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On the periodic solution for parabolic equation with Neumann boundary condition and for three dimensional time–spatial infinite strip

Dedicated to Professor Zenon Moszner with best wishes on his 60-th birthday

1. Introduction

The subject of the paper is the construction of the solution u, periodic with respect to the time variable t, of the equation

$$Pu(x,t) = 0, \quad x = (x_1, x_2),$$

where

$$P = \Delta - D_t, \quad \Delta = D^2_{x_1} + D^2_{x_2},$$

in the domain

$$D = \{(x,t),\ x_i \in (0,1),\ i = 1,2,\ t \in (-\infty,\infty)\}.$$

satisfying the initial condition

$$\lim_{t\to-\infty} \int\int_{D_1} u(x,t)dx = Q,\ Q = \text{const.} \neq 0, \tag{2}$$

where

$D_1 = \{(x,t,0): x_i \in (0,1),\ i = 1,2\}$, and the boundary condition

$$D_n u(x,t) = g(x,t) \text{ for } (x,t) \in D_2 = B(D_1) \times (-\infty,\infty), \tag{3}$$

where n is the inward normal to D_2 and $B(D_1) = \bigcup_{i,j=1}^{2} S_{i,j}$,

$$S_{1,1} = \{(x,0): x_1 = 0,\ x_2 \in (0,1)\},$$

$$S_{1,2} = \{(x,0): x_1 = 1, x_2 \in (0,1)\}.$$

$$S_{2,1} = \{(x,0): x_1 \in (0,1), x_2 = 0\},$$

$$S_{2,2} = \{(x,0): x_1 \in (0,1), x_2 = 1\},$$

and

$$g(y,s) = g_{1,i}(y_2,s) \quad \text{fo} \quad (y_2,s) \in S_{1,i} \times (-\infty,\infty),\ i=1,2,$$

$$g(y,s) = g_{2,i}(y_1,s) \quad \text{for} \quad (y_1,s) \in S_{2,i} \times (-\infty,\infty),\ i=1,2.$$

In the monograph [2], p. 101, the similar problem for the equation $(D_x^2-D_t)u(x,t) = 0$ and for the strip $D^1 = \{(x,t): x \in (0,1), t \in (-\infty,\infty)\}$ was solved. In the paper [4] the similar two dimensional problem with Dirichlet boundary condition was treated.

To the construction of the periodic solution we shall apply the convenient Green function.

We shall give the theorems on uniqueness and existence and we shall construct the periodic solution of the problem (1) - (3).

2. Some example and definition

Let

$$u(x,t) = B_o + \exp(2\pi^2 t)\cos(\pi x_1)\cos(\pi x_2), \tag{4}$$

$$B_o \neq 0,\ B_o = \text{constant}.$$

The function (4) satisfies (1) and the homogeneous Neumann boundary condition. On the other hand the function $u = 0$ has the same properties.

Consequently we obtain the example of the nonuniqueness of the (1)-(3) problem. In the sequel we shall give the class of the solutions u for which the uniqueness holds.

DEFINITION 1. *Denote by* (K) *the class of all functions* $u(x,t)$ *satisfying* (2) *such that* $u(x,t) \in C^{2,1}(D)$ *and*

$$M(t)t^{-2}\exp(\pi^2 t) \longrightarrow 0 \text{ as } t \longrightarrow -\infty,\ M(t) = \sup_{D_1}|u(x,t)|,$$

$$\lim \iint_{D_1} u(x,t)dx = Q \neq 0 \text{ as } t \longrightarrow -\infty.$$

3. Theorem on uniqueness

THEOREM 1. *If the function* $u(x,t) \in (K)$ *is the solution of the* (1)-(3) *problem, then* $u(x,t) \equiv 0$ *for* $(x,t) \in D$.

Proof. Let u_i, $i=1,2$, be the solutions of the (1)-(3) problem belonging to (K). Let

$$U(x,t) = u_1(x,t) - u_2(x,t).$$

The function $U \in (K)$ satisfies (1) and homogeneous Neumann boundary conditions. Let us consider the following limit problem

$$PU(x,t) = 0 \qquad \text{for} \qquad x \in D_1,\ t > T, \tag{5}$$

$$U(x,t)\big|_{t=T} = U(x,T), \tag{6}$$

$$D_n U(x,t) = 0 \qquad \text{for} \qquad (x,t) \in B(D_1) \times (T,\infty). \tag{7}$$

