Annales Academiae Paedagogicae Cracoviensis

Folia 45 Studia Mathematica VI

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Some properties of i-connected sets

Abstract. The so-called i-connected sets are considered. The relationships between i-connected sets in natural topology on the plane and i-connected sets in the Hashimoto topology are studied.

1. Introduction

In order to explain where the problem comes from, let us recall that if a set is connected, then its closure is connected as well. It is obvious that in the natural topology on the straight line if a set is connected, then its interior is also connected. Note that no similar fact holds for the usual Euclidean plane. The aim of this paper is to investigate the so-called i-connected sets which are connected and have nonempty connected interiors. We present some of their properties.

Moreover, we establish the connection between i-connected sets in the Euclidean plane and i-connected sets in a stronger topology – the Hashimoto topology on the plane.

2. Some properties of i-connected sets

Let X be a nonempty set and let (X, T) stand for a topological space. For any $A \subset X$ the closure of A will be denoted by $\operatorname{cl} A$ and the interior of A by $\operatorname{int} A$.

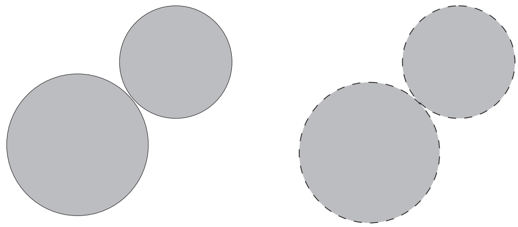
We start with the following definition.

Definition 1
Let (X, T) be a topological space. A set $A \subset X$ is said to be *i-connected* if it has a nonempty interior and both A and $\operatorname{int} A$ are connected.

Remark 1
In the natural topology on a stright line every connected set which has a nonempty interior is i-connected. Note that no similar fact holds for the Eu-

AMS (2000) Subject Classification: 28A05.

clidean plane. For instance, a set consisting of two tangent discs is connected but its interior is not (Fig. 1).



A set A is connected — The $\operatorname{int} A$ is not connected

Figure 1

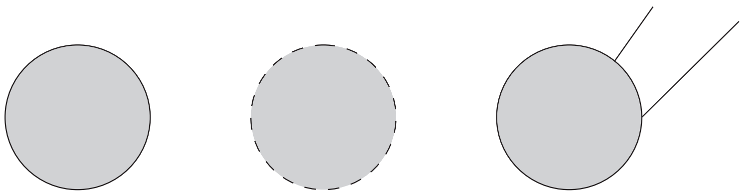
Applying the classical methods we have the following result.

Proposition
Let (X, T) be a topological space. For any set $A \subset X$ if $\operatorname{int} A$ is a connected set and $A \subset \operatorname{cl}\operatorname{int} A$, then A is i-connected.

Remark 2
For any subset $A \subset X$ in a topological space (X, T) we have the following equivalence

$$A \subset \operatorname{cl}\operatorname{int} A \iff \operatorname{cl} A = \operatorname{cl}\operatorname{int} A.$$

In particular, every regular closed set A satisfies the equality $\operatorname{cl} A = \operatorname{cl}\operatorname{int} A$.



$A = \operatorname{cl}\operatorname{int} A$ — $\operatorname{cl} A = \operatorname{cl}\operatorname{int} A$ — $\operatorname{cl} A \neq \operatorname{cl}\operatorname{int} A$

Figure 2

Theorem 1
Let (X, T) be a topological space and $A, B \subset X$ be fixed. If $\operatorname{int} A$ is connected, $\operatorname{cl} A = \operatorname{cl}\operatorname{int} A$ and B is a connected and boundary set such that $A \cap B \neq \emptyset$, then the sum $A \cup B$ is i-connected.

Proof. First, we prove that $\operatorname{int}(A \cup B) \subset \operatorname{cl} A$. To this end take an arbitrary $x \in \operatorname{int}(A \cup B)$. Then there exists a neighbourhood U_x of this point such that

$U_x \subset A \cup B$. Let V_x be an arbitrary neighbourhood of x. Notice that $U_x \cap V_x$ is a nonempty open set and $U_x \cap V_x \subset A \cup B$. Since B is a boundary set, we conclude that $U_x \cap V_x \not\subset B$. Therefore, $(U_x \cap V_x) \cap A \neq \emptyset$, so $V_x \cap A \neq \emptyset$, which implies that $x \in \operatorname{cl} A$. Since

$$\operatorname{int} A \subset \operatorname{int}(A \cup B) \subset \operatorname{cl} A = \operatorname{cl} \operatorname{int} A$$

and both $\operatorname{int} A$ and $\operatorname{cl} \operatorname{int} A$ are connected, the set $\operatorname{int}(A \cup B)$ is connected. Since the connectivity of $A \cup B$ is obvious, the proof is completed.

