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Dirichlet's problem for a generalized Jensen functional equation

Dedicated to Professor Zenon Moszner with best wishes on his 60-th birthday

1. We shall consider the functional equation

$$\phi\big((p+q)/2\big) = F\big(\phi(p),\phi(q)\big);\ p,\ q \in \mathbb{R}^n, \tag{1}$$

which becomes the Jensen equation when $F(p,q) = (p+q)/2$.

The following Dirichlet-like problem is studied in the present paper:

(D) Given a subset W of $\mathbb{R}^n$ and a continuous and bounded function $b\colon \partial W \longrightarrow \mathbb{R}$, find a continuous solution $\phi\colon \mathrm{cl}W \longrightarrow \mathbb{R}$ of (1) which coincides with the function b on the boundary ∂W of W:

$$\phi(p) = b(p),\ p \in \partial W \tag{2}$$

In some function classes Problem (D) has been dealt with by E. F. Beckenbach and L. K. Jackson [2]. However, the set of continuous solutions to (1) does not meet the conditions assumed in [2]. In this paper we are going to show a necessary and sufficient condition for the existence of a solution to (D), and to construct a solution of equation (1) via Perron's method, i.e. as supremum (or infimum) of solution sets of inequalities associated with equation (1).

2. We start with a description of sets of solutions to (1) as well as to the inequality

$$\lambda\left((p+q)/2\right) \le F\left(\lambda(p),\lambda(q)\right), \; p, \; q \in \mathbb{R}^n. \tag{3}$$

The following hypotheses are assumed throughout the paper.

(H) *The function* $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is continuous and there exists a continuous and increasing bijection* $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ *which satisfies equation* (1). *Moreover,* $W \subset \mathbb{R}^n$ *is a nonempty, convex, open and bounded set (a convex body) with the boundary* $\partial W \neq \emptyset$.

Now we quote results on the general continuous solution of equation (1) and inequality (3) (cf. [3]).

LEMMA 1. *Assume (H) to hold. The general continuous solution* $\phi\colon W \longrightarrow \mathbb{R}$ *of equation* (1) *is given by the formula*

$$\phi(p) = f\left(l(p)\right), \tag{4}$$

where $l\colon W \longrightarrow \mathbb{R}$ *is any continuous affine function, i.e.* $l\left((p+q)/2\right) = \left(l(p)+l(q)\right)/2$, p, $q \in W$: *whereas that of inequality* (3) *is given by*

$$\lambda(p) = f\left(k(p)\right), \tag{5}$$

where $k\colon W \longrightarrow \mathbb{R}$ *is any continuous convex function.*

In fact, according to [1], Ch. 5. (cf. also [4], p. 315) we have $l(p) = \sum_{i=1}^{n} a_i x_i + c$, where $p = (x_1,\ldots,x_n)$, with arbitrary reals $a_1,\ldots,a_n,c$; so that

$$l\left(\sum t_i p_i\right) = \sum t_i l\left(p_i\right), \; p_i \in W, \; 0 \le t_i \le 1, \; \sum t_i = 1. \tag{6}$$

Now we can prove a uniqueness theorem.

THEOREM 1. *Let* (H) *be satisfied. If two continuous solutions of* (1) *on* clW *satisfy the same boundary condition on* ∂W, *then they coincide in the whole set* clW.

Proof. Let $\phi_1, \phi_2 : \mathrm{cl}W \longrightarrow \mathbb{R}$ be continuous and satisfy (1) and the condition

$$\phi_1(p) = \phi_2(p) \text{ on } \partial W. \tag{7}$$

Since $\mathrm{cl}W = \mathrm{conv}(\partial W)$, we may represent any $p \in \mathrm{cl}W$ as

$$p = \sum t_i p_i,\ p_i \in \partial W,\ i = 1,\dots,n,\ 0 \le t_i \le 1,\ \sum t_i = 1. \tag{8}$$

According to Lemma 1 there are continuous affine functions l_1, $l_2 : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\phi_1(p) = f\big(l_1(p)\big),\ \phi_2(p) = f\big(l_2(p)\big),\ p \in \mathrm{cl}W.$$

Since f is injective and (7) holds, l_1 coincide with l_2 on ∂W, and (8) with (6) yield

$$\phi_1(p) = f\Big(l_1\big(\sum t_i p_i\big)\Big) = f\Big(\sum t_i l_1(p_i)\Big) = f\Big(\sum t_i l_2(p_i)\Big) = \phi_2(p).$$

REMARK 1. Similarly one proves that if λ: $\mathrm{cl}W \longrightarrow \mathbb{R}$ satisfies inequality (3) and ϕ: $\mathrm{cl}W \longrightarrow \mathbb{R}$ - equation (1), both are continuous on clW and $\phi(p) = \lambda(p)$ on ∂W, then $\lambda(p) \le \phi(p)$ on clW.