The solution of the problem (5)-(7) is of the form

$$U(x,t) = a_{o,o} + U_1(x,t) + U_2(x,t),$$

where

$$a_{o,o} = 2^2 \int_{D_1}\int U(x,T)dx = Q,$$

$$U_1(x,t,T) = \sum_{m_1=0}^{\infty} \sum_{m_2=1}^{\infty} a_{m_1,m_2}(T)\cos(m_1\pi x_1)\cos(m_2\pi x_2)$$

$$\times \exp(-(m_1^2+m_2^2)\pi^2(t-T)) = U_1^1(x,t) + U_1^2(x,t),$$

and

$$U_2(x,t,T) = \sum_{m_1=1}^{\infty} \sum_{m_2=0}^{\infty} a_{m_1,m_2}(T)\cos(m_1\pi x_1)\cos(m_2\pi x_2)$$

$$\times \exp(-(m_1^2+m_2^2)\pi^2(t-T)) = U_2^1(x,t) + U_2^2(x,t),$$

where

$$U_1^1(x,t,T) = \sum_{m_2=1}^{\infty} a_{0,m_2}(T)\cos(m_2\pi x_2)\exp(-m_2^2\pi^2(t-T)),$$

$$U_1^2(x,t,T) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} a_{m_1,m_2}(T)\cos(m_1\pi x_1)\cos(m_2\pi x_2)$$

$$\times \exp(-(m_1^2+m_2^2)\pi^2(t-T)),$$

$$U_2^1(x,t,T) = \sum_{m_1=1}^{\infty} a_{m_1,0}(T)\cos(m_1\pi x_1)\exp(-m_1^2\pi^2(t-T)),$$

$$U_2^2(x,t,T) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} a_{m_1,m_2}(T)\cos(m_1\pi x_1)\cos(m_2\pi x_2)$$

$$\times \exp(-(m_1^2+m_2^2)\pi^2(t-T)),$$

$$a_{m_1,m_2}(T) = 2^2\int_{D_1}\int U(x,T)\cos(m_1\pi x_1)\cos(m_2\pi x_2)dx_1dx_2,$$

$$m_1,m_2=0,1,\ldots\ .$$

For the function U_1^2 we have the estimation

$$|U_1^2| \le M(T) \sum_{m_1,m_2=1}^{\infty} \exp(-(m_1^2+m_2^2)\pi^2(t-T)) =$$

$$M(T) \left(\sum_{m_1=1}^{\infty} \exp(-m_1^2\pi^2(t-T)) \right)$$

$$\times \left(\sum_{m_2=1}^{\infty} \exp(-m_2^2\pi^2(t-T)) \right) = M(T)(\exp(-\pi^2(t-T))$$

$$+ \sum_{n=2}^{\infty} \exp(-n^2\pi^2(t-T)))^2 \le M(T) \left(\exp(-\pi^2(t-T)) \right.$$

$$\left. + \int_1^{\infty} 2x \exp(-\pi^2(t-T)x^2)dx \right)^2 \le M(T)(1+(\pi^2(t-T))^{-1})^2$$

$$\times \left(\exp(-\pi^2(t-T)) \right)^2 .$$

For the function U_2^2 we obtain the same estimate

$$|U_2^2| \le M(T)(1+(\pi^2(t-T))^{-1})^2 \left(\exp(-\pi^2(t-T)^{-1} \right)^2 .$$

Consequently

$$|U_1^2 + U_2^2| \le 2M(T)\left(1 + (\pi^2(t-T))^{-1}\right)^2\left(\exp(-\pi^2(t-T))\right)^2. \qquad (8)$$

Similarly we obtain the estimation

$$|U_1^1 + U_2^1| \le M(T)\left(1 - (\pi^2(t-T))^{-1}\right)^2\left(\exp(-\pi^2(t-T))\right)^2, \qquad (9)$$

By (8), (9) we obtain

$$|U(x,t,T)| \le 4M(T)\exp(\pi^2 T)\left(1 + (\pi^2(t-T))^{-1}\right)^2 \exp(-\pi^2 t). \qquad (10)$$

By (10) follows that if

$$M(T)\exp(\pi^2 T) \longrightarrow 0 \text{ as } T \longrightarrow -\infty,$$

then $U(x,t) \equiv 0$ for $(x,t) \in D$ quasi uniformly with respect to $t \in (-\infty,\infty)$.

4. Green function

Let $x_i, y_i \in (0,1)$, $i=1,2$, and let $t > s \ge 0$. Let us consider the sequences

$$x^i_{o,1} = x^i_{o,2} = x_i,\ x^i_{2n,1} = x_i + 2n, x^i_{2n+1,1}$$

$$= -x_i - 2n, x^i_{2n,2} = x_i - 2n,$$

$$x^i_{2n+1,2} = -x_i + 2n + 2,\ i=1,2,\ n=1,2,\dots,$$

Let

$$U^i_{o,j}(x_i,t;y,s) = (t-s)^{-1/2}\exp(B(t,s)(x^j_i-y_i)^2),$$

$$i=1,2, \quad j=1,2,$$

$$U^i_{n,1}(x_i,t;y_i s) = (t-s)^{-1/2}\exp(B(t,s)(x^i_{n,1}-y_i)^2),$$

$$i=1,2$$

$$U^i_{n,2}(x_i,t;y_i,s) = (t-s)^{-1/2}\exp(B(t,s)(x^i_{n,2}-y_i)^2).$$

$$i=1,2, \quad n=1,2,\ldots,$$

where $B(t,s) = (-4(t-s))^{-1}$.

