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Distance–line spaces

Dedicated to Professor Zenon Moszner with best wishes on his 60-th birthday

Summary. An axiom system is presented that characterizes some class of metric spaces in which the concept of a line (not necessarily continuous) can be defined.

0. Introduction

0.0. It is well known that in some metric spaces one can define the notion of a (straight-) line by means of the distance. In particular if a metric space is complete, convex and externally convex, then through each pair of points there passes at least one line i.e. a set of points congruent with the one-dimensional Euclidean metric space[1]. The uniqueness of the líne passing through two distinct points results from a supplementary condition, the so called "two-triple property"[2].

Similarly, in another important class of metric spaces - called elliptic spaces, one defines (in another way) closed-lines having analogical properties[3].

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[1] [1] p. 56 (th.21.1 and df.21.1); cf also [6] p.7.

[2] [1] p.56-57 (df.21.2, th.21.3).

[3] [1] p.219-221.

The purpose of this paper is to propose a still more general concept of the line in metric spaces.

I describe heree (Df.0.1) a class of spaces with such a distance function which permits to define the set of (straight-) lines.

At first I explain the meaning of the notions of the distance and of the (straight-) line.

0.1. By the distance function I understand a function with values which need not be real numbers[4] but belong to an arbitrary ordered group. Specifying, I assume that the distance function ρ in the space E satisfies the following condition

(M0) $\rho : E^2 \longrightarrow M$, where $M \subseteq G^+$, and G^+ is the nonnegative part of a fixed totally ordered abelian group $G=(G,+,0,\geq)$.

Furthermore the function ρ must have all properties of the metric, i.e. the following theorem should be valid:

THEOREM 0.1. *The distance function* ρ *satisfies condition* (M0) *and the five following conditions:*

(M1) $\forall_{X \in E}\ \rho(X,X) = 0$,

(M2) $\forall_{X,Y \in E}\ \rho(X,Y) \geq 0$,

(M3) $\forall_{X,Y \in E}\ \rho(X,Y) = \rho(Y,X)$,

(M4) $\forall_{X,Y,Z \in E}\ \rho(X,Y) + \rho(Y,Z) \geq \rho(X,Z)$,

(M5) $\forall_{X,Y \in E}\ [\rho(X,Y)=0 \Longrightarrow X=Y]$.

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[4] Cf [6] p.30; [1] p.32.

0.2. Lines are meant here as sets of points i.e. the set L of all (straight-) lines satisfies the condition

(L0) $\forall_{\lambda \in L}\ \lambda \subseteq E.$

Moreover, the line is understood, in conformity with the Theory of Incidence Structures, as a block[5] characterized by the common properties of closed[6] and open[7] lines, e.g. properties (1), (2), (3), (16) and either (18) or (20) from the 1.§ of [7] (see also [6] p.8, and [1] th.21.1 on p.56, th.83.2 on p.221).

Therefore I require, that the following theorem should hold:

THEOREM 0.2. *The set of lines* L *satisfies condition* (L0) *and the three following conditions*[8]

(L1) $\forall_{\lambda \in L}\ \exists_{X,Y}\ (X \neq Y \wedge X,Y \in \lambda),$

(L2) $\forall_{X,Y \in E}\ (X \neq Y \Rightarrow \exists^{=1}_{\lambda \in L}\ X,Y \in \lambda),$

(L3) $\forall_{\lambda \in L}\ \exists_{X \in E}\ (X \notin \lambda).$

0.3. According to the suggestion of K. Menger, who proposed to use the distance as the only primitive notion[9] one usually defines

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[5] [2] p.1.

[6] [2] p.24; [7] p.7.

[7] [7] p.10.

[8] We use here the usual logical symbols. The symbol $\exists^{=1}_{x}$ means that there exists exactly one x.

[9] See [1] p.33.

open and closed lines by means of the distance function (using two different definitions). I combine both these notions of lines (Df. 4.1, see also Df.2.1), taking as a point of departure their common metrical properties described in the following two theorems:

THEOREM 0.3. *For each* $\lambda \in L$ *and* $X, Y \in \lambda$, *such that* $X \neq Y$, *we have*

(B1) $\forall_{Z \in E} \, [\rho(X,Y) + \rho(Y,Z) = \rho(X,Z) \Longrightarrow Z \in \lambda]$,

(B2) $\forall_{Z \in E} \, [\rho(X,Z) + \rho(Z,Y) = \rho(X,Y) \Longrightarrow Z \in \lambda]$,

(B3) $\exists_{Z \in \lambda} \, [\rho(X,Z) + \rho(Z,Y) = \rho(X,Y) \wedge X \neq Z \neq Y]$.

THEOREM 0.4. *There exists a set* $M^{\pm} \subseteq G$ *such that for each* $\lambda \in L$ *and* $A \in \lambda$ *there exists a function* φ *such that*

(C0) φ *is a bijection of* λ *onto* $M^{\pm}$,

(C1) $\varphi(A) = 0$,

(C2) $\forall_{X_1, X_2 \in \lambda} \, \forall_{z \in M^{\pm}} \, [\varphi(X_1) - \varphi(X_2) = z \Rightarrow \rho(X_1, X_2) = |z|]$.

In the case of open lines we have $M^{\pm} = G$, and for closed lines we have $M^{\pm} =]-m,m]$, where $0 < m < \infty$. The set M from condition (M0), in both these cases, is equal to G^{+} and $[0,m]$, respectively.

0.4. Theorems 0.1-0.4 are valid in the affine space as well as in the projective space; they are valid in the Euclidean geometry and in the non-Euclidean geometries (hyperbolic and elliptic), and in the Minkowski space i.e., in the affine space with Minkowski's metric.

For all these metric spaces one assumes usually that G is the additive group of real numbers and ρ is continuous function. This last condition is called "completeness" of metric space[10]. In this paper I replace this condition by another, weaker, condition, which I propose to call "semi-completeness" of space E and which is expressed in the following[11]

THEOREM 0.5. *For every* $X,Y \in E$ *and* $x,y \in M$, *if the inequalities*

$$|x - y| \leq \rho(X,Y) \leq x + y$$

hold, then there exists a point Z such that

$$\rho(X,Z) = x \text{ and } \rho(Y,Z) = y.$$

This theorem excludes one-dimensional spaces from our considerations.

0.5. I shall define the distance-line space as a structure (E,ρ) which satisfies the axioms of the metric space i.e. Theorem 0.1, and which satisfies also the "semi-completeness" condition (i.e. Theorem 0.5) and some other coditions that permit to define the notion of the line verifying Theorems 0.2-04.

DEFINITION 0.1. *For a nonempty set E, the structure* (E,ρ) *is said to be a distance-line space* (DL-*space*) *iff there exists an abelian totally ordered group*

$$G = (G, +, 0, \geq),$$

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[10] [1] def.10.1 on p.28.