3. On the real line Dirichlet's problem (D) always has a solution, provided that W satisfies (H). This is no longer true when $n > 1$ and $W \subset \mathbb{R}^n$ is a convex body. We are going to prove the following

THEOREM 2. *Assume* (H) *and let* $b: \partial W \longrightarrow \mathbb{R}$ *be a continuous function. Problem* (D) *for equation* (1) *has a continuous solution* $\phi: \mathrm{cl}W \longrightarrow \mathbb{R}$ *if and only if the set*

$$G := \{(p,x) \in \mathbb{R}^{n+1} : p \in \partial W,\ f(x) = b(p) \in \mathbb{R}\} \tag{9}$$

is a subset of an n-dimensional hyperplane M *in* $\mathbb{R}^{n+1}$, *given by*

$$M = \{(p,x) \in \mathbb{R}^{n+1} : x = a(p) + c\}, \tag{10}$$

where $c \in \mathbb{R}$, *and* $a(p)$ *is a linear functional on* $\mathbb{R}^n$.

Proof. Assume that (D) has a solution $\phi: \mathrm{cl}W \longrightarrow \mathbb{R}$. Thus condition (2) is satisfied and, by (4),

$$\phi(p) = f\big(l(p)\big) = b(p),\ p \in \partial W, \tag{11}$$

where $l(p) = a(p) + c$, and $a: \mathbb{R}^n \longrightarrow \mathbb{R}$ is a linear functional, $c \in \mathbb{R}$ is a constant. Thus the set G given by (9) is contained in the set M given by (10) with $a(p)$ and c just determined. For, take $(p,x) \in G$, then $f(x) = b(p) = f\big(l(p)\big)$ i.e., $x = l(p) = a(p) + c$, since, by (H), f is a bijection; and (p,x) belongs to M, according to (10).

On the other hand, if the set G is contained in an n-dimensional hyperplane M in $\mathbb{R}$ of form (10), then by (9) we have for $p \in \partial W$

$$x = a(p) + c \text{ whenever } f(x) = b(p). \tag{12}$$

Thus $l: \mathbb{R}^n \longrightarrow \mathbb{R}$, $l(p) = a(p) + c$, is an affine functional and (12) yields

$$b(p) = f(x) = f\big(l(p)\big), \quad p \in \partial W. \tag{13}$$

This l produces the function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $\phi(p) = f\big(l(p)\big)$, $p \in \mathbb{R}^n$, whose restriction to cl$W$ satisfies (D), cf. Lemma 1 and (13).

We conclude the Section with a lemma on solutions of (1) with values prescribed on vertices of a simplex in $\mathbb{R}^n$.

LEMMA 2. *Let* (H) *be fulfilled and let* $S := \langle p_0,\dots,p_n\rangle \subset W$ *be an n-dimensional simplex. For every system* $b_0,\dots,b_n$ *of reals there is a unique continuous solution* $\phi_S : W \longrightarrow \mathbb{R}$ *of equation* (1) *that satisfies the condition*

$$\phi_S(p_i) = b_i, \quad i = 0,1,\dots,n. \tag{14}$$

Proof. By Lemma 1 the required solution of (1) is of form (4), i.e., for $p = \big(x_1,\dots, x_n\big) \in W$, (15)

$$\phi_S(p) = f\big(l(p)\big) = f\big(a_1x_1 + \dots + a_nx_n + c\big),$$

with some reals $a_1,\dots,a_n,c$. By (14) we get, with $p_i = \big(x_{i1},\dots,x_{in}\big)$,

$$x_{i1}a_1 + \dots + x_{in}a_n + c = f^{-1}(b_i), \quad i = 0,1,\dots,n,$$

which is equivalent to:

$$\begin{cases} \big(x_{j1} - x_{01}\big)a_1 + \dots + \big(x_{jn} - x_{01}\big)a_n = f^{-1}\big(b_j\big) - f^{-1}\big(b_0\big), \\ \qquad j = 1,\dots,n, \\ x_{01}a_1 + \dots + x_{0n}a_n + c = f^{-1}\big(b_0\big). \end{cases}$$

