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On certain properties of invariant subsets in bioperands

Dedicated to Professor Zenon Moszner with best wishes on his 60-th birthday

In the present paper we deal with certain properties of invariant subsets in a bioperand over semigroups. According to [1] we quote the following definitions.

DEFINITION 1. *A left operand* ${}_SM$ *over a semigroup* S *is said to be a non-empty set* M *endowed with a mapping* $S \times M \ni (s, a) \longrightarrow sa \in M$ *such that*

$$\forall\ x \in M,\ \forall\ s_1, s_2 \in S : s_1(s_2x) = (s_1s_2)x.$$

The definition of a right operand M_T over a semigroup T is analogous.

DEFINITION 2. *A bioperand* ${}_SM_T$ *over semigroups* S *and* T *is called a non-empty set* M *which is both a left operand* ${}_SM$ *and a right operand* M_T, *and the following condition holds*:

$$\forall\ x \in M,\ \ \forall\ s \in S,\ \forall\ t \in T : (sx)t = s(xt).$$

The set M *is called a fibre of the bioperand* ${}_SM_T$ (cf. [2]).

DEFINITION 3. *A non-empty subset* $N \subset M$ *is said to be a left [right] invariant subset of a left [right] operand* ${}_SM$ $[M_T]$ *if* $SN \subset N$ $[NT \subset N]$.

The definitions of left and right invariant subsets in a bioperand ${}_SM_T$ are analogous.

As immediate consequences of these definitions we obtain the following properties of invariant subsets in a bioperand ${}_SM_T$:

1. The fibre M is an invariant subset in the bioperand ${}_SM_T$.

2. Let $(N_i : i \in I)$ be any non-empty family of invariant subsets in the bioperand ${}_SM_T$.

The union $\bigcup (N_i : i \in I)$ is an invariant subset in the bioperand ${}_SM_T$.

The non-empty intersection $\bigcap (N_i : i \in I)$ is an invariant subset in the bioperand ${}_SM_T$.

Left and right invariant subsets in the bioperand ${}_SM_T$ have analogous properties.

Assume that D denotes the family of all invariant subsets in the bioperand ${}_SM_T$ together with the empty set ϕ.

The family D has the following properties:

(i) $\phi \in D$ and $M \in D$,

(ii) if $N_1, N_2 \in D$, then $N_1 \cup N_2 \in D$,

(iii) if $(N_i : i \in I)$ is any family of subsets belonging to D, then $\bigcap (N_i : i \in I) \in D$.

It follows immediately from properties (i) - (iii) that by means of the family D we can generate the topology in the set M. The family D is a family of closed subsets in the space M.

This topology we shall call a topology generated by the family of invariant subsets in the bioperand ${}_SM_T$.

Let D_l $[D_r]$ denote the family of all left [right] invariant subsets in the bioperand ${}_SM_T$. Let the empty set ϕ belongs to $D_l[D_r]$.
Using the family $D_l[D_r]$ we can generate the topology in the set M.

Let the topology in the bioperand ${}_SM_T$ be generated by the family D of invariant subsets. Let A be any subset of the fibre M in the bioperand ${}_SM_T$.

A closure $\bar{A}$ of the subset A we define as follows:

$$\bar{A} := \bigcap (N \in D : A \subset N).$$

Analogously we define a closure $(\bar{A})_l$ $[(\bar{A})_r]$ of the subset A in the topological space M generated by the family D_l $[D_r]$.

For the sake of simplicity we introduce some convention. Let S be any semigroup and an element 1 is not in the set S. An operation in the semigroup S we extend to an operation in the set $S \cup \{1\}$ as follows:

$$1 \cdot 1 = 1, \qquad 1 \cdot s = s \cdot 1 = s,$$

for any $s \in S$.
The semigroup will be denote by S^1.
In the bioperand ${}_{S^1}M_{T^1}$ we put

$$1x = x1 = x$$

for any $x \in M$.

Let A be a subset of the bioperand ${}_SM_T$, we put

$$SA := \{z \in M : z = s\,x,\ s \in S,\ x \in A\}.$$

Analogously we define subsets AT, SAT.

Let us notice that $S^1A = SA \cup A$, $AT^1 = AT \cup A$, $S^1A\,T^1 = SAT \cup SA \cup AT \cup A$.

THEOREM 1. *Let* A *be any subset of the fibre* M *of the bioperand* ${}_SM_T$.
Then the sets $(\bar{A})_l$, $(\bar{A})_r$, $\bar{A}$ *have the following form:*

(a) $(\bar{A})_l = S^1A$,

(b) $(\bar{A})_r = AT^1$,

(c) $\bar{A} = S^1A\,T^1$.

Proof. **(a)** Let us notice that the set S^1A is a left invariant subset in the bioperand ${}_SM_T$. Indeed, $S(S^1A) = (SS^1)A \subset S^1A$. Since every left invariant subset in the bioperand ${}_SM_T$ containing A contains as well the subset S^1A, so $(\bar{A})_l = S^1A$.

Analogously we can prove equalities (b) and (c).

If A is a non-empty subset of the fibre M of the bioperand ${}_SM_T$, then $\bar{A}$ $\left[(\bar{A})_l,\ (\bar{A})_r\right]$ is called a [left, right] invariant subset in the bioperand ${}_SM_T$ generated by the subset A.