[11] Cf [3] p.653 (A2 and A3).

and a subset M *of* G *such that the following axioms hold:*

(DL0) $\rho:E^2 \longrightarrow M$, *where* $M = [0,m]$ *and either* $0 \neq m \in G^+ = \{x\in G: x\geq 0\}$ *or* $m = \infty$ *i.e.* $M = G^+$. *The set* $M_o \stackrel{df}{=}]0,m[\neq \emptyset$.

(DL1) *Let*[12]

$$X,Y \in E, \tag{0.1}$$

$$x,y \in M. \tag{0.2}$$

There exists a Z *such that*[13]

$$XZ = x \quad \text{and} \quad YZ = y \tag{0.3}$$

iff $|x - y| \leq XY \leq x + y$.

(DL2) *If* (0.1) *and* (0.2) *hold* $X \neq Y$, $XY + x = y$, *then there exists at most one* Z *satisfying condition* (0.3).

(DL3) *For each points* X,Y,Z,U *if* $Y \neq Z$, $XY = ZU$, $YU > |XY - XU|$ *and* $XY + YZ = XZ \notin M_o$, *then the following equivalence holds:* $(XU = YZ \Longleftrightarrow YU \notin M_o)$.

1. Consequences of (DL0) and (DL1)

We shall show that from the single axiom (DL1) and conditions (DL0), it results

THEOREM 1.0. *The structure* (E,ρ) *with* (DL0)-(DL3) *is a pseudo--metric space*[14] *i.e. it satisfies axioms* (M1)-(M4).

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[12]Compare [4] Ax.1a) on p.283.

[13]Instead of $\rho(X,Y)$, we write XY.

[14][5] p.169.

Proof. (M2) results directly from (DL0).

Let XX=x and hence $x \geq 0$. Therefore, we have $|x-0| \leq XX \leq x+0$, and hence, by (DL1), there exists a point Z such that XZ=x and XZ=0, which implies x=0 i.e. axiom (M1).

Now let YX=y. Putting Z=X, we obtain from (DL1): $|XX-y| \leq XY \leq XX+y$. Thus, by (M2) and (M1), we have $y \leq XY \leq y$ i.e. XY=y=YX, and hence (M3) holds too.

Finally, (M4) results directly from (DL1) and (M3).■

THEOREM 1.1. $|XZ - YZ| \leq XY$.

This theorem results directly from (DL1).■

THEOREM 1.2. $\forall_{X \in E} \forall_{z \in M} \exists_{y \in E} \; XY = z$.

To prove this, we put in (DL1) Y=X and we obtain $|z-z| \leq XX \leq z+z$, what implies the existence of Y.■

We define the betweenness relation of three points as follows:

DEFINITION 1.1. *We said that the point* Y *lies between the points* X *and* Z *and we write* $\beta(X,Y,Z)$ *or* X|Y|Z *iff* XY+YZ = XZ.

Directly from this definition we obtain

THEOREM 1.3. $X|Y|Z \Longleftrightarrow Z|Y|X$,

THEOREM 1.4. X|X|Y.

It is easy to prove

THEOREM 1.5. *If* $XY \leq x \in M$, *then there exists a* Z *such that* X|Y|Z *and* XZ = x.

Proof. We put y=x-XY and we obtain $0 \leq x-y=XY \leq x+y$, and hence the existence of Z results from (DL1).■

Similarly we prove

THEOREM 1.6. *If* $0 \leq x \leq XY$, *then there exists a* Z *such that* X|Z|Y *and* XZ = x.

Proof. We put y=XY-x and we obtain $|x-y| \leq XY=x+y$, and hence the existence of Z results from (DL1).■

To prove some other properties of β it is useful to define a relation characterizing the order of 4 points on a line:

DEFINITION 1.2. $X|Y|Z|U \overset{df}{\Longleftrightarrow} XY + YZ + ZU = XU$.

Directly from this definition we get:

THEOREM 1.7. $X|Y|Z|U \Longrightarrow U|Z|Y|X$,

THEOREM 1.8. $X|Y|Z \wedge X|Z|U \Longrightarrow X|Y|Z|U$.

The implication converse to the latter is also true:

THEOREM 1.9. $X|Y|Z|U \Longrightarrow X|Y|Z \wedge X|Z|U$.

Proof. From the hypothesis we obtain (1)) $XY + YZ + ZU = XU$, and from (M4) the inequalities: (2) $XZ+ZU \geq XU$ and (3) $XY+YZ \geq XZ$. From (3) and (1) we obtain $XU=(XY+YZ)+ZU \geq XZ+ZU$, and hence, by (2), the equality (4) $XZ+ZU=XU$. From (4) and (1) we infer that (5) $XZ=XY+YZ$. Equalities (4) and (5) give us the thesis.■

Now we shall define two different notions of the half-line (which coincide in the case $m = \infty$). To define the second one it is more convenient to use the set

$$M_1 \overset{df}{=} M_o \cup \{0\}.$$

DEFINITION 1.3. *If*

$$XY \in M_o, \tag{1.1}$$

then

a) *the set* $\overrightarrow{XY} \overset{df}{=} \{Z \in E : X|Z|Y \vee X|Y|Z\}$ *is called a closed half-line* XY,

b) *the seet* $\overline{XY}^{c} \overset{df}{=} \{Z \in \overrightarrow{XY} : XZ \in M_1\}$ *is called an open half-line* XY.

Directly from this definition we have

THEOREM 1.10. *If* (1.1) *holds, then*

a) $X,Y \in \overline{XY}^{\subset} \subseteq \overrightarrow{XY}$,

b) $Z \in \overrightarrow{XY} \Longleftrightarrow |XZ - XY| = YZ$.

Using (DL2) we shall prove, in section 2, two theorems (Th.2.3 and Th.2.8), which can be proved without (DL2) but only in the following restricted form:

THEOREM 1.11. (Cf Th.2.3) *If* (1.1) *holds and* $x \in M$, *then there exists at least one* $Z \in \overrightarrow{XY}$ *such that* $XZ = x$.

Proof. For two cases: $x \geq XY$ and $x < XY$, it suffices to use Theorems 1.5 and 1.6 respectively.■

THEOREM 1.12. (Cf Th.2.8) $\forall_{x \in M_o} \exists_{y \in M_1} \; y > x$.

Proof. For m finite we can put $y=m$, for $m=\infty$, $y=x+x$.■

THEOREM 1.13. $\exists_{x \in M_o} \; x+x \in M$.

Proof. The existence of $y \in M_o$ results from (DL0). If $2y \notin M$, then $M=[0,m]$, where $m<\infty$, and in this case it suffices to put $x=m-y$.■

2. Consequences of (DL0)-(DL2)

K. Menger has stressed the importance of the so called "two-triple property"[15] in the construction of an Euclidean line in distance geometries, observing that this property excludes the existence of a "fourchette" [M7]. We adopt here a certain form of the two-triple property as a new axiom: (DL2). This axiom together with (DL1) implies not only the two-triple property (Th.2.7) but

[15] [6] p.8; [1] p.56.

also Menger's[16] "absence de lentilles" (Th.2.2), and the unicity of point on a half-line (Th.2.3).