Since the vectors $p_j - p_0$, $j = 1,\dots,n$, are linearly independent, the above system has a unique solution $a_1,\dots,a_n,c$, so that formula (15) determines the unique continuous solution $\phi_s : W \longrightarrow \mathbb{R}$ of equation (1) satisfying (14).

4. For construction of the solution of Dirichlet's problem (D) one may use solutions of inequalities associated with equation (1). The solutions are called lower, resp. upper functions.

DEFINITION 1. *Let* $W \subset \mathbb{R}^n$ *be a convex body with nonempty boundary* ∂W *and let* $b: \partial W \longrightarrow \mathbb{R}$ *be a continuous function. A function* $\lambda: \mathrm{cl}W \longrightarrow \mathbb{R}$ *(resp.* $\mu: \mathrm{cl}W - \mathbb{R}$*) is said to be a lower (upper) function for the function* b *if it satisfies the conditions:*

(i) λ *(resp.* μ*) is continuous on* clW,

(ii) $\lambda(p) \le b(p)$ *on* ∂W ($\mu(p) \ge b(p)$ *on* ∂W),

(iii) λ *satisfies inequality* (3) *on* W $\big(\mu((p+q)/2) \ge F(\mu(p),\mu(q))$ *on* W$\big)$.

The set of all lower (resp. upper) functions for b will be denoted by L_b (resp. U_b). The following properties of lower and upper functions are easily established.

LEMMA 3. *Assume* (H) *to hold.*

(a) *If* $\lambda \in L_b$ *and* $\mu \in U_b$, *then* $\lambda(p) \le \mu(p)$ *on* clW.

(b) *If* $\lambda_1, \lambda_2 \in L_b$, *then the function* $\lambda(p) = \max\{\lambda_1(p), \lambda_2(p)\}$ *(defined on* clW*) is also a lower function. This assertion is true for upper function with „max" replaced by „min".*

(c) *If* ϕ *is a solution of* (1) *satisfying* $\phi(p) = b(p)$ *on* ∂W, *and* $p_* = \min\{b(p),\ p \in \partial W\}$, *then the constant fuction* $\lambda_*(p) = \phi(p_*)$, $p \in \mathrm{cl}W$, *belongs to* L_b. *Replacing here* p_* *by* $p^* = \max\{b(p),\ p \in \partial W\}$ *we get an upper function* λ^* *for* b.

The last lemma we need corresponds to Lemma 2.

LEMMA 4. *Let* (H) *be fulfilled and let* b: $\partial W \longrightarrow \mathbb{R}$ *be a continuous function. If* $\lambda \in L_b$, *then there exists a closed convex set* $A \subset W$ *and a function* ϕ_A: $A \longrightarrow \mathbb{R}$ *which solves Problem* (D) *on* A *with the boundary function*

$$b_A(p) := \lambda(p),\ p \in \partial A. \tag{16}$$

Moreover, the function

$$\lambda_A(p) := \begin{cases} \lambda(p),\ p \in \mathrm{cl}\,W\setminus A, \\ \phi_A(p),\ p \in A \end{cases} \tag{17}$$

is also a lower function (for $b(p) := \lambda(p)$, $p \in \partial W$).

Proof. Take a simplex $S = \langle p_0,\dots,p_n \rangle \subset W$. By Lemma 2 there is a unique solution ϕ_S: $W \longrightarrow \mathbb{R}$ of equation (1) satisfying (14) with $b_i := \lambda(p_i)$, where $i = 0,1,\dots,n$. Let us put

$$B = \{p \in \mathrm{cl}W\colon \lambda(p) = \phi_S(p)\};\ A = \mathrm{clconv}B.$$

This A is a closed convex subset of W. In view of Remark 1 we have $\lambda(p) \le \phi_S(p)$ for $p \in A$, and $\lambda(p) = \phi_S(p)$ for $p \in \partial A \subseteq A$. Further, the set (9) (with $p \in \partial A$ and $b = b_A$ defined by (16)) satisfies the condition of Theorem 2. Thus there exists a solution ϕ_A of the Problem (D) on A.