Let

$$G_i(x_i,t;y_i,s) = U^i_{o,1}(x_i,t;y_i,s)$$

$$+ \sum_{n=1}^{\infty}\left\{U^i_{n,1}(x_i,t;y_i,s) + U^i_{n,2}(x_i,t;y_i,s)\right\}, \quad i=1,2,$$

for $s < t$ and $G_i(x_i,t;y_i,s) = 0$ for $s \geq t$, $i=1,2$,

and

$$G(x,t,y,s) = G_1(x_1,t;y_1,s)G_2(x_2,t;y_2,s).$$

By [1], G is the Green function for the equation (1) and Neumann boundary data for the domain D.

LEMMA1. *If* $t > s$, $x_1 \in (0,1)$, *then:*

$$G_1(x_1,t,0,s) = \sum_{i=1}^{3} Z_i(x_1,t,s), \tag{11}$$

where

$$Z_1(x_1,t,s) = 2(t-s)^{-1/2}\exp(B(t,s)x_1^2),$$

$$Z_2(x_1,t,s) = 2(t-s)^{-1/2}$$

$$\times \sum_{n=0}^{\infty} \exp(B(t,s)(x_1+2n+2)^2),$$

$$Z_3(x_1,t,s) = 2(t-s)^{-1/2} \sum_{n=0}^{\infty} \exp(B(t,s)(x_1-2n-2)^2),$$

and $G_1(x_1,t;0,s) = 0$ for $s \geq t$,

$$G_1(x_1,t,1,s) = S_1(x_1,t,s) + S_2(x_1,t,s) + S_3(x_1,t,s), \tag{12}$$

where

$$S_1(x_1,t,s) = 2(t-s)^{-1/2}\exp(B(t,s)(-x_1+1)^2),$$

$$S_2(x_1,t,s) = 2(t-s)^{-1/2} \sum_{n=0}^{\infty} \exp(B(t,s)(-x_1+2n+1)^2),$$

$$S_3(x_1,t,s) = 2(t-s)^{-1/2} \sum_{n=0}^{\infty} \exp(B(t,s)(x_1+2n+2)^2)$$

for $s < t$,

and $G_1(x_1,t;1,s) = 0$ for $s \geq t$,

$$G_2(x_2,t,0,s) = \sum_{1}^{3} V_i(x_2,t,s),$$

where

$$V_1(x_2,t,s) = 2(t-s)^{-1/2}\exp(B(t,s)x_2^2),$$

$$V_2(x_2,t,s) = 2(t-s)^{-1/2} \sum_{n=0}^{\infty} \exp(B(t,s)(x_2+2n+2)^2),$$

$$V_3(x,t,s) = 2(t-s)^{-1/2} \sum_{n=0}^{\infty} \exp(B(t,s)(x_2-2n-2)^2),$$

and $G_2(x_2,t;0,s) = 0$ for $s \geq t$,

$$G_2(x_2,t.1.s) = \sum_{1}^{3} W_i(x_2,t,s),$$

where

$$W_1(x_2,t,s) = 2(t-s)^{-1/2}\exp(B(t,s)(-x_2+1)^2),$$

$$W_2(x_2,t,s) = 2(t-s)^{-1/2} \sum_0^{\infty} \exp(B(t,s)(-x_2+2n+1)^2),$$

$$W_3(x_2,t,s) = 2(t-s)^{-1/2} \sum_0^{\infty} \exp(B(t,s)(x_2+2n+2)^2),$$

and $G_2(x_2,t;1,s) = 0$ for $s \geq t$.

5. Green potentials

Let us consider the following Green potentials

$$u_1(x,t) = A \int_{-\infty}^{t} \int_0^1 g_{1,1}(y_2,s)G_1(x_1,t;0,s)$$

$$\times\, G_2(x_2,t;y_2,s)dy_2ds = \sum_1^2 u_1^i(x,t),$$

where

$$u_1^i(x,t) = A \int_{-\infty}^{t} \int_0^1 g_{1,1}(y_2,s)Z_i(x_1,t,s)$$

$$\times\, G_2(x_2,t;y_2,s)dy_2ds, \quad i=1,2,3.$$

Let

$$u_1^1(x,t) = u_1^{1,1}(x,t) = u_1^{1,2}(x,t),$$

where

$$u_1^{1,1}(x,t) = 2A \int_{-\infty}^{t} \int_{0}^{1} g_{1,1}(y_2,s)(t-s)^{-1/2}$$

$$\times \exp(B(t,s)x_1^2)\exp(B(t,s)(x_2-y_2)^2 dy_2 ds,$$

$$u_1^{1,2}(x,t) = 2A \int_{-\infty}^{t} \int_{0}^{1} g_{1,1}(y_2,s)(t-s)^{-1/2}$$

