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Randomly $C_n \cup C_m$ graphs

Abstract. A graph G is said to be a randomly H graph if and only if any subgraph of G without isolated vertices, which is isomorphic to a subgraph of H, can be extended to a subgraph F of G such that F is isomorphic to H. In this paper the problem of randomly H graphs, where $H = C_n \cup C_m$, $m \neq n$, is discussed.

1. Introduction

In 1951 Ore [12] studied arbitrarily traceable graphs, which were later referred to as randomly eulerian graphs. This concept was later extended by Chartrand and White [5], and Erickson [8]. In 1968 Chartrand and Kronk [2] introduced and characterized the concept of randomly hamiltonian graphs. Analogous questions were studied in [4], [6], [7], and [12].

In 1986 Chartrand, Oellermann, and Ruiz [3] generalized these concepts and introduced the term 'randomly H graph' as follows: Let G be a graph containing a subgraph H without isolated vertices. Then G is called a randomly H graph if whenever F is a subgraph of G without isolated vertices that is isomorphic to a subgraph of H, then F can be extended to a subgraph H_1 of G such that H_1 is isomorphic to H.

The graph G shown in Figure 1 is not randomly P_4 since the subgraph F of G cannot be extended to a subgraph of G isomorphic to P_4, while the graph $K_{3,3}$ is randomly P_4 as well as randomly C_4.

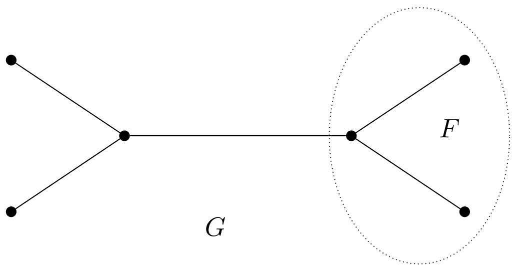

Figure 1

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Every nonempty graph is randomly K_2, while every graph G without isolated vertices is a randomly G graph. K_n is randomly H for every $H \subseteq K_n$. The graph $K_{3,3}$ is randomly H for every subgraph H of $K_{3,3}$ (see [3], Theorem 1).

The requirement that both H and F are without isolated vertices follows from [3]. That is why we consider that both H and F are free of isolated vertices.

2. Preliminaries

The general question is 'For what classes of graphs H is it possible to characterize all those graphs G that are randomly H?'.

In [10] the characterization of randomly $K_{r,s}$ graphs was given, but in terms of H-closed graphs. In [1] Alavi, Lick, and Tian studied randomly complete n-partite graphs and characterized them.

The problem of characterization of randomly H graphs, where H is r-regular graph on p vertices, was given by Tomasta and Tomová (see [14]). In general, the characterization of such graphs seems to be difficult. However, there exist several results for some special values of r and p.

Theorem A (see Sumner [13])
Let H be a 1-regular graph on $2p$ vertices. A graph G on $2p$ vertices is randomly H (perfect matchable) if and only if

1. $G = K_{2p}$, *or*
2. $G = K_{p,p}$, *or*
3. $G = H$.

This is a list of results about randomly 2-regular connected graphs, which means randomly C_n graphs.

Theorem B (see Tomasta and Tomová [14])
Let G be a p-vertex graph which is randomly C_n, $n > 4$, $p > n$. Then $G = K_p$.

Theorem C (see Chartrand, Oellermann, and Ruiz [3])
A graph G is randomly C_3 if and only if each component of G is a complete graph of order at least 3.

Theorem D (see Chartrand, Oellermann, and Ruiz [3] and also Híc [10])
A graph G is randomly C_4 if and only if

1. $G = K_p$, *where* $p \geq 4$, *or*
2. $G = K_{r,s}$, *where* $2 \leq r \leq s$.

THEOREM E (see Chartrand, Oellermann, and Ruiz [3])
A graph G is randomly C_n, $n \geq 5$, if and only if

1. $G = K_p$, *where* $p \geq n$, *or*
2. $G = C_n$, *or*
3. $G = K_{\frac{n}{2},\frac{n}{2}}$ *and* n *is even.*

The following is a list of results about randomly 2-regular disconnected graphs, more specifically randomly $2C_n = C_n \cup C_n$ graphs.