Moreover, $S \subseteq A$. For, by (16) and (14), $p_i \in B$, $i = 0,1,\dots,n$. and $S = \mathrm{conv}\ \{p_0,\dots,p_n\} \subseteq \mathrm{conv}B \subseteq A$. Thus, because of the uniqueness (Theorem 1) the function ϕ_A is an extension of ϕ_S onto the set A.

It remains to prove that λ_A given by (17) represents a lower function for $b = \lambda|_{\partial W}$. Indeed, conditions (i) and (ii) (with $b = b_A$ given by (16)) are obviously satisfied by λ_A. It remains to check whether λ_A fulfils inequality (3) on W. For, by Lemma 1 and property (iii) of λ from Definition 1 there is a continuous convex function $k\colon W \longrightarrow \mathbb{R}$ such that $\lambda(p) = f(k(p))$, $p \in W$. Further, again by Lemma 1, the solution ϕ_S of (1) is given by $\phi_S(p) = f(l(p))$, $p \in W$, where $l\colon W \longrightarrow \mathbb{R}$ is a continuous affine function. Thus we may write

$$\lambda_A(p) = f(h(p)),\ p \in W \tag{18}$$

where

$$h(p) = \begin{cases} k(p), & p \in W\setminus A, \\ l(p), & p \in A. \end{cases}$$

Since $\lambda(p) \le \phi_S(p)$ for $p \in A$, and f is an increasing bijection, we have $k(p) \le l(p)$ for $p \in A$ and $h(p) = \max\{k(p), l(p)\}$ for $p \in W$. Therefore h is a convex function and function (18) satisfies inequality (3) on W.

5. Now we can construct a solution of equation (1) by using Perron's method.

THEOREM 3. *Let* (H) *be fulfilled and let* $b\colon \partial W \longrightarrow \mathbb{R}$ *be a continuous function. There exists a simplex* $S = \langle p_0,\dots,p_n\rangle \subseteq W$ *such that the function*

$$\phi\colon \mathrm{cl}W \longrightarrow \mathbb{R};\qquad \phi(p) = \sup\{\lambda(p)\colon \lambda \in L_b\}, \tag{19}$$

satisfies equation (1) *in* S, *and* $\phi(p_i) = b(p_i)$, $i = 0,1,\dots,n$.

Proof. By (c) and (a) of Lemma 2 the set L_b of functions is bounded above by the constant function $\lambda^* \in U_b$. Therefore function (19) is well-defined. Let $S \subset W$ be an n-dimensional simplex with vertices $p_o,\dots,p_n$, and let $\Phi_S\colon W \longrightarrow \mathbb{R}$ be the solution of (1) spoken about in Lemma 2, with $b_i = b(p_i)$ in (14).

By (19) there is a $\lambda_1 \in L_b$ such that

$$\lambda_1(q) > \phi(q) - 1, \ q \in \mathrm{cl}W. \tag{20}$$

In view of Lemma 4 there is a closed convex set $A_1 \subset W$, $S \subseteq A_1$, and a continuous solution $\phi_1\colon A_1 \longrightarrow \mathbb{R}$ of equation (1), $\phi_1|_S = \Phi_S|_S$, which is determined by the values of the function λ_1 on ∂A_1 (cf. (16)).

Put

$$\psi_1(p) = \begin{cases} \lambda_1(p), & p \in \mathrm{cl}W\setminus A_1, \\ \phi_1(p), & p \in A_1 \end{cases}$$

(cf. 17). According to Remark 1 we have $\psi_1(p) \geq \lambda_1(p)$, $p \in \mathrm{cl}W$, and, by (20), $\psi_1(q) > \phi(q) - 1$, $q \in \mathrm{cl}W$. Now, we take a lower function $\lambda_2 \in L_b$ for which

$$\lambda_2(q) \geq \phi(q) - 1/2, \ q \in \mathrm{cl}W, \tag{21}$$

and we put

$$u_2(p) = \max\{\lambda_2(p), \psi_1(p)\}, \ p \in \mathrm{cl}W. \tag{22}$$

By Lemma 3(b) we know that $u_2 \in L_b$. We use again Lemma 4 to find a closed convex set $A_2 \subset W$, a solution $\phi_2 \colon A_2 \longrightarrow \mathbb{R}$ of equation (1), determined by the values of u_2 on ∂A_2. The function