Furthermore axiom (DL2) permits us to show that the set M_o has neither maximal nor minimal elements (Th.2.8) and permits also to prove that DL-space is a metric space (Th.2.0).

A direct corollary from (DL2) is

COROLLARY 2.1. *If*

$$X \neq Y, \tag{2.1}$$

$$XZ_1 = XZ_2, \tag{2.2}$$

$$X|Y|Z_k \qquad \textit{for } k=1,2, \tag{2.3}$$

then $Z_1 = Z_2$.

THEOREM 2.0. *The DL-space is a metric space.*

Proof. In consequence of Th.1.0, only condition (M5) is to be checked. Let (1) $XY=0$ and (2) $x \in M_o$. By Th.1.2. there is a Z_1 such that (3) $XZ_1=x$. Thus $|x-x|<XZ_1<x+x$ and hence, by (DL1), there exists a Z_2 such that (4) $XZ_2=x$ and (5) $Z_1Z_2=x$.

From (DL1) it follows that $|YX-Z_kX|\leq YZ_k\leq YX+Z_kX$ for $k=1,2$, and hence, by (1), (3), (4), we obtain $x\leq YZ_k\leq x$, and finally (6) $YZ_k=x$. From (1) we have $XY+x=x$, and hence from Cor.2.1 it results that if $X\neq Y$, then there exists at most one Z_k such that $XZ_k=YZ_k=x$. Therefore, from (3), (4), (6), we obtain $Z_1=Z_2$. But from (2),(5) we get $Z_1\neq Z_2$, and hence the supposition $X\neq Y$ is false.■

Theorem 2.0 implies

THEOREM 2.1. *If* $X|Y|X$, *then* $X = Y$.

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[16] [6] p.7-8.

Proof. By the definition we obtain XY+YX=XX, i.e. 2.XY=0. Since G is an ordered group, it has no cyclic elements[17], and we obtain the equality XY=0, i.e., by (M5), X=Y.■

THEOREM 2.2. *If* (2.2) *holds,*

$$XY \in M_1, \tag{2.4}$$

$$X|Z_k|Y \qquad \textit{for } k=1,2, \tag{2.5}$$

then $Z_1 = Z_2$.

Proof. By Th.1.12 there is a u∈M such that u>XY. By Th.1.5, there exists a point U such that XU=u and X|Y|U, and hence $X|Z_k|Y|U$. Therefore, $UZ_1=u-XZ_1=u-XZ_2=UZ_2$, and also $U|Y|Z_k$ for k=1,2. Finally, by Cor.2.1, we obtain $Z_1=Z_2$.■

THEOREM 2.3. *If*

$$XY \in M_o, \tag{2.6}$$

then for every x ∈ M *there exists exactly one point* Z *such that* $Z \in \overrightarrow{XY}$ *and*

$$XZ = x, \tag{2.7}$$

Proof. We get the existence of Z from Th.1.11. The uniqueness, for the cases x<XY and x≥XY, results from Th.2.2 and Cor.2.1, respectively.■

Directly from this theorem we obtain

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[17]Cf [8] th.7.4.1 on p.130.

COROLLARY 2.2. *If* (2.6) *holds, then for every* $x \in M_1$ *there exists exactly one point* Z *such that* (2.7) *and* $Z \in \overline{XY}^{\subset}$ *hold.*

THEOREM 2.4. *If* (2.6) *holds,*

$$XZ_1 \leq XZ_2, \tag{2.8}$$

$$Z_k \in \overrightarrow{XY} \qquad \textit{for } k=1,2, \tag{2.9}$$

then either $X|Y|Z_1|Z_2$ *or* $X|Z_1|Y|Z_2$ *or* $X|Z_1|Z_2|Y$ *and hence always* $X|Z_1|Z_2$.

To prove this theorem, we can use two following lemmas:

LEMMA 2.1. *Relations* (2.1), (2.3), (2.8), *imply* $X|Y|Z_1|Z_2$.

LEMMA 2.2. *Relations* (2.4), (2.5), (2.8), *imply* $X|Z_1|Z_2|Y$.

Proof of Lem.2.1. From Th.1.6 it results, that there exists such a Z_o, that $Y|Z_o|Z_2$ and $YZ_o=YZ_1=XZ_1-XY$. Therefore we have $X|Y|Z_o|Z_2$, and hence $X|Y|Z_o$. On the other hand $XZ_o=XY+YZ_o=XY+YZ_1=$ $=XZ_1$, and hence, by Cor. 2.1, we obtain $Z_o=Z_1$, and finally $X|Y|Z_1|Z_2$.■

Proof of Lem.2.2. From Th.1.6 it results, that there exists such a Z_o, that $X|Z_o|Z_2$ and $XZ_o=XZ_1$. Therefore we have $X|Z_o|Z_2|Y$, and hence $X|Z_o|Y$. Thus, by Th.2.2, we obtain $Z_o=Z_1$, and finally $X|Z_1|Z_2|Y$.■

Proof of Th.2.4. If $XY \leq XZ_1$, we obtain the thesis by Lem.2.1. If $XZ_1 \leq XY \leq XZ_2$, then, by Th.1.8, $X|Z_1|Y|Z_2$, and hence $X|Z_1|Z_2$. If $XZ_2 \leq XY$, then the thesis results from Lem.2.2.■

Directly from Lem.2.1 we obtain

COROLLARY 2.3. *If* (2.1), (2.3) *and* $Y \neq Z_1$ *hold, then* $Z_2 \in \overline{YZ_1}^{\subset}$.

THEOREM 2.5. *If*

$$XY_k \in M_o \qquad \textit{for } k=1,2, \tag{2.10}$$

$$Y_2 \in \overrightarrow{XY_1}, \tag{2.11}$$

then $\overrightarrow{XY_1} = \overrightarrow{XY_2}$

Proof. From (2.10) we infer that $Y_2 \in \overrightarrow{XY_1}$ $\Longleftrightarrow$ $[X|Y_1|Y_2 \vee X|Y_2|Y_1]$ $\Longleftrightarrow$ $Y_1 \in \overrightarrow{XY_2}$. Therefore it suffices to prove that $\overrightarrow{XY_1} \subseteq \overrightarrow{XY_2}$. In fact, if $Z \in \overrightarrow{XY_1}$, then from (2.11) and Th.2.4, we have either $X|Z|Y_2$ or $X|Y_2|Z$, and hence $Z \in \overrightarrow{XY_2}$.■

COROLLARY 2.4. *From* $XY_1 \in M_o$ *and* $Y_2 \neq X$ *it results also that* $\overline{XY_1}^{\subset} = \overline{XY_2}^{\subset}$.

