FELIKS BARAŃSKI, JAN LUCHTER

The temperature vawes in the $(n+1)$-dimensional half time–space

Dedicated of Professor Zenon Moszner with best wishes on his 60-th birthday

1. Introduction. The subject of the paper is the construction of the periodic solution of the equation

$$Pu(x,t) = 0,\ x = (x_1,\dots,x_n),\ P = \Delta - D_t, \Delta = \sum_1^n D^2_{x_i}, \tag{1}$$

in the domain

$$D = \left\{(X,t): X = (x_1,\dots,x_{n-1}) \in E_{n-1},\ x_n > 0,\ t \in (-\infty,\infty)\right\},$$

satisfying the boundary condition

$$u(X,0,t) = h(X,t) \text{ for } (X,t) \in D_1 = \left\{(X,t): X \in E_{n-1}, t \in (-\infty,\infty)\right\}. \tag{2}$$

We shall prove that if h is periodic with respect to t, then the solution u(x,t) is also periodic with respect to t.

In the monograph [3], p. 222, the similar problem for the domain $D = \{(x,t): x \in (0,\infty),\ t \in (-\infty,\infty)\}$, was treated. The periodic solution u is called the temperature wave. In the monograph [1], pp.102-104, the similar problem for the domain $D = \{(x,t): x\in(-a,a),\ t \in (-\infty,\infty)\}$ was considered.

To the construction of the solution of the problem (1), (2), we shall apply the convenient Green function G.

2. Some example.

Let

$$u(x,t) = \left(\exp\left(a \sum_{i=1}^{n} x_i\right)\right) \cos(t+ax_i),\ a = (-2\sqrt{\pi})^{-1}.$$

The function u is the solution of the equation (1) and is periodic with respect to t.

Indeed. Let

$$P_{x_i,t} = D^2_{x_i} - D_t,\ U_i(x_i,t) = \exp(ax_i)\cos(t+ax_i),\ i=1,\dots,n,$$

$$w(x,t) = \prod_{i=1}^{n} U_i(x_i,t),$$

$$w_1(x,t) = \prod_{i=2}^{n} U_i(x_i,t),\ w_n(x,t)$$

$$= \prod_{l=1}^{n-1} U_l(x_l,t), \quad w_l(x,t) = \prod_{\substack{k=1\\ k\neq l}}^{n} U_k(x_k,t),$$

$i=1,\ldots,n.$

We have

$$P_{x_l,t} U_l(x_l,t) = 0, \quad i=1,\ldots,n,$$

$$Pw(x,t) = \sum_{l}^{n} \left[(D^2_{x_l} - D_t)U_l(x_l,t)\right] w_l(x,t) = 0.$$

REMARK. For $x_n = 0$ we obtain the function

$$u(X,0,t) = \left(\exp\left(a \sum_{l}^{n-1} x_l\right)\right)\left(\prod_{l}^{n-1} \cos(t+ax_l)\right)\cos t,$$

which is periodic with respect to t.

3. Green function.

Let $y = (y_1,\ldots,y_n)$, $\qquad Y = (y_1,\ldots,y_{n-1})$.
It is known that the function

$$G(x,t;y,s) = G_1(x,t;y,s) - G_2(x,t;y,s),$$

where

$$G_1(x,t;y,s) = (t-s)^{-n/2}\exp(B(t,s)r^2(X,Y)) \exp(B(t,s)(x_n-y_n)^2),$$

$$G_2(x,t;y,s) = (t-s)^{-n/2}\exp(B(t,s)r^2(X,Y)) \exp(B(t,s)(x_n+y_n)^2),$$

$$r^2(X,Y) = \sum_{1}^{n-1} (x_i - y_i)^2,$$

$$B(t,s) = (-4(t-s))^{-1} \qquad \text{for } s < t,$$

and

$$G_1(x,t;y,s) = G_2(x,t;y,s) = 0 \qquad \text{for } s \geq t,$$

is the Green function for the equation (1), for the domain D, and for Dirichlet boundary data.

4. Uniqueness theorem.

Denote by (U) the class of all functions $u \in C^{2,1}(D)$, such that $\| u \|_D \leq C$, C being a positive constant.

THEOREM 1. *If the functions* u_1, $u_2 \in$ (U) *are the solutions of the* (1), (2) *problem, then* $u_1 \equiv u_2$ *in* D.