THEOREM F (see Híc and Pokorný [11])
A graph G is randomly $2C_3$ if and only if

1. $G = K_p$, $p \geq 6$, *or*
2. $G = K_{p_1} \cup K_{p_2} \cup \ldots \cup K_{p_n}$, *where* $n \geq 2$, $p_i = 3$ *or* $p_i \geq 6$.

THEOREM G (see Híc and Pokorný [11])
A graph G is randomly $2C_{2n+1}$, where $n \geq 2$, if and only if

1. $G = 2C_{2n+1}$, *or*
2. $G = 2K_{2n+1}$, *or*
3. $G = C_{2n+1} \cup K_{2n+1}$, *or*
4. $G = K_p$, $p \geq 2(2n+1)$.

THEOREM H (see Híc and Pokorný [11])
A graph G is randomly $2C_4$ if and only if

1. $G = K_{r,s}$, *where* $4 \leq r \leq s$, *or*
2. $G = 2C_4$, *or*
3. $G = 2K_4$, *or*
4. $G = C_4 \cup K_4$, *or*
5. $G = K_p$, *where* $p \geq 8$.

THEOREM I (see Híc and Pokorný [11])
A graph G is randomly $2C_{2n}$, where $n \geq 3$, if and only if

(i) $G = 2K_{2n}$, *or*
(ii) $G = 2C_{2n}$, *or*
(iii) $G = 2K_{n,n}$, *or*
(iv) $G = C_{2n} \cup K_{n,n}$, *or*
(v) $G = C_{2n} \cup K_{2n}$, *or*

(vi) $G = K_{n,n} \cup K_{2n}$, *or*

(vii) $G = K_{2n,2n}$, *or*

(viii) $G = K_p$, $p \geq 4n$.

This paper deals with randomly 2-regular graphs H, where $H = C_n \cup C_m$, $n \neq m$ (both components of H are circuits).

All the terms used in this paper can be found in [9]. Especially, if H is a subgraph of G, we will use $G - H = \langle V(G) - V(H) \rangle$ to denote the induced subgraph of the graph G with the vertex set $V(G) - V(H)$.

3. Results

Lemma 1

Let G be a disconnected randomly $C_n \cup C_m$ graph, where $3 \leq n < m$. Then G has two components. Moreover, one of the components has n vertices and the other one has m vertices.

Proof. First, we will prove that G has two components.

a) Let G have k components, where $k > 2$. Let us construct a subgraph H of G which consists of three edges which belong to three different components of G. The subgraph H must be isomorphic to some subgraph of $C_n \cup C_m$. However, the subgraph H cannot be extended to $C_n \cup C_m$, a contradiction.

b) Let G have two components. Now we will prove that one of the components of G has n vertices and the other one has m vertices. We will discuss four different cases.

1. Obviously none of the components has less than n vertices. Moreover, one of the components has at least m vertices.

2. Let one of the components of G have k vertices, $k > m$. Let us construct a subgraph $H_1 = P_{m-2} \cup P_3$ of the component. Let H_2 be a subgraph of the other component of G which is isomorphic to P_2 . Then $H_1 \cup H_2$ should be isomorphic to a subgraph of $C_n \cup C_m$, but it cannot be extended to $C_n \cup C_m$, a contradiction. Thus none of the components of G has more then m vertices.

3. Let both components of G have m vertices. Let us construct a subgraph $H_1 = P_{n-\lfloor \frac{n}{2} \rfloor} \cup P_{\lfloor \frac{m}{2} \rfloor}$ of the first component of G and a subgraph $H_2 = P_{m-\lfloor \frac{m}{2} \rfloor} \cup P_{\lfloor \frac{n}{2} \rfloor}$ of the second component of G. Then $H_1 \cup H_2$ must be isomorphic to a subgraph of $C_n \cup C_m$, but it cannot be extended to $C_n \cup C_m$, a contradiction.

4. Let one of the components of G has k vertices, where $n < k < m$. According to parts 1 and 2 of this proof the other component of G has m vertices. Let us construct a subgraph $F = P_k \cup P_n$ of G, where P_k is a subgraph of the component of G with k vertices. Then F ought to be isomorphic to a subgraph of $C_n \cup C_m$, but it cannot be extended to $C_n \cup C_m$, a contradiction.

According to a) and b), G has two components. Moreover, one of them has n vertices and the other one has m vertices.

