n = m = 2$. In this way, we obtain two conditions from condition (5):

(a') $Z_1^1 + Z_1^1 \subset Z_1^1$,

(b') $Z_2^1 + Z_2^1 \subset Z_2^1$

and two conditions to be verified from condition (6):

(c') $Z_1^1 + Z_1^1 \cap Z_2^0 \subset Z_1^1 \cap Z_2^0$,

(d') $Z_2^1 + Z_2^1 \cap Z_1^0 \subset Z_2^1 \cap Z_1^0$.

Clearly, conditions (a) and (a'), (b) and (b') are identical. If we assume that condition (3) holds ($Z_1 \cup Z_2 = \mathbb{R}(2)$), then we have $Z_1^1 \cap Z_2^0 = Z_2^0$ and $Z_2^1 \cap Z_1^0 = Z_1^0$, and, since addition is commutative, condition (c') corresponds precisely with condition (d), and so does (d') with (c), although they differ by notation. Therefore, if we assume that condition (3) is satisfied, then condition (iii) of Theorem 1 may be treated as a generalization of the system of conditions (a), (b), (c) and (d) from [9] for the case of n, m being arbitrarily chosen natural numbers, independent of each other.

Remark 8
It is easily seen that if in Lemma 1 and Theorem 1 we delete the assumption that

$$Z_1 \cup \ldots \cup Z_m = \mathbb{R}(n)$$

and define

$$Z_i^1 := Z_i \qquad \text{and} \qquad Z_i^0 := \left(\bigcup_{j=1}^{m} Z_j \right) \setminus Z_i$$

for $i = 1, \ldots, m$, then both Lemma 1 and Theorem 1 remain valid.

4. Another properties of the systems satisfying (3) and (4)

We start from the following

Definition 2
Let $C \subset \mathbb{R}(n)$ be a cone over $\mathbb{Q}$. Denote:

$\langle C \rangle$	–	the linear subspace of $\mathbb{R}^n$ over the field $\mathbb{R}$ generated by C;
$\overline{C}$	–	the closure of the set C in $\langle C \rangle$;
C^*	–	the interior of the set C in $\langle C \rangle$;
$\operatorname{int} C$	–	the interior of the set C in $\mathbb{R}^n$.

Now we will be investigated another properties of the systems $Z_1, \ldots, Z_m$ satisfying conditions (3) and (4).

Theorem 2
Let a system $Z_1, \ldots, Z_m$ satisfy conditions (3) *and* (4). *If there exist $k, l \in \{1, \ldots, m\}$ such that $k \neq l$ and $(Z_k \cap Z_l)^* \neq \emptyset$, then*

$$Z_k \cap \langle Z_k \cap Z_l \rangle = Z_l \cap \langle Z_k \cap Z_l \rangle.$$

Proof. Let $x \in (Z_k \cap Z_l)^*$. Then there exists $r > 0$ such that the ball

$$K(x, r) \subset Z_k \cap Z_l \subset \langle Z_k \cap Z_l \rangle.$$

For $z \in Z_l \cap \langle Z_k \cap Z_l \rangle$ there exists $q \in \mathbb{Q}_+$ such that $||qz|| < r$, so

$$x + qz \in K(x, r) \subset Z_k \cap Z_l,$$

and, by condition (iv) of Theorem 1 (the condition equivalent to (4) when condition (3) is assumed), because of the fact that $x \in Z_l^1$ and $qz \in Z_l^1$ we obtain $qz \in Z_k^1$. Since $qz \in \langle Z_k \cap Z_l \rangle$, $z \in Z_k \cap \langle Z_k \cap Z_l \rangle$ (for Z_k is a cone over $\mathbb{Q}$), which gives $Z_l \cap \langle Z_k \cap Z_l \rangle \subset Z_k \cap \langle Z_k \cap Z_l \rangle$; consequently, by symmetry, we get

$$Z_l \cap \langle Z_k \cap Z_l \rangle = Z_k \cap \langle Z_k \cap Z_l \rangle.$$

Notice that if $\operatorname{int}(Z_k \cap Z_l) \neq \emptyset$, then Theorem 2 yields $Z_k = Z_l$, which results in the following

Corollary 2
If a system $Z_1, \ldots, Z_m$ satisfies conditions (3) *and* (4)*, then for all $k, l \in \{1, \ldots, m\}$ $Z_k = Z_l$ or $Z_k \cap Z_l$ is a set with empty interior in $\mathbb{R}^n$.*

Remark 9
If the sets Z_1, Z_2 satisfy conditions (3) and (4), then $\operatorname{int}(Z_1 \cap Z_2) \neq \emptyset$ if and only if $Z_1 = Z_2 = \mathbb{R}(n)$ (the only possible division of $\mathbb{R}(n)$ into two equal sets whose union is $\mathbb{R}(n)$).

Let us make the following definition

Definition 3
For every subset $\{l_1, \ldots, l_p\} \subset \{1, \ldots, n\}$ we define the set

$$B_{l_1,\ldots,l_p} := \{(x_1, \ldots, x_n) \in \mathbb{R}(n) : \; x_{l_1} = \ldots = x_{l_p} = 0\},$$

and then we define the set

$$\mathbb{B} := \{B_{l_1,\ldots,l_p} : \; \{l_1, \ldots, l_p\} \subset \{1, \ldots, n\}\}.$$

Notice that for every $B \in \mathbb{B}$ the set $\mathbb{R}(n) \setminus B$ is a cone over $\mathbb{R}$.