$$\exp(B(t,s)x_1^2)\left(\sum_1^{\infty} \exp(B(t,s)(x_{n,1}^2-y_2)^2)\right.$$

$$\left.+ \exp(B(t,s)(x_{n,2}^2-y_2)^2)\right) dy_2 ds.$$

Let

$$u_2(x,t) = A \int_{-\infty}^{t} \int_{0}^{1} g_{1,2}(y_2,s)G_1(x_1,t;1,s)$$

$$\times G_2(x_2,t;y_2,s)dy_2 ds = \sum_1^3 u_2^1(x,t),$$

where

$$u_2^1(x,t) = 2A \int_{-\infty}^{t} \int_{0}^{1} g_{1,2}(y_2,s)S_1(x_1,t,s)$$

$$\times\ G_2(x_2,t;y_2,s)dy_2 ds, \quad i=1,2,3,$$

and

$$u_3(x,t) = A \int_{-\infty}^{t} \int_{0}^{1} g_{2,1}(y_1,s)G_2(x_2,t;0,s)$$

$$\times\ G_1(x_1,t;y_1,s)dy_1 ds = \sum_{1}^{3} u_3^i(x,t),$$

where

$$u_3^i(x,t) = 2A \int_{-\infty}^{t} \int_{0}^{1} g_{2,1}(y_1,s)V_i(x_2,t,s)$$

$$\times\ G_1(x_1,t;y_1,s)dy_1 ds, \quad i=1,2,3,$$

and

$$u_4(x,t) = A \int_{-\infty}^{t} \int_{0}^{1} g_{2,2}(y_1,s)G_2(x_2,t;1,s)$$

$$\times\ G_1(x_1,t;y_1,s)dy_1 ds = \sum_{1}^{3} u_4^i(x,t),$$

where

$$u_4^i(x,t) = 2A \int_{-\infty}^{t} \int_{0}^{1} g_{2,2}(y_1,s)W_i(x_2,t,s)$$

$$\times\ G_1(x_1,t;y_1,s)dy_1 ds, \quad i=1,2,3,$$

and $A = (8\pi)^{-1}$.

6. Some definitions and lemmas

In the sequel by C, C_1 we shall denote the positive constants

Let

$$Q_1\{(y_2,s):\ y_2 < 0,\ s \in (-\infty,\infty)\},$$

$$Q_2\{(y_2,s):\ y_2 > 1,\ s \in (-\infty,\infty)\},$$

$$Q_3\{(y_1,s):\ y_1 < 0,\ s \in (-\infty,\infty)\},$$

$$Q_4\{(y_1,s):\ y_1 > 1,\ s \in (-\infty,\infty)\}.$$

DEFINITION 2. *Denote by* (g) *the class of all functions* g(y,s) *continuous and bounded for* (y,s) ∈ D.

LEMMA 3. *If* $z \geq 0$, k *is a positive constant, then*

$$z^k \exp(-z^2) \leq C \tag{13}$$

We omit te simple proof.

By [3], p. 498, we obtain

LEMMA 4. *If* $g_{i,j} \in (g)$, i,j=1,2, *then*

$$Pu_i(x,t) = 0 \text{ for } (x,t) \in D_1,\ i=1,2,3,4.$$

LEMMA 5. *For every* $(x,t) \in D_2$, *we have*

$$I(x,t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} (t-s)^{-2}\ x_1 \exp(B(t,s)x_1^2)\exp(B(t,s)(x_2-y_2)^2)dy_2 ds = \frac{\pi}{8}.$$

Proof. By [3], p. 447, we have

$$I(x,t) = \int_{-\infty}^{t} (t-s)^{-3/2} x_1 \exp((B(t,s)x_1^2) I_1(x_2,t) ds,$$

where

$$I_1(x,t) = \int_{-\infty}^{\infty} (t-s)^{-1/2} \exp(B(t,s)(x_2-y_2)^2) dy_2 = \sqrt{\pi}.$$

Thus

$$I(x,t) = \sqrt{\pi} \int_{-\infty}^{t} x_1 (t-s)^{-3/2} \exp(B(t,s)x_1^2) ds.$$

Applying in the integral I the change of the integral variable

$$z = \frac{1}{2} x_1 (t-s)^{-1/2} \tag{14}$$

we get

$$I(x,t) = \frac{\pi}{4} \int_{0}^{\infty} \exp(-z^2) dz = \frac{\pi}{8} .$$

LEMMA 6. *If* $g_{1,1} \in (g)$, *then*

$$D_{x_1} u_1^{1,1}(x,t) \longrightarrow g_{1,1}(x_2^{\circ}, t_o)$$

as

$$(x,t) \longrightarrow (0,x_2^o,t_o)x_2^o \in (0,1),\ t_o < t.$$

Proof. We have

$$D_{x_1}u_1^{1,1}(x,t) = A\int_{-\infty}^{t} x_1(t-s)^{-3/2}\exp(B(t,s)x_1^2)$$

$$\times \int_{-\infty}^{\infty} \bar{g}_{1,1}(y_2,s)(t-s)^{-1/2}$$

$$\times \exp(B(t,s)(x_2-y_2)^2)dy_2ds$$

$$= \int_{-\infty}^{t}\int_{-\infty}^{\infty} [g_{1,1}(x_2^o,t_o) + \bar{g}_{1,1}(y_2,s) - \bar{g}_{1,1}(x_2,t_o)]$$