Now we are ready to introduce the definition of an i-connected topological space.

Definition 2
A topological space (X, T) is said to be *i-connected* if every connected set in (X, T) which has a nonempty interior is i-connected.

Remark 3
The straight line with a natural topology is an i-connected space but the Euclidean plane is not i-connected.

Remark 4
The property of being an "i-connected space" is a topological property.

Before stating the next theorem we have to introduce some notations.

Let (X, T) be a topological space and let a nonempty set $A \subset X$ be fixed. We denote by (A, T_A) the topological subspace of (X, T) with $T_A = \{U \cap A : U \in T\}$.

Let $\operatorname{cl}_A M$ and $\operatorname{int}_A M$ stand for the closure and the interior of $M \subset A$ in (A, T_A), respectively.

We see at once that the i-connectivity of a set in the sense of Definition 1 is not equivalent to the i-connectivity of the topological subspace induced on that set.

Now, let us quote the following lemma which is a consequence of Lemma 6.1 in [2].

Lemma 1
Let (X, T) be a topological space and let $A \subset X$. Then $M \subset A$ is connected in (A, T_A) if and only if M is connected in (X, T).

Theorem 2
Let (X, T) be a topological space and let a set $A \subset X$ be nonempty and such that $\operatorname{cl} A = \operatorname{cl} \operatorname{int} A$. If the space (X, T) is i-connected, then the subspace (A, T_A) is i-connected.

Proof. Let a set $M \subset A$ with a nonempty interior be connected in (A, T_A). According to Lemma 1, M is connected in (X, T). Now, we prove that the condition $\operatorname{int}_A M \neq \emptyset$ implies that $\operatorname{int} M \neq \emptyset$. Indeed, as $\operatorname{int}_A M \neq \emptyset$, therefore there exists an $x \in \operatorname{int}_A M$ and $U_x \in T$ such that $U_x \cap A \subset M$ and $U_x \cap A \neq \emptyset$; consequently $U_x \cap \operatorname{int} A \subset \operatorname{int} M$. Since $U_x \cap A \neq \emptyset$ and, by the assumption, $\operatorname{cl} \operatorname{int} A \subset \operatorname{cl} A$, we conclude that $U_x \cap \operatorname{int} A \neq \emptyset$ and finally $\operatorname{int} M \neq \emptyset$. From this and from the fact that (X, T) is i-connected we obtain that $\operatorname{int} M$ is connected in (X, T).

The inclusions $\operatorname{int} M \subset \operatorname{int}_A M \subset M$ imply that $\operatorname{int}_A M$ is connected in (X, T) and, by Lemma 1, connected in (A, T_A) which proves the theorem.

We can observe that the above theorem means that the i-connectivity is hereditary with respect to the property $\operatorname{cl} A = \operatorname{cl} \operatorname{int} A$, in particular, with respect to open sets.

3. The i-connected sets in the Hashimoto topology

Let (X, T) and (X, T') be topological spaces such that $T \subset T'$. It is easy to check that for any set $A \subset X$ we have:

(1) the interior of the set A in (X, T) is contained in the interior of A in (X, T');

(2) the closure of A in (X, T') is contained in the closure of A in (X, T);

(3) if $M \subset X$ is connected in the stronger topology (X, T'), then M is connected in the weaker topology (X, T).

Here and in what follows $(\mathbb{R}^2, T_d)$ denotes the Euclidean plane, i.e.,

$$T_d = \left\{U \subset \mathbb{R}^2 :\ \forall\, x \in U\ \exists\, r > 0 \quad K(x, r) \subset U\right\},$$

where $K(x, r)$ stands for the open ball centered at x and with the radius r in the Euclidean metric space.

From now on $(\mathbb{R}^2, T^*)$ stands for the Euclidean plane with the Hashimoto topology, i.e.,

$$T^* = \{U \backslash F \subset \mathbb{R}^2 :\ U \in T_d \ \text{ and } \ \mu(F) = 0\},$$

where μ denotes the Lebesgue measure in $\mathbb{R}^2$.

For simplicity we write $\operatorname{int} A$ ($\operatorname{cl} A$) for the interior (the closure) of A in the space $(\mathbb{R}^2, T_d)$ and $\operatorname{int}_* A$ ($\operatorname{cl}_* A$) for the interior (the closure) of A in $(\mathbb{R}^2, T^*)$.

Since $T_d \subset T^*$, by (1)-(3), we have:

(4) $\forall\, A \subset \mathbb{R}^2 \quad \operatorname{int} A \subset \operatorname{int}_* A$,

(5) $\forall\, A \subset \mathbb{R}^2 \quad \operatorname{cl}_* A \subset \operatorname{cl} A$,

(6) if $M \subset \mathbb{R}^2$ is connected in the Hashimoto topology $(\mathbb{R}^2, T^*)$, then M is connected in $(\mathbb{R}^2, T_d)$.