Proof. Let

$$U(x,t) = u_1(x,t) - u_2(x,t)$$

We have

$$PU(x,t) = 0 \qquad \text{for } (x,t) \in D, \tag{3}$$

$$U(x,t) = 0 \qquad \text{for } (x,t) \in D_1 \tag{4}$$

Let us consider the following Cauchy problem for the equation (3), with the initial condition

$$U(x,t)\Big|_{t=T} = U(x,T) \qquad \text{for } x \in E_n^+. \tag{5}$$

By [2], vol. I. p. 458, the solution of the problem (3), (5) is of the form

$$U(x,t) = A \int_{E_n^+} U(y,T)G(x,t;y,T)dy = I_1 + I_2,$$

where

$$I_1(x,t) = A \int_{E_{n-1}} \int_{y_n>0} U(y,T)(t-T)^{-n/2+1}$$

$$\times \exp(B(t,T)r^2(X,Y))(t-T)^{-1/2} \exp(B(t,T)(x_n-y_n)^2)dYdy_n,$$

$$I_2(x,t) = -A \int_{E_{n-1}} \int_{y_n>0} U(y,T)(t-T)^{\frac{-n}{2}+1}$$

$$\exp(B(t,T)r^2(X,Y)) \ (t-T)^{-1/2}\exp(B(t,T)(x_n+y_n)^2)dYdy_n,$$

$$A = (2\sqrt{\pi})^{-n}.$$

The solution of the problem (3), (5) is unique.

We have

$$|I_1 + I_2| \le AC \int_{E_{n-1}} (t-T)^{\frac{-n}{2}+1} \exp(B(t,T)r^2(X,Y))dY \, [J_1 + J_2],$$

where

$$J_1 = \int_{y_n>0} (t-T)^{-1/2}\exp(B(t,T)(x_n-y_n)^2)dy_n,$$

$$J_2 = -\int_{y_n>0} (t-T)^{-1/2}\exp(B(t,T)(x_n+y_n)^2)dy_n.$$

Applying in the integrals J_1, J_2 the change of the integral variable

$$x_n - y_n = 2\sqrt{t-T}\ z,\ x_n + y_n = 2\sqrt{t-T}\ z$$

respectively we obtain

$$[J_1 + J_2] = \int_{x_n(t-T)^{-1/2}}^{x_n(t-T)^{-1/2}} \exp(-z^2)dz = J_3$$

Since

$$\int_{E_{n-1}} (t-T)^{\frac{-n}{2}+1} \exp(B(t,T)r^2(X,Y))dY = (2\sqrt{\pi})^{\frac{n}{2}-1}$$

and $J_3 = 0$ for $T = -\infty$, thus

$$U(x,t) = 0 \text{ and } _1 \equiv u_2 \text{ in } D(t_1,t_2) = \{(x,t): x\in E_{n-1}, t\in(t_1,t_2)\}$$

5. Green potential

Let us consider the Green potential

$$u(x,t) = \int_{-\infty}^{t} \int_{E_{n-1}} h(Y,s) D_{y_n} G(x,t;,0,s) dY ds =$$

$$= A \int_{-\infty}^{t} \int_{E_{n-1}} h(Y,s) K(x,t;Y,s) dY ds, \qquad (6)$$

where

$$K(x,t;Y,s) = (t-s)^{-(n-1)/2} \exp(B(t,s) r^2(X,Y) x_n (t-s)^{-3/2}$$

$$\times \exp(B(t,s) x_n^2), \text{ for } s < t,$$

$$K(x,t;Y,s) = 0 \text{ for } s \geq t.$$

In the sequel we shall prove that the function u given by formula (6) is the solution of the problem (1), (2).

6. Some denotations and lemmas

DEFINITION 2. Denote by (h) the class of all functions $h \in C(D)$ and bounded in D.

In the sequel by C, C_1 we shall denote the convenient positive constants.