$$\times K(x,t,y,s)dy_2ds = J_1+,J_2,$$

where

$$K(x,t,y,s) = x_1(t-s)^{-3/2}\left(\exp(b(t,s)x_1^2)\right)[(t-s)^{-1/2}$$

$$\times \exp(B(t,s)(x_2-y_2)^2)] \qquad \text{for } s < t,$$

and $K(x,t;y,s) = 0$ for $s \geq t$,

$$J_1(x,t) = g_{1,1}(x_2^o,t_o)AI(x,t) = g_{1,1}(x_2^o,t_o),$$

$$J_2(x,t) = A\int_{-\infty}^{t}\int_{-\infty}^{\infty} r(x_2^o,t_o,y_2,s)K(x,t,y,s)dy_2ds,$$

$$r(x_2^o,t_o;y_2,s) = \bar{g}_{1,1}(y_2,s) - g_{1,1}(x_2^o,t_o).$$

We shall estimate J_2. By continuity of $g_{1,1}$ at the point (x_2^o,t_o), for every positive number b, there exists the number h, and the neighboorbood of the point (x_2^o,t_o) o the form

$$O(x_2^o,t_o) = \{(y_2,s):\ y_2 \in (x_2^o-h,x_2^o+h),\ s \in (t_o-h,t_o+h)\},$$

such tnat

$$|r(x_2^o,t_o,y_2,s)| \leq b \qquad \text{for } (y_2,s) \in O(x_2^o,t_o). \tag{15}$$

Let $t \in (t_o - \frac{h}{2},\ t_o + \frac{h}{2})$. The inequality (15) also for $s \in (t_o-h,t_o+h)$ holds. Let

$$J_2 = J_2^1 + J_2^2 + J_2^3,$$

where

$$J_2^1(x,t) = A \int_{t_{o-h}}^{t_{o+h}} \int_{x_2^o-h}^{x_{2+h}^o} r(x_2^o,t_o,y_2,s)K(x,t;y,s,dy_2 ds,$$

$$J_2^2(x,t) = A \int_{-\infty}^{t_o-h} \int_{x_2^o-h}^{x_2^o+h} r(x_2^o,t_o,y_o,s)K(x,t;y,s)dy_2,ds,$$

$$J_2^3(x,t) = A \int_{-\infty}^{t_o+h} \int_{|x_2^o-y_2|\geq h} r(x_2^o,t_o;y_2,s)K(x,t;y,s)dy_2ds.$$

We have the estimations

$$|J_2^1| AbI = b,$$

and if $M = \sup_{D_2}|g|$, then

$$|J_2^2| \leq 2AM \int_{-\infty}^{t_o-h} \int_{x_2^o-h}^{x_2^o+h} (t-s)^{-3/2}x_1\left(\exp(B(t,s)x_1^2)\right)(t-s)^{-1/2}$$

$$\times \exp(B(t,s)(x_2-y_2)^2)dy_2ds \leq 2AM \int_{-\infty}^{t_o-h} (t-s)^{-3/2}x_1\exp(B(t,s)x_1^2)$$

$$\times \int_{-\infty}^{\infty}(t-s)^{-1/2}\exp(B(t,s)(x_2-y_2)^2)dy_2ds \leq I^1,$$

where

$$I^1 = C_1 \int_{-\infty}^{t_o-h} (t-s)^{-3/2}x_1\exp(B(t,s)x_1^2)ds.$$

Applying (14) we obtain

$$I^1 = C_1 \int_0^a \exp(-z^2)dz,$$

where

$$a = \frac{1}{2}\, x_1 (t - t_o + h).$$

Hence

$$I^1 \longrightarrow 0 \text{ as } x_1 \longrightarrow 0 \text{ and } J_2^2 \longrightarrow 0 \text{ as } x_1 \longrightarrow 0.$$

For the integral J_2^3 we obtain the estimate

$$|J_2^3| \leq 2AM \int_{-\infty}^{t} (t-s)^{-3/2} x_1 \exp(B(t,s)x_1^2) \exp\left(\frac{1}{2}\, B(t,s)h^2\right)$$

$$\times \left[\int_{-\infty}^{\infty} (t-s)^{-1/2} \exp(B(t,s)(x_2 - y_2)^2) dy_2\right] ds \leq C_3 I^2,$$

where

$$I^2 = \int_{-\infty}^{t} (t-s)^{-3/2} \exp(B(t,s)\tfrac{1}{2} h^2) ds$$

$$= \frac{1}{h} \int_{-\infty}^{t} h(t-s)^{-3/2} \exp\left(\frac{B(t,s)}{2}\, h^2\right) ds \,.$$