Lemma 2
If $U \subset \mathbb{R}^2$ *is an open set in* $(\mathbb{R}^2, T_d)$*, then* $\operatorname{cl} U = \operatorname{cl}_* U$*.*

Proof. If $U = \emptyset$, then the assertion is obvious. Assume that $U \neq \emptyset$. According to (5), it is enough to prove that $\operatorname{cl} U \subset \operatorname{cl}_* U$. Let $x \in \operatorname{cl} U$ and $V_x \in T^*$ be a neighbourhood of x. There exist a neighbourhood $U_x \in T_d$ and $F \subset \mathbb{R}^2$ such that $\mu(F) = 0$ and $V_x = U_x \backslash F$. Then $V_x \cap U = (U_x \cap U) \backslash F$. As $U_x \cap U$ is nonempty and open in $(\mathbb{R}^2, T_d)$, we have $\mu(U_x \cap U) > 0$ and finally $(U_x \cap U) \backslash F \neq \emptyset$ which proves our claim.

The main result reads now as follows.

Theorem 3
If a set $A \subset \mathbb{R}^2$ *has a nonempty and connected interior in the space* $(\mathbb{R}^2, T_d)$ *and* $\operatorname{cl} A = \operatorname{cl} \operatorname{int} A$*, then* A *is i-connected in the space* $(\mathbb{R}^2, T^*)$*.*

Proof. We start with the observation that if $\operatorname{int} A$ is open and connected in $(\mathbb{R}^2, T_d)$, then it is connected in $(\mathbb{R}^2, T^*)$. Indeed, suppose that $\operatorname{int} A$ is disconnected in $(\mathbb{R}^2, T^*)$. Then, according to the assumption that $\operatorname{int} A \in T^*$, there exist two nonempty sets $U_1, U_2 \in T_d$ and $F_1, F_2 \subset X$ such that $\mu(F_1) = 0 = \mu(F_2)$, $\operatorname{int} A = (U_1 \backslash F_1) \cup (U_2 \backslash F_2)$ and $(U_1 \backslash F_1) \cap (U_2 \backslash F_2) = \emptyset$. Hence $\operatorname{int} A \subset U_1 \cup U_2$ and $U_1 \cap U_2 \subset F_1 \cup F_2$. A reasoning similar to that in the proof of Lemma 2 shows that $U_1 \cap U_2 = \emptyset$ which means that $\operatorname{int} A$ is disconnected in $(\mathbb{R}^2, T_d)$, contrary to the assumption.

By (4), we have

$$\operatorname{int} A \subset \operatorname{int}_* A \subset A \subset \operatorname{cl} A.$$

Moreover, in the view of the assumption and Lemma 2, we get

$$\operatorname{cl} A = \operatorname{cl} \operatorname{int} A = \operatorname{cl}_* \operatorname{int} A$$

and, consequently,

$$\operatorname{int} A \subset \operatorname{int}_* A \subset A \subset \operatorname{cl}_* \operatorname{int} A.$$

Since $\operatorname{int} A$ and $\operatorname{cl}_* \operatorname{int} A$ are connected in $(\mathbb{R}^2, T^*)$, the sets $\operatorname{int}_* A$ and A are connected in $(\mathbb{R}^2, T^*)$, and the proof is completed.

Note that every set satisfying the assumptions of the above theorem is i-connected in $(\mathbb{R}^2, T_d)$. Therefore we obtain the following corollary.

Corollary
If a set $A \subset \mathbb{R}^2$ *is i-connected in the space* $(\mathbb{R}^2, T_d)$ *and* $\operatorname{cl} A = \operatorname{cl} \operatorname{int} A$*, then* A *is i-connected in the space* $(\mathbb{R}^2, T^*)$*.*

Remark 5
The assumption that $\mathrm{cl}\, A = \mathrm{cl}\,\mathrm{int}\, A$ is essential. In fact, the example below (Fig. 3) shows that there exist sets which are i-connected in $(\mathbb{R}^2, T_d)$ but are not i-connected in $(\mathbb{R}^2, T^*)$.

Note that the set from Figure 3 is not connected in the Hashimoto topology because it is the union of two nonempty disjoint and closed subsets that are represented by the disc and the line segment without one of the end points.

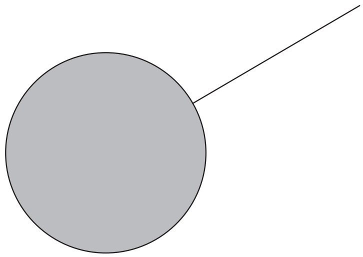


Figure 3

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Received: 20 November 2006; final version: 22 January 2007; available online: 5 March 2007.