$$\psi_2(p) = \begin{cases} u_2(p), & p \in \mathrm{cl}W \setminus A_2, \\ \phi_2(p), & p \in A_2 \end{cases}$$

satisfies, cf. Remark 1 and (22) with (21),

$$\psi_2(p) \geq u_2(p) > \phi(p) - 1/2, \quad p \in \mathrm{cl}W.$$

Moreover, $S \subseteq A_2$ and $\phi_2|_S = \Phi_S|_S$.

We continue this procedure by induction and arrive to an increasing sequence $(\psi_n)_{n\in\mathbb{N}}$ of functions, defined as follows

$$\psi_n(p) := \begin{cases} u_n(p), & p \in \mathrm{cl}W \setminus A_n, \\ \phi_n(p), & p \in A_n. \end{cases} \tag{23}$$

where:

$u_n(p) = \max\,\{\lambda_n(p), \psi_{n-1}(p)\}, \; p \in \mathrm{cl}W,$

$\lambda_n \in L_b$ is chosen to satisfy $\lambda_n(q) > \phi(q) - 1/n, \; q \in \mathrm{cl}W,$

A_n is the set from Lemma 4 such that there exists a continuous solution $\phi_n \colon A_n \longrightarrow \mathbb{R}$ of equation (1), determined by the values of u_n on ∂A_n, and we have $S \subseteq A_n$ and $\phi_n|_S = \Phi_S|_S$.

The functions ψ_n are continuous and (as lower functions, cf. Lemma 4) satisfy inequality (3) on W. From the inequalities

$$\phi(p) - 1/n \leq \psi_n(p) \leq \phi(p), \quad p \in \mathrm{cl}W, \; n \in \mathbb{N},$$

we see that

$$\phi(p) = \lim_{n\to\infty} \psi_n(p), \tag{24}$$

uniformly on clW so that ϕ is continuous on clW.

Now, since the simplex S is contained in all the sets A_n, $n \in \mathbb{N}$, from (24) and (23) we conclude that the restriction of ϕ to S is the restriction of Φ_S to S, too. Thus ϕ satisfies equation (1) on S and the condition $\phi(p_i) = b(p_i)$ on the vertices p_i of S.

REMARK 2. Similarly one can prove that, under assumptions of Theorem 3 there exists a simplex $\hat{S}$ contained in W such that the function $\hat{\phi}(p) = \inf\ \{\mu(p),\ \mu \in U_b\}$, $p \in$ clW, satisfies equation (1) in $\hat{S}$, and coincides with b on the vertices of $\hat{S}$.

REMARK 3. We owe to Professor P. Volkmann the following example illustrating Theorem 3.

Consider the Jensen equation (1) (i.e. $F(x,y) = (x+y)/2$), take $n=3$, $W = [0,1]^2$, put $p_0 = (0,0)$, $p_1 = (1,0)$, $p_2 = (1,1)$, $p_3 = (0,1)$, and define the function $b : \partial W \longrightarrow \mathbb{R}$ as follows: $b(p_0) = b(p_2) = 0$, $b(p_1) = b(p_3) = 1$, and b is linear on all the sides of the square and is continuous on ∂W.

The function ϕ given by (19) is now the maximal convex function (cf. (3) with our F) satisfying (2), i.e. the function

$$\phi(x,y) = \begin{cases} x - y, & (x,y) \in \langle p_0,p_1,p_2\rangle, \\ y - x, & (x,y) \in \langle p_0,p_2,p_3\rangle \end{cases}$$

The simplex S such that ϕ satisfies equation (1) on S (i.e., is affine) is either the triangle $S = \langle p_0,p_1,p_2\rangle$ or $S = \langle p_0,p_2,p_3\rangle$.

Similarly, the function $\hat{\phi}$ from Remark 2 is given by

$$\hat{\phi}(x,y) = \begin{cases} x + y, & (x,y) \in \langle p_0, p_2, p_3 \rangle, \\ 2 - x + y, & (x,y) \in \langle p_1, p_2, p_3 \rangle \end{cases}$$

and $\hat{S}$ is one of the triangles occurring in this definition.

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