Lemma 2
Let G be a disconnected randomly $C_n \cup C_m$ graph, where $3 \le n < m$. Then

(i) $G = C_n \cup C_m$ *, or*

(ii) $G = K_n \cup C_m$ *, or*

(iii) $G = K_{\frac{n}{2},\frac{n}{2}} \cup C_m$ *, where n is even.*

Proof. Let G be a disconnected randomly $C_n \cup C_m$ graph. According to Lemma 1, G has two components with n and m vertices. Obviously, one of the components is randomly C_n and the other one is randomly C_m . According to Theorem D and Theorem E, the first component can be C_n , K_n , or $K_{\frac{n}{2},\frac{n}{2}}$, where n is even, and the other component can be C_m , K_m , or $K_{\frac{m}{2},\frac{m}{2}}$, where m is even. We will prove that the second component can be neither K_m , nor $K_{\frac{m}{2},\frac{m}{2}}$. Let us construct a subgraph $F = C_n$ of this component. Then F is also a subgraph of G which is isomorphic to a subgraph of $C_n \cup C_m$, but it cannot be extended to $C_n \cup C_m$, a contradiction.

Lemma 3
Let G be a connected randomly $C_n \cup C_m$ graph, where $3 \le n < m$. If $|V(G)| > m + n$, then G is a complete graph.

Proof. Let H be a subgraph of G isomorphic to C_n . Let $G' = G - H$. Obviously G' is randomly C_m . We will prove that G' is complete. Since $|V(G')| > m$, according to Theorem B, $G' = K_p$, $p > m$. Now we will prove that $G'' = \langle V(H) \rangle$ is complete, too. Let $H' = C_n$ be a subgraph of G'. If $G''' = G - H'$, then $G'' \subseteq G'''$. According to Theorem B, G''' is complete. Then G'' is complete, too. Finally, we will prove that for every $u \in V(G')$, $v \in V(G'')$ the graph G contains the edge $\{u, v\}$. Let us choose $u - v$ path on m vertices. Since both G' and G'' are complete and G is connected, the path always exists and can be extended to a graph which is isomorphic to $C_n \cup C_m$ only if we add the edge $\{u, v\}$ to the path. Since both u and v are arbitrary vertices, G is complete.

Lemma 4
Let G be a connected randomly $C_n \cup C_m$ graph, where $4 \le n < m$, $|V(G)| = m + n$, and both m and n are even. If G contains a proper subgraph which is isomorphic to $K_{\frac{m+n}{2},\frac{m+n}{2}}$, then G is a complete graph.

Proof. Let $V(K_{\frac{m+n}{2},\frac{m+n}{2}}) = \{u_1, u_2, \ldots, u_{\frac{m+n}{2}}\} \cup \{v_1, v_2, \ldots, v_{\frac{m+n}{2}}\}$. Let $\{u_i, u_j\} \in E(G)$ and $\{u_i, u_j\} \notin E(K_{\frac{m+n}{2},\frac{m+n}{2}})$. Let v_k , v_t be arbitrary vertices that belong to the different partition set than u_i and u_j. Let us construct the path $v_k, u_i, u_j, v_s, u_s, \ldots, v_r, u_r, v_t$ of the length m. Since G is randomly $C_n \cup C_m$, the path can be extended to C_m only if we add the edge $\{v_k, v_t\}$. Since both v_k and v_t are arbitrary vertices, $\{v_k, v_t\} \in E(G)$ for every k, t. If we use a similar method with the edge $\{v_i, v_j\} \in E(G)$, we will prove that G is a complete graph.

Lemma 5

Let G be a connected randomly $C_n \cup C_m$ graph, where $3 \le n < m$, $|V(G)| = m + n$. Then

(i) $G = K_{\frac{m+n}{2}, \frac{m+n}{2}}$ *if m and n are even, or*

(ii) $G = K_{m+n}$.

Proof. Let H be a subgraph of G isomorphic to C_n. Let $G' = G - H$. Obviously G' is randomly C_m. We will discuss three cases.

1. If m is odd, then according to Theorem E we have $G' = C_m$ or $G' = K_m$. We will prove that G' cannot be C_m. Assume the contrary. Let G' be isomorphic to C_m. Then $V(G') = \{v_1, v_2, \ldots, v_m\}$ and $E(G') = \{\{v_i, v_{i+1}\};\ i = 1, 2, \ldots, m-1\} \cup \{\{v_m, v_1\}\}$. Since G is connected, there exists an edge $\{u, v\}$, where $u \in V(H)$, $v \in V(G')$. Without loss of generality we may assume that $v = v_1$. Let us construct the path $u, v_1, v_2, \ldots, v_{m-1}$. This path can be extended to C_m only by adding the edge $\{v_{m-1}, u\}$. Now let us construct the path $v_m, v_{m-1}, u, v_1, v_2, \ldots, v_{m-3}$. This path can be extended to C_m only by adding $\{v_{m-3}, v_m\}$. So G' is not isomorphic to C_m, a contradiction. Then $G' = K_m$. If we choose a subgraph C_n of G' and we use similar ideas that we used in the proof of Lemma 3, we will prove that G is complete.