THEOREM 2.6. *If* (2.6) *and* (2.9) *hold, then* $Z_1Z_2 = |XZ_1 - XZ_2|$.

Proof. Omitting the trivial cases: $X=Z_1$, $X=Z_2$, $Z_1=Z_2$, we can assume that one from distances XZ_k, for example XZ_1, belongs to M_o. Therefore, by Th.2.5 we obtain $\overrightarrow{XY} = \overrightarrow{XZ_1}$ and hence the thesis results from Th.1.10b).■

THEOREM 2.7. *If* $X|Y|Z$, $Y|Z|U$, $Y \neq Z$ *and* $XZ+ZU \in M$, *then* $X|Y|Z|U$.

Proof. From Th.1.5, it results the existence of U' such that $X|Z|U'$ and $XU'=XZ+ZU$. Thus $ZU'=ZU$ and $X|Y|Z|U'$. Therefore $Y|Z|U'$ i.e. $YU'=YZ+ZU'$ and hence $YU'=YU$. Finally, by Cor. 2.1, we obtain $U=U'$, i.e. $X|Y|Z|U$.■

Now we shall prove

THEOREM 2.8. *The set* M_o *has neither the minimal element, nor the maximal element.*

To prove it we need the following lemmas:

LEMMA 2.3. *If*

$$x \in M_o \tag{2.12}$$

and

$$2x \in M_1, \tag{2.13}$$

then

$$\exists_y \; 0 < y < x. \tag{2.14}$$

LEMMA 2.4. *If* $2a = m$, *then there exists an* $x \in M_o \setminus \{a\}$.

LEMMA 2.5. *Hypothesis* (2.12) *implies* (2.14).

Proof of Lem.2.3. By Th.1.2 and Th.1.6 there are points X,Y,Z such that XZ=2x, X|Y|Z and XY=x. Thus, we have YZ=x and |x-x|<XY<x+x. The existence of a point U such that (Fig.1)

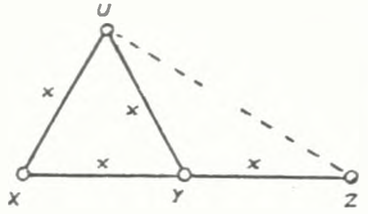

Fig. 1

(1) XU = YU = x results from (DL1), and from (M4) we obtain UZ≤UY+YZ=2x. Therefore it is easy to see that (2) UZ < 2x, because the equality UZ=2x implies Z|Y|U and hence, by Cor.2.1, X=U, contrary to (1). The supposition UZ=x leads to a contradiction too, because in this case we obtain X|U|Z, and by Th.2.2, U=Y, contrary to (1). Thus (3) UX≠x. Let us suppose that (2.14) does not hold. Therefore UZ≥x, and from (2), (3) we obtain x<UZ<2x. In this case, putting y=UZ-x we obtain 0<y<x, and hence (2.14) holds.■

Proof of Lem.2.4. Suppose the lemma is false i.e. M_o=⟨a⟩, and hence (1) M=⟨0,a,2a⟩. The existence of X,Y such that (2) XY=a results from Th.1.2. Therefore we have |a-a|<XY<a+a and |2a-2a|<XY<2a+2a, and hence there exist Z and U such that (3) XZ=YZ=a, (4) XU=YU=2a. By Th.1.6 there exists a V such, that (5) U|V|Y and (6) UV=a, and hence (7) YV=a (Fig. 2). From (4) and (6), we obtain (8) V≠X.

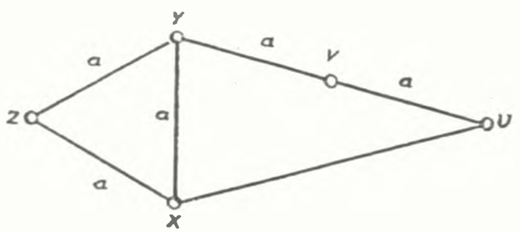

Fig. 2

Now we shall prove successively that:

(9) XV=2a, (10) V|Y|X, (11) Z≠U,V, (12) ZV=a, (13) ZU=a.

In fact, the supposition $XV \neq 2a$ implies, by (1) and (8), that $XV=a$ and hence $U|V|X$, what implies (by (5) and Cor. 2.1) the equality $X=Y$, contrary to (2). Thus (9) holds.

From (9), (7) and (2) we obtain (10), and from (9), (3) and (4) we obtain (11).

The supposition $ZV \neq a$ implies, by (1) and (11), that $ZV=2a$, and hence $V|Y|Z$, what, together with (10), implies (by Cor.2.1) the equality $X=Z$, contrary to (3). Thus (12) holds.

Finally, the supposition $ZU \neq a$ implies, by (1) and (11), that $ZU=2a$, and hence $U|V|Z$, what implies (by (5) and Cor.2.1) the equality $Z=Y$, contrary to (3). Thus (13) holds.

From (3), (4), (13) we infer that $U|Z|X$ and $U|Z|Y$, and hence (by Cor.2.1) $X=Y$, contrary to (2).■

Proof of Lem.2.5.

a) If $2x \in M_1$, then the thesis results from Lem.2.3.

b) If $2x \notin M$, i.e. $2x>m$, it suffices to put $y=m-x$ to obtain (2.14).

c) Assume now that $2x=m$. From Lem.2.2 it results that there exists $z \in M_o$ such that $z \neq x$. If $z<x$, we put $y=z$, if $z>x$, we put $y=m-z$.■

Proof of Th.2.8.

a) From Lem.2.5 it results that M_o has not any minimal element.

b) If $M=G^+$, then, G being a totally ordered group, M has not any maximal element.

If $M \neq G^+$ i.e. $M=[0,m]$ where $0<m<\infty$, then for every x belonging to M_o we put $x'=m-x$. By Lem.2.5 there is a $y' \in M_o$ such that $0<y'<x'$. Putting $y=m-y'$ we obtain $x<y<m$, and hence the maximal element of M_o does not exist.■

Directly from Th. 2.8 and Th.1.6 we obtain

COROLLARY 2.5. *If* $X \neq Y$, *then there exists such a* Z, *that* $X|Z|Y$ *and* $X \neq Z \neq Y$.

Let us assume now that the following three conditions hold (Fig.3):

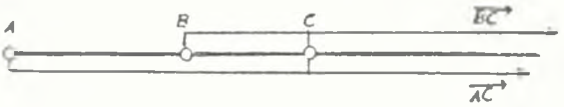

Fig. 3

$$A|B|C, \tag{2.15}$$

$$AC \in M_1, \tag{2.16}$$

$$B \neq C. \tag{2.17}$$

If we restrict ourselves to the case $m=\infty$ (in which the lines are open), then conditions (2.15)–(2.17) imply that the half-line $\overrightarrow{AC}$ is equal to the set of points X which satisfy the condition

$$A|X|B \vee X \in \overrightarrow{BC}. \tag{2.18}$$

In the general case, this equality does not hold and we have only

THEOREM 2.9. *If* (2.15)-(2.17), *then for every* X *such that* $X \in \overrightarrow{AC}$ *we have* (2.18).