Applying in I^2 the transformation

$$z = h(2\sqrt{2}\ (t-s))^{-1}$$

we obtain

$$I^2 = C_4 h^{-1} \int_0^{\infty} \exp(-z^2)dz = C_5 h^{-1}.$$

Since

$$C_3 x_1 I^2 \longrightarrow 0 \text{ as } x_1 \longrightarrow 0,$$

thus

$$J_2^3 \longrightarrow 0 \text{ as } x_1 \longrightarrow 0.$$

Finally

$$J_2 \longrightarrow 0 \text{ as } (x,t) \longrightarrow (0,x_2^2,t)_o,\ x_2^o \in (0,1).$$

LEMMA 7. *If* $g_{1,1} \in (g)$, *then*

$$D_{x_1} u_1^{1,2}(x,t) \longrightarrow 0,$$

as

$$(x,t) \longrightarrow (0,x_2^o,t_o),x_2^o \in (0,1),$$

Proof. We have

$$D_{x_1} u_1^{1,2}(x,t) = A \int_{-\infty}^{t} \int_0^1 g_{1,1}(y_2,s)x_1(t-s)^{-2}\exp(B(t,s)x_1^2)$$

$$\times\Big\{\exp(B(t,s)(x_2+y_2)^2) - \exp(B(t,s)(1-x_2+1-y_2)^2)$$

$$+\sum_{2}^{\infty}\left[\exp(B(t,s)(x_{n,1}^2-y_2)^2)+\exp(B(t,s)(x_{n,2}^2-y_2)^2)\right]\Big\}\, dy_2\, ds.$$

For $x_2 \in (0,1), y_2 \in [0,1]$, the inequalities

$$\exp(B(t,s)(x_2+y_2)^2) \le \exp(B(t,s)x_2^2),$$

$$\exp(B(t,s)(1-x_2+1-y_2)^2) \le \exp\ (B(t,s)\bar{x}_2^2),$$

$\bar{x}=1-x_2$ hold.

Consequently

$$|u_1^{1,2}| \le J^1 + J^2 + J_3^1,\ J_3^1,$$

where

$$J^1(x,t) = A\int_{-\infty}^{t}\int_0^1 g_{1,1}(y_2,s)x_1\exp(B(t,s)x_1^2)(t-s)^{-2}$$

$$\times \exp(B(t,s)x_2^2)dy_2ds = A\int_{-\infty}^{t}\int_0^1 g_{1,1}(y_2,s)x_1\exp(B(t,s)x_1^2)x_2^{-2}$$

$$\times \exp\left(\frac{1}{2}B(t,s)x_2^2\right)\left[\frac{x_2}{(t-s)^{1/2}}\exp\left(\frac{1}{2}B(t,s)x_2^2\right)\right]$$

$$\times \left[\frac{x_2}{(t-s)^{3/2}}\exp\left(-\frac{1}{2}B(t,s)x_2^2\right)\right]dy_2ds,$$

$$J^2(x,t) = J^1(x_1,\bar{x}_2,t)$$

$$J_3^1(x,t) = A\int_{-\infty}^{t}\int_0^1 g_{1,1}(y_2,s)x_1\exp(B(t,s)x_1^2)$$

$$\times \sum_{2}^{\infty} \left[\frac{x_{n,i}^2-y_2}{(t-s)^{1/2}} \exp(\frac{1}{2} B(t,s)(x_{n,i}^2-y_2)^2)\right]$$

$$\times \left[\frac{x_{n,i}^2-y_2}{(t-s)^{3/2}} \times \exp(\frac{1}{2} B(t,s)(x_{n,i}-y_2)^2)\right]\frac{1}{(x_{n,i}^2-y_2)^2} \, dy_2 ds,$$

$$i=1,2.$$

By Lemmas 3,7 and by [3], p. 474, we obtain the estimations

$$|J^i| \le Cx_1 \longrightarrow 0 \text{ as } x_1 \longrightarrow 0, \ i=1,2,$$

$$|J_3^i| \le Cx_1\left(\sum_{n=1}^{\infty} n^{-2}\right) \longrightarrow 0 \text{ as } x_1 \longrightarrow 0, \ i=1,2.$$

LEMMA 8. *If* $g_{1,i} \in (g)$, *then*

$$D_{x_1}(u_1^2(x,t) + u_1^3(x,t)) \longrightarrow 0,$$

as $(x,t) \longrightarrow (0,x_1^o,t_o)$, $x_2^o \in (0,1)$.

Proof. We have

$$D_{x_1}(Z_2(0,t_o,s) + Z_3(0,t_o,s)) = 0.$$

By Lemmas 4,6,7 we obtain

LEMMA 9. *If* $g_{1,1} \in (g)$, *then*

$$D_{x_1} u_1(x,t) \longrightarrow g_{1,1}(x_2^o, t_o)$$

as $(x,t) \longrightarrow (0, x_2^o, t_o)$, $x_2^o \in (0,1)$.