Let

$$K_1(x_n,t,s) = x_n (t-s)^{-3/2} \exp(B(t,s) x_n^2)$$

and

$$I(x_n,t) = \int_{-\infty}^{t} K_1(x_n,t,s) ds.$$

LEMMA 1. *For every* $x_n > 0$, $t\epsilon(-\infty,\infty)$

$$I(x_n,t) = 2\sqrt{\pi}.$$

Proof. Applying in the integral I the change of the integral variable

$$z = \frac{1}{2} x_n^{(t-s)^{-1/2}}, \ z \in (0,\infty), \tag{7}$$

we obtain

$$I(x_n,t) = 4 \int_0^t \exp(-z^2)dz = 2\sqrt{\pi}.$$

Let

$$K_2(X,t,Y,s) = (t-s)^{-(n-1)/2}\exp(B(t,s)r^2(X,Y)) \qquad \text{for } s < t$$

and

$$K_2(X,t;Y,s) = 0 \qquad \text{for } s \geq t$$

Let us consider the integral

$$J(X,t) = \int_{E_{n-1}} K_2(X,t;Y,s)dY.$$

LEMMA 2. *For every* $(X,t) \in D_1$ *we have*

$$J(X,t) = (2\sqrt{\pi})^{n-1}. \tag{8}$$

Proof. By transformation

$$v_i = \frac{1}{2}(t-s)^{-1/2}(y_i - x_i), \quad i=1,\dots,n-1,$$

we obtain (8).

By Lemmas 1,2 and by (6), (8) we obtain

LEMMA 3. *For every* $(x,t) \in D$ *we have*

$$A \int_{-\infty}^{t} \int_{E_{n-1}} K(x,t;Y,s)\,dY\,ds = 1. \tag{9}$$

Let b and N(b) be the positive numbers and

$$H(Y,s) = h(Y,s) - h(X_o,t_o)$$

and $X_o = (x_1^o,\dots,x_{n-1}^o)$ be an arbitrary point belonging to E_{n-1}, $t_o \in (-\infty,\infty)$.

Let us consider the following sets

$$Q_1(X_o,t_o,N(b)) = \{(Y,s):\ |y_i - x_i^o| < N(b),$$

$$i=1,\dots,n-1,\ s\in(t_o-N(b),t_o+N(b))\}.$$

$$Q_2(X_o,t_o,N(b)) = \{(Y,s):\ |y_i - x_i| \ge N(b),$$

$$i=1,\dots,n-1,\ s\in(-\infty,t_o+N(b))\},$$

$$Q_3(X_o,t_o,N(b)) = \{(Y,s):\ |y_i-x_i| < N(b),$$

$i=1,\dots,n-1,\ s\in(-\infty,t_o-N(b))\}.$

Let

$$u = u_1 + u_2,$$

where

$$u_1(x,t) = \sum_1^3 u_1^i(x,t),$$

$$u_1^i(x,t) = A \int_{Q_i} H(Y,s)K(x,t,Y,s)dYds,\ i=1,2,3,$$

$$u_2(x,t) = h(X_o,t_o)\ A \int_{-\infty}^{t} \int_{E_{n-1}} K(x,t,Y,s)dYds$$

and u is given by (6)

7. Properties of the potential (6)

LEMMA 4. *If* $h\in(h)$, $(x,t)\in D$, *then the potential* u *defined by* (6) *satisfies the following conditions*

1° $Pu(x,t) = 0$ *for* $(x,t)\in D$, 2° $u(x,t) \longrightarrow h(X_o,t_o)$ as $(x,t) \longrightarrow (X_o,t_o)$.

Proof. Ad 1°.By [1], p. 498, we obtain the assertion 1°.

Ad 2°. We have the identity

$$u(x,t) = A \int_{-\infty}^{t} \int_{E_{n-1}} (h(Y,s) - h(X_o,t_o) +$$

$$h(X_o,t_o))K(x,t,Y,s)dYds = u_1(x,t) + u_2(x,t).$$

By Lemma 3 we obtain

$$u_2(x,t) = h(X_o,t_o). \tag{10}$$

Now we shall estimate u_1. By continuity of the function h at the point (X_o,t_o) for every positive number b there exists the positive number N(b) and the nieighbourhood Q_1 such that for (Y,s) belonging to Q_1, the inequality

$$|H(Y,s)| \geq b$$

holds. Consequently by Lemma 3 we obtain

$$|u_1^1(x,t)| \leq bA \int_{-\infty}^{t} \int_{E_{n-1}} K(x,t,Y,s)dYds = b \text{ for } (x,t) \in D. \tag{11}$$