Introducing such notions as an interior of a set, a boundary of a set, a derived set and using well-known theorems of the topology we can obtain many properties of [left, right] invariant subsets in the bioperand ${}_SM_T$.
For example we shall consider the interior IntA of a set A, the boundary FrA of a set A, the derived set A^d in the bioperand ${}_SM_T$ with respect to the topology generated by the family D_l.

THEOREM 2. *Let* A *be any subset of the fibre* M *of the bioperand* ${}_SM_T$. *Let the topology in the bioperand* ${}_SM_T$ *be generated by the family* D_l.

Then the interior IntA *and the boundary* FrA *are the following subsets*:

(a) $\mathrm{Int}A = \{x \in A : x \notin S(M\setminus A)\}$,

(b) $\mathrm{Fr}A = \big(A \cap S(M\setminus A)\big) \cup \big(SA \cap (M\setminus A)\big)$.

Proof. **(a)** From the definition of the interior of a set we have $\mathrm{Int}A = M\setminus(\overline{M\setminus A})_1$. Since $(\overline{M\setminus A})_1 = S(M\setminus A) \cup (M\setminus A)$ the eqality $\mathrm{Int}A = M\setminus[S\ (M\setminus A) \cup (M\setminus A)]$ holds. Whence the equality (a) is satisfied.

(b) From the definition of the boundary of a set we have $\mathrm{Fr}A = (\bar{A})_1 \cap (\overline{M\setminus A})_1$. Hence $\mathrm{Fr}A = S^1A \cap S^1(M\setminus A) = (SA\cup A) \cap \big(S(M\setminus A) \cup (M\setminus A)\big) = \big(SA \cap S\ (M\setminus A)\big) \cup \big(A \cap S(M\setminus A)\big) \cup (SA \cap (M\setminus A)\big)$. Let us notice that $SA \cap S(M\setminus A) \subset \big(A \cap S(M\setminus A)\big) \cup \big(SA \cap (M\setminus A)\big)$. Indeed, if $x \in SA \cap S(M\setminus A)$, then $x \in SA$ and $x \in S(M\setminus A)$. In the case when $x \in A$ we obtain that $x \in A \cap S(M\setminus A)$. In the case when $x \notin A$ we obtain that $x \in M\setminus A$, so $x \in SA \cap (M\setminus A)$. Hence $\mathrm{Fr}A = \big(A \cap S(M\setminus A)\big) \cup \big(SA \cap (M\setminus A)\big)$.
Therefore the equality (a) is satisfied.

The definition of an accumulation point of a set implies the following corollary.

COROLLARY 1. *Let* A *be any subset of the fibre* M *of the bioperand* ${}_SM_T$. *A point* $x \in M$ *is the accumulation point of the subset* A *if and only if* $x = s\ y$ *for same elements* $s \in S$ *and* $y \in A\setminus\{x\}$.

The derived set A^d is a set of all accumulation points of the subset A.

DEFINITION 4. *A [left, right] invariant subset* N *in the bioperand* ${}_SM_T$ *is called minimal, if there is no proper subset of the set* N *being a [left, right] invariant subset in the bioperand* ${}_SM_T$.

COROLLARY 2. *A subset* L *is a minimal left invariant subset in the bioperand* ${}_SM_T$ *if and only if* $(\langle\bar{x}\rangle)_l = L$ *for any* $x \in L$.

An analogous property we have for minimal [right] subsets in the bioperand ${}_SM_T$.

THEOREM 3. *Let* L *be a minimal left invariant subset in the bioperand* ${}_SM_T$.
Then for any $t \in T$ *the subset* Lt *is a minimal left invariant subset in the bioperand* ${}_SM_T$.

Proof. Let $z \in Lt$ then $z = x\ t$, $x \in L$. From Corollary 2 we have $S^1x = L$. Therefore $\left(\langle\bar{z}\rangle\right)_l = S^1z = S^1(x\ t) = (S^1x)t = Lt$. Hence by Corollary 2 it follows that Lt is a minimal left invariant subset in the bioperand ${}_SM_T$.

THEOREM 4. *Let* ${}_SM_T$ *be a bioperand, and* N *its minimal invariant subset containing at least one minimal left invariant subset in the bioperand* ${}_SM_T$.
Then the subset N *is a union of all minimal left invariant subsets in the bioperand* ${}_SM_T$ *contained in the subset* N.

Proof. Let K be the union of all minimal left invariant subsets in the bioperand ${}_SM_T$ contained in the set N.
We shall prove that K = N. The subset K is evidently a left invariant subset in the bioperand ${}_SM_T$. We shall show that K is a right invariant subset in the bioperand ${}_SM_T$, i.e. $KT \subset K$. Let $z \in K\ T$ then $z = x\ t$, where $x \in K$, $t \in T$. By the definition of the subset K it follows that there exists a minimal left invariant subset $L \subset N$ such that $x \in L$. From Theorem 3 it follows that Lt is a minimal left invariant subset in the bioperand ${}_SM_T$. Furthermore,

$Lt \subset N$ and so $Lt \subset K$. Whence $z = x\,t \in K$. Therefore K is an invariant subset in the bioperand ${}_SM_T$ contained in the subset N, i.e. $K = N$.

REFERENCES

[1] Clifford A., Preston G., *The Algebraic Theory of Semigroups*, vol. 2, Mir Publishers, Moscow, 1972 (Russian).

[2] Tabor J., *Algebraic objects over a small category*, Diss. Math., 155 (1978).