LEMMA 10. *If* $g_{1,1} \in (g)$, *then*

$$D_{x_1} u_1(x,t) \longrightarrow 0$$

as $(x,t) \longrightarrow (x_1^o, 0, t_o)$, $x_1^o \in (0,1)$.

Proof. Since $G_2(0, t_o, y_2, s) = 0$ thus we get the assertion of the Lemma 10.

Similarly we obtain

LEMMA 11. *If* $g_{1,1} \in (g)$, *then*

$$D_{x_1} u_1(x,t) \longrightarrow 0$$

as $(x,t) \longrightarrow (x_1^o, 1, t_o)$, $x_1^o \in (0,1)$.

By Lemmas 10,11 we obtain

LEMMA 12. *If* $g_{1,1} \in (g)$, *then*

$$D_{x_1} u_1(x,t) \longrightarrow 0$$

as $x \longrightarrow (x_1^o, x_1^o) \in S_{1,2} \cup S_{2,2} \cup S_{2,1}$,

$$t \longrightarrow t_o$$

Analogously we obtain

LEMMA 13. *If* $g_{i,j} \in (g)$, $i,j=1,2),(i,j) \neq (1,1)$, *then*

$$D_{x_1} u_2(x,t) \longrightarrow g_{1,2}(x_2^o, t_o)$$

as $(x,t) \longrightarrow (1, x_2^o, t_o)$, $x_2^o \in (0,1)$,

$$D_{x_1} u_2(x,t) \longrightarrow 0$$

as $x \longrightarrow x_o \in S_{1,1} \cup S_{1,2} \cup S_{2,2}$, $t \longrightarrow t_o$,

$$D_{x_2} u_3(x,t) \longrightarrow g_{2,1}(x_1^o, t_o)$$

as $(x,t) \longrightarrow (x_1^o, 0, t_o)$, $x_1^o \in (0,1)$.

$$D_{x_2} u_3(x,t) \longrightarrow 0 \text{ as } x \longrightarrow (x_1^o, x_2^o) \in S_{1,1} \cup S_{1,2} \cup S_{2,1},\quad t \longrightarrow t_o,$$

$$D_{x_2} u_4(x,t) \longrightarrow g_{2,2}(x_1^o, t_o) \text{ as } (x,t) \longrightarrow (x_1^o, 1, t_o),\quad x_1^o \in (0,1),$$

$$D_{x_2} u_4(x,t) \longrightarrow 0 \text{ as } x \longrightarrow (x_1^o, x_2^o) \in S_{1,1} \cup S_{1,2} \cup S_{2,1},\quad t \longrightarrow t_o.$$

7. Existence theorem

By Lemmas 1-13 and Theorem 1 we obtain

THEOREM 2. *If* $g \in (g)$, *then the function*

$$u(x,t) = Q - \sum_{1}^{4} u_i(x,t)$$

is the unique solution of the problem (1), (2), (3) *belonging to the class* (K).

8. Periodic solutions

LEMMA 14. *If the function* $u \in (K)$ *is the solution of the equation* (1), *satisfying the condition* (3), *the conditions*

$$u(x,0) = u(x,p),\ p \text{ a positive number},\ x \in D_1, \tag{16}$$

$$\int\int_{D_1} u(x,0)dx = Q, \tag{17}$$

then the condition

$$\int_0^p \int_{B(D_1)} g(x,t)dS_x dt = 0 \tag{18}$$

holds.

Proof. Applying Green formula for the integral

$$Lu = \int_0^p \int\int_{D_1} Pu(x,t)dxdt = 0$$

we obtain

$$\int_0^p \int_{B(D_1)} D_n u(x,t)dS_x dt - \iint_{D_1}(u(x,p) - u(x,0))dx_1 dx_2 = 0. \tag{19}$$

By (3) and (19) we get (18).

Conversely. If (17), (18), (19) hold, then (16) holds.

9. The problem (Ia)-(Id) and its uniqueness

Let us consider the following problem

$$Pw(x,t) = 0 \text{ for } (x,t) \in D_p =$$

$$\{(x,t):\ x \in D_1,\ t \in (C,p]\},\ w \in C^{2,1}(D_p \cap C^{1,1}(\bar{D}_p), \tag{Ia}$$

$$w(x,0) = 0 \text{ for } x \in D_1, \tag{Ib}$$

$$D_n w(x,t) = g(x,t) \text{ for } x \in B(D_1),\ t \in (0,p], \tag{Ic}$$

$$\iint_{S(T)} D_n g(x,t)dS_x dt = 0,\ S(T) = \{(x,T):\ x \in B(D_1),\ t \in (0,p)\} \tag{Id}$$

10. Uniqueness of the (Ia)-(Id) problem

THEOREM 3. *If* W_i, i=1,2, *are the solutions of the class* $C^{2,1}(D) \cap C^{1,1}(\bar{D})$, *of the problem* (Ia)-(Id), *then* $w_1 \equiv w_2$ in $\bar{D}_p$.