Proof. Neglecting the trivial case, we can assume that $A \neq B$. In this case $\overrightarrow{AC}=\overrightarrow{AB}$ (by Th.2.5). Therefore each point belonging to $\overrightarrow{AC}$ satisfies one from the following relations either (a) $A|X|B$ or (b) $A|B|X$.

It is obvious that for (a) the thesis holds, and for (b), by Lem.2.1, we obtain either $B|C|X$ or $B|X|C$ and finally $X \in \overrightarrow{BC}$.■

Now we shall study the concept of line. In the case $m=\infty$, the line by two different points B,C can be defined as the union

$$\overrightarrow{BC} \cup \overrightarrow{CB}. \tag{2.19}$$

In the general case however, there exist points on the line passing by B and C, which do not belong to the set (2.19). One can prove only the following

THEOREM 2.10. *If* (2.15)-(2.17) *hold and* $A \neq B$ (Fig.3), *then* $\overrightarrow{AB} \subseteq \overrightarrow{BC} \cup \overrightarrow{CB}$. (2.20)

Proof. Let $X \in \overrightarrow{AB}$. From Th.2.5 and Th.2.9 we obtain (2.18), and if $A|X|B$, then $A|X|B|C$, and hence $X \in \overrightarrow{CB}$.■

COROLLARY 2.6. *If* (2.17), $AB \in M_o$ *and* $C \in \overline{AB}^c$, *then* (2.20) *holds.*
In the general case we can define lines as follows

DEFINITION 2.1. *If* $A \neq B$, *then*

$$l(A,B) \overset{df}{=} \{X \in E\colon \exists_{Y}[A|Y|B \wedge A \neq Y \neq B \wedge X \in \overline{YA}^c \cup \overline{YB}^c]\}.$$

Directly form this definition we obtain

THEOREM 2.11. *If* $A \neq B$, *then*

a) $l(A,B) = l(B,A)$,

b) $A,B \in l(A,B)$,

c) *if* $A|X|B$, *then* $X \in l(A,B)$.

Basing on Cor. 2.5 and Th. 2.9 one proves

THEOREM 2.12. *If* $AB \in M_o$, *then* $\overrightarrow{AB} \subseteq l(A,B)$.

One can also prove that hypotheses (2.15)-(2.17) imply the inclusion $l(A,B) \subseteq l(A,C)$, but not the converse one. To obtain the latter we must add a new axiom, e.g. (DL3).

3. Consequences of (DL0)-(DL3)

The last axiom of the DL-space: (DL3) is superfluous if $m = \infty$ i.e. if $M = G^+$, because its hypotheses are never satisfied. In other words, all the theorems below can be proved only under axioms (DL1), (DL2) and

$$\text{(DL0')} \quad \rho : E^2 \longrightarrow M = G^+ = \{x \in G : x \geq 0\} \text{ and } M_0 = G^+ \setminus \{0\}.$$

In this case the notions: $\overrightarrow{AB}^+$ and $\overrightarrow{AB}^c$ are equivalent, and the line l(B,C) can be defined as the set (2.19).

However, if we assume that m can be finite, then we need the supplementary axiom: (DL3).

Directly from this axiom we obtain (Fig. 4).

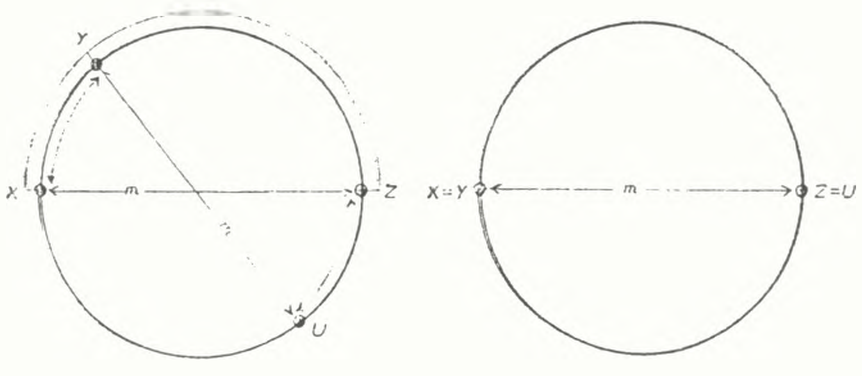

Fig. 4

COROLLARY 3.1. *If*

$XZ = m$ (i.e. $XZ \notin M_1$), (3.1)

$Y \neq Z$, (3.2)

$$X|Y|Z, \tag{3.3}$$

$$XY = ZU, \tag{3.4}$$

$$Y|Z|U, \tag{3.5}$$

then

$$XU = YZ \tag{3.6}$$

and

$$X|U|Z. \tag{3.7}$$

Proof. Omitting the trivial case (Fig.4b) we can assume that (1) $X \neq Y$ (Fig.4a). From (3.3)-(3.5) we obtain (2) $XZ = YU$, and hence (by (3.1)), (3) $YU = m \notin M_o$. On the other hand, from (3.1)-(3.3) and (1) we obtain $0 \neq XY \neq m$ and hence (4) $YU = m > |XY - XU|$. From (3.1)-(3.4), (3) and (4) we get, by (DL3), thesis (3.6). From (2), (3.6) and (3.5) we obtain (3.7).■

COROLLARY 3.2. *If* (3.1)-(3.4), *and* (3.6), *then either* $X|U|Y$ *or* $Y|U|Z$ *or* $Y|Z|U$.

Proof. From (3.1), (3.3), (3.4) and (3.6) we have (1) $YZ + ZU = m$. By (DL3) we obtain from hypotheses the disjunction: either (a) $YU \notin M_o$ or (b) $YU \leq |XY-XU|$.

In the case (a) (Fig.5a), omitting the trivial case: $Y = U$, we obtain $YU = m$ and hence, from (1), the thesis $Y|Z|U$.