2. Similarly, if n is odd, then G is complete, too.

3. Let both m and n be even. According to Theorem E we have $G' = C_m$, $G' = K_m$, or $G' = K_{\frac{m}{2}, \frac{m}{2}}$. It is easy to prove that G' cannot be C_m. In case $G' = K_m$ we can prove that G is complete. Let us consider that $G' = K_{\frac{m}{2}, \frac{m}{2}}$. Let $G'' = \langle V(H) \rangle$. Note that G is randomly $C_n \cup C_m$. If we choose a subgraph $H' = C_m$ of G', then according to Theorem E it must be $G'' = C_n$, or $G'' = K_n$, or $G'' = K_{\frac{n}{2}, \frac{n}{2}}$. Using similar ideas as in the part 1 of this proof we can prove that G'' cannot be C_n. If $G'' = K_n$, then G is complete. Now let us assume that $G'' = K_{\frac{n}{2}, \frac{n}{2}}$. Let the vertex sets of G' and G'' be $V(G') = \{u_1, u_2, \ldots, u_{\frac{m}{2}}\} \cup \{v_1, v_2, \ldots, v_{\frac{m}{2}}\}$ and $V(G'') = \{w_1, w_2, \ldots, w_{\frac{n}{2}}\} \cup \{t_1, t_2, \ldots, t_{\frac{n}{2}}\}$. As G is a connected randomly $C_m \cup C_n$ graph, there exists at least one edge which connects a vertex of G' with a vertex of G''. Let us denote this edge $\{u_i, w_j\}$. We will prove that for every $r \in \{1, 2, \ldots, \frac{m}{2}\}$ and $s \in \{1, 2, \ldots, \frac{n}{2}\}$, $\{v_r, t_s\} \in E(G)$. Let us consider a path of the length m in G' and G'' that starts in v_r, ends in t_s, and contains the edge $\{u_i, w_j\}$. This path always exists. Since G is randomly $C_m \cup C_n$, the path

can be extended to C_m only by adding the edge $\{v_r, t_s\}$. Since r and s were arbitrary, we proved that every vertex from $\{v_1, v_2, \ldots, v_{\frac{m}{2}}\}$ is connected with every vertex from $\{t_1, t_2, \ldots, t_{\frac{n}{2}}\}$. If we repeat a similar procedure with the edge $\{v_r, t_s\}$ we can prove that every vertex from $\{u_1, u_2, \ldots, u_{\frac{m}{2}}\}$ is connected with every vertex from $\{w_1, w_2, \ldots, w_{\frac{n}{2}}\}$. It means that if G is randomly $C_n \cup C_m$ and both m and n are even, then $K_{\frac{m+n}{2},\frac{m+n}{2}} \subseteq G \subseteq K_{m+n}$. According to Lemma 4, $G = K_{\frac{m+n}{2},\frac{m+n}{2}}$ or $G = K_{m+n}$.

The following theorem summarizes the characterization of randomly $C_n \cup C_m$ graphs. It is easy to prove that each of the graphs that are mentioned in the theorem is randomly $C_n \cup C_m$. The rest of the theorem follows from Lemma 1-5.

THEOREM 1
A graph G is randomly $C_n \cup C_m$, where $3 \leq n < m$ if and only if

(i) $G = C_n \cup C_m$, *or*

(ii) $G = K_n \cup C_m$, *or*

(iii) $G = K_{\frac{n}{2},\frac{n}{2}} \cup C_m$ *where n is even, or*

(iv) $G = K_{\frac{m+n}{2},\frac{m+n}{2}}$ *where both m and n are even, or*

(v) $G = K_p$, *where $p \geq m + n$.*

Conclusion

In the paper a characterization of randomly H graphs where $H = C_n \cup C_m$ is given. The case of 2-regular randomly H graphs, where H is a 2-regular graph which contains more than two components, remains open.

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