Proof. Let

$$W(x,t) = w_1(x,s) - w_2(x,s).$$

Let $D_p(t) = \{(x,s): x \in D_1, s \in (0,t]\}$. We have the identity

$$W(x,s)PW(x,s) = 0 . \tag{20}$$

Integrationg the identity (20) over $D_p(t)$ we obtain

$$J = \int_0^t \iint_{D_1} W(x,s)dxds = - \int_0^t \iint_{D_1} (\text{grad } W(x,s))^2 dxds$$

$$+ \iint_{S(t)} W(x,s)D_n W(x,s)dS_x ds - \frac{1}{2} \iint_{D_1} W^2(x,s)\Big|_0^t dx = 0.$$

by the last formula we obtain the assertion of the Theorem 3.

10. Solution of the (Ia)-(Id) problem

Similarly as theorem 2 we can prove

THEOREM 4. *If* $g \in (g)$, *then the function*

$$w = \sum_1^4 w_i,$$

where

$$w_1(x,t) = A \int_0^t \int_0^1 g_{1,1}(y_2,s)G(x,t,0,y_2,s)dy_2 ds,$$

$$w_2(x,t) = A \int_0^t \int_0^1 g_{1,2}(y_2,s)G(x,t,1,y_2,s)dy_2ds,$$

$$w_3(x,t) = A \int_0^t \int_0^1 g_{2,1}(y_1,s)G(x,t,y_1,0,s)dy_1ds,$$

$$w_4(x,t) = A \int_0^t \int_0^1 g_{2,2}(y_1,s)G(x,t,y_1,1,s)dy_2,s)dy_2ds,$$

is the unique solution of the (Ia)-(Id) *problem.*

11. The problem (IIa)-(IIc)

Let us consider the following problem

$$Pv(x,t) = 0 \text{ for } (x,t) \in D_p,\ v \in C^{2,1}(D_p) \cap C^{1,1}(\bar{D}_p), \tag{IIa}$$

$$D_n v(x,t) = 0 \text{ for } (x,t) \in S, \tag{IIb}$$

$$v(x,0) = u(x,0) \text{ for } x \in D_1, \tag{IIc}$$

and by (IIc) the function v is unknown.

Properties of the function v

LEMMA 15. *If the functions* w,v *are the solutions of the problems* (Ia)-(Id) and (IIa)-(IIc), *respectively, then the function* u = w = v *is the solution of the problem* (1), (3), (16), (17).

Proof. By (IIb), (IIc) we have $\iint_{D_1} v(x,p)dx = \iint_{D_1} u(x,0)dx = Q$.

Let

$$v(x,t) = Q + \sum_{m_1+m_2\geq 1}^{\infty} b_{m_1 m_2} \exp(-(m_1^2+m_1^2)\pi^2 t)\cos(m_1\pi x_1)\cos(m_2\pi x_2), \qquad (21)$$

where

$$b_{m_1,m_2} = 4 \iint_{D_1} u(x,0)\cos(m_1\pi x_1)\cos(m_2\pi x_2)dx, \qquad m_1+m_2\geq 1.$$

By (21) and (IIc) the function u(x,0) is of the form

$$u(x,0) = Q + \sum_{m_1+m_2\geq 1}^{\infty} b_{m_1,m_2} \cos(m_1\pi m_2)\cos(m_2\pi x_2). \qquad (22)$$

The coefficients $b_{m_1 m_2}$ are unknown.

By (Ib), (Ic), (Id), (IIb), (IIc) we have

$$v(x,p) + w(x,p) = v(x,0) = w(x,0) = u(x,0). \qquad (23)$$

For the function w(x,0) we have

$$w(x,0) = w(x,p) = c_{o,o} + \sum_{m_1+m_2\geq 1} c_{m_1,m_2} \cos(m_1\pi m_2)$$

$$\times\cos(m_2\pi x_2), \qquad (24)$$

where

$$c_{o,o} = 4 \iint_{D_1} w(x,0)dx = 0, \quad c_{m_1,m_2} = \iint_{D_1} w(x,p)\cos(m_1\pi x_1)\cos(m_2\pi x_2)dx.$$

By (21)-(24) we have

$$Q + \sum_{m_1+m_2\geq 1} b_{m_1,m_2} \exp(-(m_1^2+m_1^2)\pi^2 p)\cos(m_1\pi x_1)$$

$$\times \cos(m_2\pi x_2) + \sum_{m_1+m_2\geq 1} \cos(m_1\pi x_1)\cos(m_2\pi x_2) = \tag{25}$$

$$Q + \sum_{m_1+m_2\geq 1} b_{m_1,m_2} \cos(m_1\pi x_1)\ \cos(m_2\pi x_2).$$

By (25) we obtain

$$b_{m_1,m_2} = (1-\exp(-(m_1^2+m_2^2)\pi^2 p)^{-1} c_{m_1,m_2}, m_1+m_2\geq 1.$$

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