In the case (b) (Fig.5b), Th.1.1 yields $YU = |XY-XU|$. Omitting the trivial case: $X = Y$ i.e. $Z = U$, we obtain from Th.1.10b the condition $U \in \overrightarrow{XY}$. Taking into account (3.3) and (3.1), we arrive by

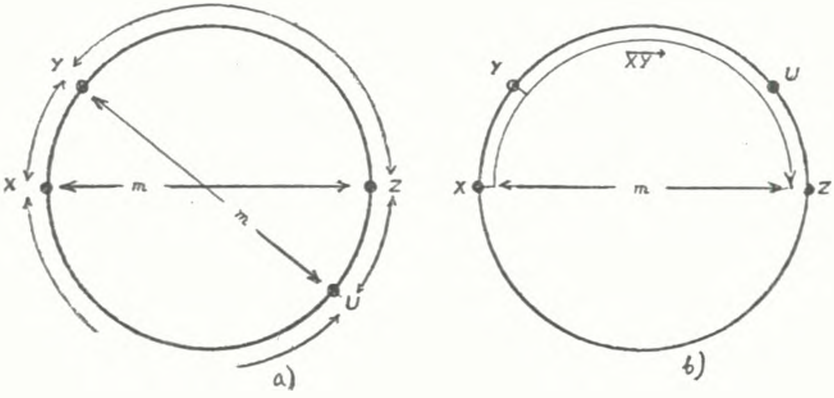

Fig. 5

Th.2.4, at the following disjunction: either X|Y|U|Z or X|U|Y|Z or X|U|Z|Y, which implies the thesis: either X|U|Y or Y|U|Z.■

THEOREM 3.1. *Conditions* (3.1)-(3.3) *and* (3.5) *imply* (3.7).

Proof. The existence of a point V such that Y|U|V and YV = *m* results from Th.1.5 (Fig.6). Therefore we have (1) Y|Z|U|V, and hence Y|Z|V and ZV = *m* - YZ = XZ - YZ = XY. Finally, by Cor.3.1, we have X|V|Z, and hence (3.7) results from (1). ■

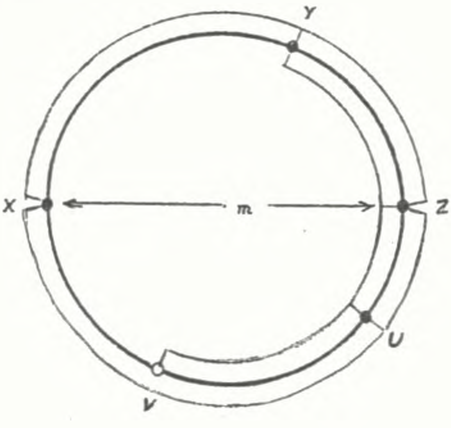

Fig. 6

THEOREM 3.2. *If* (3.2), (3.3) *and* (3.5) *hold, then there exists such a* V *that* (Fig.7)

$X|Z|V,$ (3.8)

$Z|V|U,$ (3.9)

$V|U|X.$ (3.10)

Proof. If $X|Z|U$ holds (Fig.7a), then it suffices to put $V = U$. Let us assume now that $X|Z|U$ does not hold (Fig.7b), and hence, by Th.2.7, $XZ + ZU \notin M$ i.e. we have (1) $m < XZ + ZU$. From (1) we see that $m < \infty$ and hence there exists such a V that we have (3.8) and (2) $XV = m$. Therefore $X|Y|Z|V$, and hence (3) $Y|Z|V$ and (4) $X|Y|V$ hold.

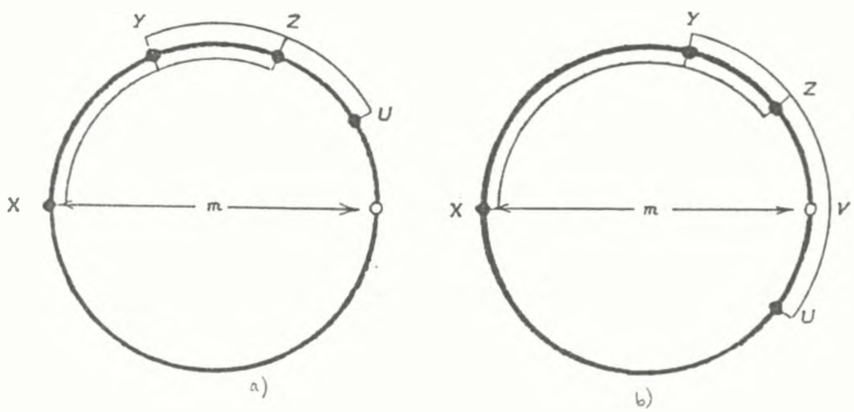

Fig. 7

On the other hand (1), (3,3) and (3.5) imply $XY + YU > m$, and hence $YU > m - XY = XV - XY = YV$. Therefore (3) and (3.5) imply, by Lem. 2.1, $Y|Z|V|U$, and hence (5) $Y|V|U$, (6) $Y \neq V$ and (3.9) hold.

Finally, by Th. 3.1, we obtain from (2),, (4)-(6), thesis (3.10). ■

We shall prove yet

THEOREM 3.3. *If* $A_1|B|A_2$ *and* $A_k \neq B$ *for* $k=1,2$, *then* $l(A_1,A_2) = \overrightarrow{BA_1} \cup \overrightarrow{BA_2}$.

To prove it we need the following lemmas:

LEMMA 3.1. *If* (Fig.8)

$$A_1|B_1|B_2|A_2, \tag{3.11}$$

$$A_k \neq B_k \qquad \textit{for } k=1,2, \tag{3.12}$$

$$X \in \overrightarrow{B_1A_1}, \tag{3.13}$$

then $X \in \overrightarrow{B_2A_1} \cup \overrightarrow{B_2A_2}$.

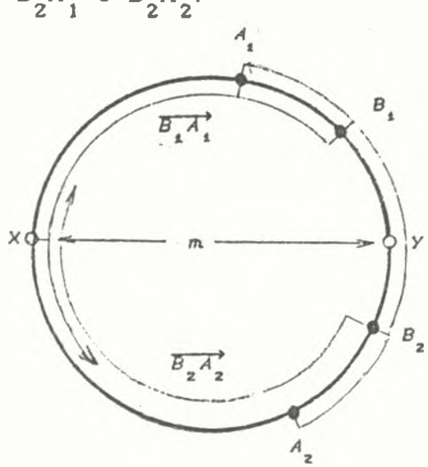

Fig. 8

LEMMA 3.2. *If* (Fig.9)

$$A_1A_2 \notin M_1 \qquad (\textit{i.e. } A_1A_2 = m), \tag{3.14}$$

then

$$X \in l(A_1,A_2) \tag{3.15}$$

if and only if

$$A_1|X|A_2. \tag{3.16}$$

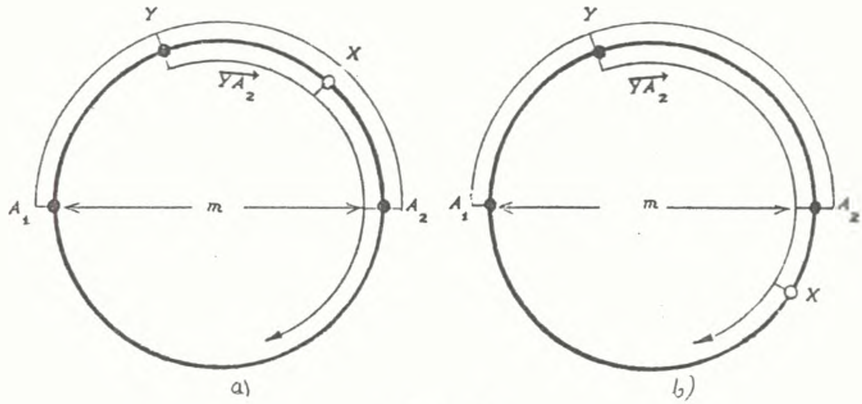

Fig. 9

LEMMA 3.3. *If* (3.14), (3.16) *hold and*

$$A_1|C|A_2, \tag{3.17}$$

$$A_k \neq C \qquad \textit{for } k=1,2, \tag{3.18}$$

then

$$X \in \overrightarrow{CA_1} \cup \overrightarrow{CA_2}. \tag{3.19}$$

Proof of Lem.3.1. Let us assume that $X \notin \overrightarrow{B_2A_1}$ (Fig.8). Therefore from (3.13) we obtain $B_1 \neq B_2$, and hence, by (3.11), neither

$$B_2|B_1|X \tag{1}$$

nor

$A_1|X|B_1$ hold. (2)