Lemma 2
If a set $\emptyset \neq Z \subset \mathbb{R}(n)$ is a cone over $\mathbb{Q}$, then there exists a subset $\{l_1, \ldots, l_p\}$ of the set $\{1, \ldots, n\}$ such that $Z \subset B_{l_1,\ldots,l_p}$ and exists $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n) \in Z$ such that $\overline{x}_k > 0$ for every $k \in \{1, \ldots, n\} \setminus \{l_1, \ldots, l_p\}$.

Proof. Let K be the family of all the subsets of the set $\{1, \ldots, n\}$ which satisfy the condition

$$\forall\, \mathbf{k} = \{k_1, \ldots, k_\nu\} \in K \quad Z \subset B_{k_1,\ldots,k_\nu}\,.$$

The set K is non-empty, for $Z \subset \mathbb{R}(n) = B_\emptyset$, so $\emptyset \in K$. Obviously,

$$Z \subset B_{l_1,\ldots,l_p}\,,$$

where $L = \{l_1, \ldots, l_p\} = \bigcup_{\mathbf{k} \in K} \mathbf{k}$.

Take $x = (x_1, \ldots, x_n) \in Z$. Let $M = \{m_1, \ldots, m_t\}$ be such a subset of the set $\{1, \ldots, n\} \setminus L$ that

$$\forall\, j \in \{1, \ldots, t\} \;\; x_{m_j} = 0 \qquad \text{and} \qquad \forall\, s \in (\{1, \ldots, n\} \setminus L) \setminus M \;\; x_s > 0.$$

Observe that if the set $M \neq \emptyset$, since the set $Z \not\subset B_{m_j}$, for every $j \in \{1, \ldots, t\}$ there exists $y^{m_j} = (y_1^{m_j}, \ldots, y_n^{m_j}) \in Z$ such that $y_{m_j}^{m_j} > 0$. Define

$$\overline{x} := \begin{cases} x + \sum_{j=1}^{t} y^{m_j} & \text{if } M \neq \emptyset, \\ x & \text{if } M = \emptyset, \end{cases}$$

which finishes the proof.

Theorem 3
If a system $Z_1, \dots, Z_m$ satisfies conditions (3) *and* (4) *and if there exists such $k \in \{1, \dots, m\}$ that $Z_k = \mathbb{R}(n)$, then $Z_i \in \mathbb{B}$ for every $i \in \{1, \dots, m\}$.*

Proof. Fix an arbitrary $i \in \{1, \dots, m\}$ for which $Z_i \neq \emptyset$. Lemma 2 guarantees the existence of $\{l_1, \dots, l_p\} \subset \{1, \dots, n\}$ such that $Z_i \subset B_{l_1,\dots,l_p}$ and

$$\exists\, \overline{x} = (\overline{x}_1, \dots, \overline{x}_n) \in Z_i : \ \forall\, k \in \{1, \dots, n\} \setminus \{l_1, \dots, l_p\} \quad \overline{x}_k > 0.$$

Let $z = (z_1, \dots, z_n) \in B_{l_1,\dots,l_p}$. Then, for every $k \in \{1, \dots, n\} \setminus \{l_1, \dots, l_p\}$ there exists $q_k \in \mathbb{Q}_+$ such that

$$\overline{x}_k > q_k z_k$$

and for every $j \in \{l_1, \dots, l_p\}$

$$\overline{x}_j = z_j = 0.$$

Denote

$$q = \min\{q_j : \ j \in \{1, \dots, n\} \setminus \{l_1, \dots, l_p\}\}.$$

Then, $\overline{x} - qz \in \mathbb{R}(n) = Z_k$, $qz \in \mathbb{R}(n) = Z_k$ and $\overline{x} = (\overline{x} - qz) + qz \in Z_i$. By condition (iv) of Theorem 1 (equivalent to (4) when (3)) is assumed), we obtain $qz \in Z_i$, therefore $z \in Z_i$. We have shown that $B_{l_1,\dots,l_p} \subset Z_i$, and hence, because $Z_i \subset B_{l_1,\dots,l_p}$ we obtain

$$Z_i = B_{l_1,\dots,l_p},$$

which proves the theorem.

Remark 10
Z. Moszner in [9] (see Theorem 1) proved that every function $f = (f_1, \dots, f_p)$: $\mathbb{R}(p) \longrightarrow \mathbb{R}(p)$ satisfying condition (1) with $n = m = p$ satisfies the condition

$$\forall\, x, y \in \mathbb{R}(p) : \quad f(x+y) = f(x) \cdot f(y), \tag{9}$$

if and only if there exists $k \in \{1, \dots, p\}$ for which $f_k \neq 0$ on $\mathbb{R}(p)$, or, when we use the “language of cones", if there exists $k \in \{1, \dots, p\}$ such that $Z_k = \mathbb{R}(p)$.

More exactly, for $k = 1, \dots, p$ let M_k be subsets of $\{1, \dots, p\}$ such that M_j is empty for at least one index $j \in \{1, \dots, p\}$. Let

$$Z_k := \{x \in \mathbb{R}(p) : \ \forall i \in M_k \ \ x_i = 0\},$$

i.e. $Z_k \in \mathbb{B}$. Finally, let $a_k : \mathbb{R}^p \longrightarrow \mathbb{R}$ be additive functions. It was shown in [9], Corollary 2, that all solutions of equation (9) are of the form

$$f_k(x) = \begin{cases} \exp a_k(x) & \text{for } x \in Z_k\,, \\ 0 & \text{for } x \in \mathbb{R}(p) \setminus Z_k\,. \end{cases}$$

Thus our Theorem 3 is a natural generalization expressed in the "language of cones" of Corollary 2 in [9] to the case of functions $f\colon \mathbb{R}(n) \longrightarrow \mathbb{R}(m)$ with n and m possibly distinct.

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