Let $t \in \left[t_o - \frac{1}{2} N(b),\ t_o + \frac{1}{2} N(b)\right]$. For u_1^2 we have the estimation

$$|u_1^2(x,t)| \leq \|H\|\ A \int_{Q_2} K(x,t,Y,s)dYds$$

$$\leq C \int_{-\infty}^{t_o+N(b)} \prod_{l=1}^{n-1} \int_{-\infty}^{\infty} (t-s)^{-1/2}$$

$$\times \exp(B(t,s)(x_l-y_l)^2)dy_l\ K_1(x_n,t,s)ds$$

$$\leq C_1 \int_{-\infty}^{\infty} K_1(x_n,t,s)ds = N^1(x_n,t).$$

Applying in the integral N^1 the transformation (7) we obtain

$$N^1(x,t) = C_2 \int_0^{z_1} \exp(-z^2)dz,$$

where $z_1 = \frac{1}{2} x_n \left(t-t_o - \frac{1}{2} N(b)\right)$. Consequently

$$u_1^2(x,t) \longrightarrow 0 \text{ as } (x_n,t) \longrightarrow (0,t_o). \tag{12}$$

For u_1^3 we have the estimation

$$|u_1^3(x,t)| \le C_3 \int_{-\infty}^{t_o-N(b)} K_1(x_n,t,s)ds = N^2(x_n,t).$$

Applying in N^2 the transformation (7) we obtain

$$N^2 = C_4 \int_0^{z_2} \exp(-z^2)dz, \ z_2 = \frac{1}{2} x_n \left(t-t_o + \frac{1}{2} N(b)\right).$$

Consequently

$$u_1^3(x,t) \longrightarrow 0 \text{ as } (x_n,t) \longrightarrow (0,t_o). \tag{13}$$

Finally by (10) - (13), we obtain the assertion 2°.

8. Existence theorem

By Theorem 1 and Lemmas 1-4 we obtain

THEOREM 2. *If* $h \in (h)$, *then the function* u *defined by formula* (6) *is the unique solution of the problem* (1), (2).

Proof. By maximum principle we have

$$|u(x,t)| \leq \sup_{D_1} |h(x,t)| = C \text{ for } (x,t) \in D,$$

and

$$\|u\|_D \leq C.$$

By Theorem 1 the solution of the problem (1), (2) is unique.

9. Periodic solution of the problem (1), (2)

THEOREM 3. *If* $h \in (h)$ *and* h *is periodic with respect to* t, *i.e.* $h(x,t+p) = h(x,t)$ *for* $(x,t) \in D$, *then*: 1° *the solution* u *of the problem* (1), (2) *is periodic*, 2° *the derivatives* $D_t^i\, u(x,t)$ *are also periodic*, 3° *the function* $\Delta u(x,t)$ *is periodic with respect to* t.

Proof. Ad 1°. By theorems 1,2 there exists the unique solution $u(x,t)$ of the problem (1), (2) i.e.

$$Pu(x,t) = 0 \text{ for } (x,t) \in D, \; u(X,t) = h(X,t) \text{ for } (X,t) \in D_1.$$

Let $U(x,t) = u(x,t+p)$. The function U satisfies the equation

$$PU(x,t) = Pu(x,t+p) = 0 \text{ for } (x,t) \in D$$

and the boundary condition

$$U(x,t) = h(x,t+p) = h(x,t).$$

Thus

$$U(x,t) = h(x,t) \text{ for } (x,t) \in D.$$

So

$$PU(x,t) = 0 \text{ for } (x,t) \in D$$

and

$$U(x,t) = h(x,t) \text{ for } (x,t) \in D_1.$$

By Theorem 1 we obtain

$$U(x,t) = u(x,t+p) = u(x,t) \text{ for } (x,t) \in D.$$

Ad 2°. By 1°, for i=1, we have

$$D_t u(x,t) = \lim_{k \to 0} k^{-1}[u(x,t+k) - u(x,t)]$$

$$= \lim_{k \to 0} k^{-1}[u(x,t+p+k) - u(x,t+p)]$$

$$= D_t u(x,t+p) \text{ for } (x,t) \in D.$$

Similarly we obtain

$$D_t^2 u(x,t) = D_t^2 u(x,t+p)$$

and by induction we obtain

$$D_t^i u(x,t) = D_t^i u(x,t+p) \text{ for } (x,t) \in D.$$

Ad 3°. By 1°, 2° and (1) we get

$$\Delta u(x,t+p) = D_t u(x,t+p) = D_t u(x,t) = \Delta u(x,t) \qquad \text{for } (x,t) \in D.$$

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