From (3.13) and (2) we infer that (3) $B_1|A_1|X$. From (3.11), (3.12) and (3) we infer by means of Th.3.2 (Fig.8 and Fig.7b), that there exists such a Y that (4) $X|B_1|Y$, (5) $B_1|Y|B_2$ and (6) $Y|B_2|X$ hold. From (4) and (1) we obtain (7) $Y \neq B_2$, and (3.11) and (5) yield $Y|B_2|A_2$ what, by (6), (7), (3.12) and Cor.2.3, implies the thesis $X \in \overline{B_2A_2}^{c} \subseteq \overrightarrow{B_2A_2}$.■

Proof of Lem.3.2. The assumption (3.16) implies (3.15), by Th.2.11c. Let us assume that (3.15) holds, and hence (by the definition) there exists such a Y that (1) $A_1|Y|A_2$, (2) $Y \neq A_k$ for k=1,2 and either $X \in \overrightarrow{YA_1}$ or $X \in \overrightarrow{YA_2}$.

It suffices to study only one of these cases. Let us assume that $X \in \overrightarrow{YA_2}$, i.e. either $Y|X|A_2$ or $Y|A_2|X$. In the first case (Fig. 9a) from Th.1.8 we obtain $A_1|Y|X|A_2$ and hence (3.16) holds. In the other (Fig.9b) thesis (3.16) results from (1), (2) and (3.14) by Th.3.1.■

Proof of Lem 3.3. Omitting the trivial case we assume that $X \neq A_k$ for k=1,2, and hence there exists a D such that (1) $A_2D=A_1C$ and $D \in \overrightarrow{A_2X}$ (Fig.10). On the other hand, from (3.16) we obtain $A_1 \in \overrightarrow{A_2X}$, and hence, by Th.2.4, taking into account (3.14), we have either (2a) $A_2|X|D|A_1$ or (2b) $A_2|D|X|A_1$ and finally (3) $A_2|D|A_1$. From (3), (1) and (3.17) we obtain (4) $A_1D = A_2C$. In the case (2a) we put $A = A_2$, $B = A_1$, and in the case (2b) we put $A = A_1$, $B = A_2$. In both cases we obtain (5) $A|X|D|B$. Conditions (3.14), (3.18), (3.17), (1), (4), give us the hypotheses of Cor.3.2: $AB = m$, $C \neq A$, $A|C|B$, $AC = BD$. $AD = BC$ (Fig.10a,b,c), and hence we get either (6a) $B|D|C$ or (6b) $C|D|A$ or (6c) $C|A|D$. Taking into account condition (5), we obtain the following results:

In the case (6a) from Lem. 2.1 it results that either $B|C|X|A$ or $B|X|C|A$, and hence (7) $[C|X|A \vee C|X|B]$.

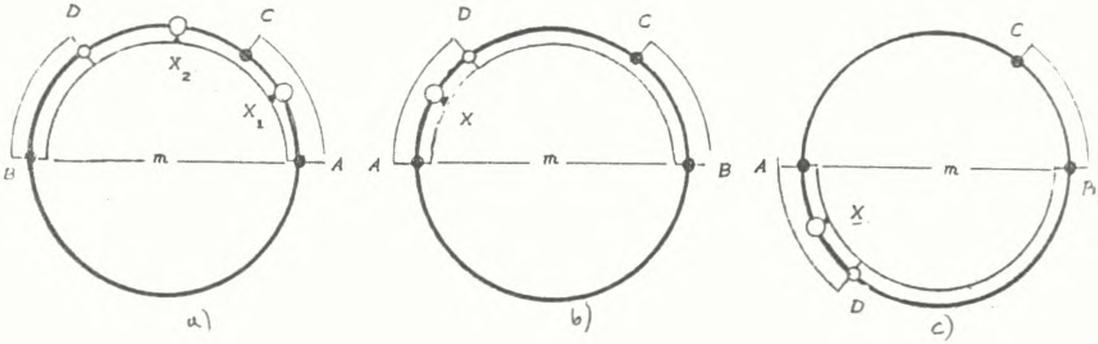

Fig. 10

In the case (6b) from Th.1.8 it results that $C|D|X|A$ and hence

$$C|X|A \tag{8}$$

In the case (6c) from Th.1.8 it results that $C|A|X|D$ and hence

$$C|A|X \tag{9}$$

Each of conditions (7)-(9) implies thesis (3.19).■

Proof of Th.3.3. The inclusion $\overrightarrow{BA_1} \cup \overrightarrow{BA_2} \subseteq l(A_1,A_2)$ results directly from Df.2.1.

Let us assume that (1) $X \in l(A_1A_2)$.

If (3.14) holds, then by Lem.3.2 we obtain (3.16), and Lem.3.3 implies (2) $X \in \overrightarrow{BA_1} \cup \overrightarrow{BA_2}$.

Let us assume now that (3.14) does not hold, i.e. $A_1A_2 \in M_1$. From Df.2.1 it results, that there exists such a C that (3.17)-(3.19) hold, and Lem.2.2 implies either

$$\text{(a)} \quad A_1|B|C|A_2 \quad \text{or} \quad \text{(b)} \; A_1|C|B|A_2.$$

In the case (a) if $X \in \overrightarrow{BA_1}$, then (2) results from Lem.3.1; if $X \in \overrightarrow{BA_2}$, then (2) results from Th. 2.9.

In a similar manner we obtain (2) in the case (b).■

To prove that there is the unique line passing through two distinct points, we need the two lemmas:

LEMMA 3.4. *If* (3.17) *and* $A_1 \neq C$, *then* $l(A_1,A_2) = l(A_1,C)$.

Proof. Let D be such a point that $A_1|D|C|A_2$ and $A_1 \neq D \neq C$. From Th.3.3 $l(A_1,A_2) = \overrightarrow{DA_1} \cup \overrightarrow{DA_2}$ and $l(A_1,C) = \overrightarrow{DA_1} \cup \overrightarrow{DC}$. Taking into account the equality $\overrightarrow{DA_2} = \overrightarrow{DC}$, we obtain the thesis. ■

LEMMA 3.5. *If* $B \neq A \neq C$ *and* $C \in l(A,B)$, *then* $l(A,B) = l(A,C)$.

Proof. By Definition 2.1, there exists such a Y that (1) $A|Y|B$, (2) $A \neq Y \neq B$, (3) $C \in \overrightarrow{YA} \cup \overrightarrow{YB}$. From Lem.3.4 we obtain (4) $l(A,B) = l(A,Y)$.

If $A|C|B$, then the thesis results from Lem.3.4; otherwise (1) and (3) imply that either $Y|A|C$ or $Y|B|C$ holds.

In the case where $Y|A|C$ we have from Lem.3.4 that $l(A,Y) = l(Y,C) = l(A,C)$ and the thesis results from (4).

In the case where $Y|B|C$ it results from (1) and Th.3.2 that there exists such a point D that $A|B|D$ and $D|C|A$, and hence, by Lem.3.4, $l(A,B) = l(A,D) = l(A,C)$.■

THEOREM 3.4. *If* $A \neq B$, $C \neq D$ *and* $C,D \in l(A,B)$, *then* $l(C,D) = l(A,B)$.

Proof. If D = A, then the thesis results directly from Lem.3.5. If D ≠ A, then from Lem.3.5 we obtain successively $l(A,B) = l(A,D) = l(C,D)$. ■

4.Properties of lines in DL-space

We have started this paper with the description of properties which should be satisfied by a distance and lines in the DL-space i.e. with Theorems 0.1-0.5. Now we shall prove that all these theorems actually result from our system of axioms.

It is obvious that Th.0.5, results directly from axiom (DL.1).

Theorem 0.1 was proved in section 2, as Theorem 2.0.

The set of lines L is defined as follows

DEFINITION 4.1. $L \overset{df}{=} \{\lambda \subseteq E: \exists_{A,B\in E} [A \neq B \wedge \lambda = l(A,B)]\}$.

Condition (L1) (from Th.0.2) results directly from this definition and from Th.2.11b.

Condition (L2) results from Definitions 2.1, 4.1 and from Th.2.11b and Th. 3.4.

To prove (L3) it suffices to use

LEMMA 4.1. *If* (Fig.11)

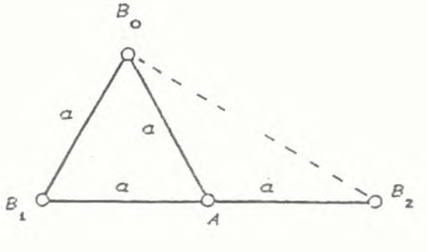

Fig. 11

$AB_k = a \in M_0$ *for* $k=0,1,2$, (4.1)

$B_0B_1 = a$, (4.2)

$$B_1|A|B_2, \tag{4.3}$$

then

$$B_0 \notin l(B_1,B_2). \tag{4.4}$$

Proof. To prove that the assumption $B_0 \in l(B_1,B_2)$ leads to a contradiction, it suffices (by Th.3.3) to investigate the two cases:

a) $B_0 \in \overrightarrow{AB_1}$. Then (by Th.2.3), $B_0 = B_1$, contrary to (4.2).

b) $B_0 \in \overrightarrow{AB_2}$. Then (by Th. 2.3), $B_0 = B_2$. Therefore from (4.1) and (4.3) it results $B_1B_0 = 2a$, contrary to (4.2). ■

Proof of (L3). Let $\lambda \in L$. Th.1.13 yields the existence of an $a \in M$ such that

$$a \neq 0 \text{ and } 2a \in M. \tag{4.5}$$

We fix on λ two points: B_1, B_2 such that $B_1B_2 = 2a$ and also two other points A, B_0 which satisfy the hypotheses of Lem. 4.1, and hence we obtain (4.4).■

To prove Th. 0.3 it suffices to see that (B1) results directly from Th.2.12, (B2) - from Th.2.11c, and (B3) - from Cor. 2.5 and from Th. 2.11c.

To prove Th.0.4, we need two lemmas.

LEMMA 4.2. *If* (4.3) *holds,*

$$A \neq B_k \quad \textit{for } k=1,2, \tag{4.6}$$

$$X_k \in \overrightarrow{AB_k} \quad \textit{for } k=1,2, \tag{4.7}$$

$$AX_1 + AX_2 \in M, \tag{4.8}$$

then $X_1|A|X_2$.

Proof. Omitting trivial cases we can assume that for k=1,2 $A \neq X_k$, and hence $AX_k \in M_0$. Thus there exist: (1) $z \in M_0$ such that (2) $z \leq AX_1$, (3) $z \leq AB_1$, (4) $z + AX_2 \in M$, and a point Z such that (5) $Z \in \overrightarrow{AB_1}$ and (6) $AZ= z$. From (3), (5) and (4.3) we obtain $B_1|Z|A|B_2$, and hence (7) $Z|A|B_2$.

One the other hand, from (4.7) we obtain two possibilities: either (a) $A|X_2|B_2$ or (b) $A|B_2|X_2$. In the case (a) from (7) we obtain (8) $Z|A|X_2$, by Th.1.8. In the case (b) from (7), (4), (6) and (4.6), we obtain (8), by Th. 2.7.

From (5) and (4.7) we infer successively that $\overrightarrow{AZ} = \overrightarrow{AB_1}$, $X_1 \in \overrightarrow{AZ}$, and finally, by (2), $X_1|Z|A$. Therefore, from (4.8) and (8) we obtain, by Th.2.7, the thesis. ■

LEMMA 4.3. *If* (4.3) *and* (4.1) *hold for* k=1,2 *and if*

$$X \in \overrightarrow{AB_k} \quad \textit{for } k=1,2, \tag{4.9}$$

then $AX \notin M_0$.

Proof. a) Let $AX \leq a$. Therefore from (4.9) we obtain for k=1,2: $A|X|B_k$, hence $X|A|X$, and finally, by Th.2.1. $X = A$ i.e. $AX \notin M_0$.

b) Now let $AX > a$. In this case from (4.9) we obtain for k=1,2: $A|B_k|X$, since, by (4.1) and (4.3), B_1 and B_2 are distinct, the thesis results from (4.3) and Th.2.2. ■

Now let us define $M^{\pm}$ as follows

DEFINITION 4.2. $M^{\pm} \overset{df}{=} \{x \in G: x \in M \vee -x \in M_0\}$.

Proof of Th.0.4. On a fixed line λ we fix apart from A, a point B_1, such that $AB_1 = a$, where (4.5) holds. Next we fix a new point B_2, such that $AB_2 = a$ and (4.3) holds.

It is easy to see that

$$\lambda = l(B_1,B_2) = \overrightarrow{AB_1} \cup \overrightarrow{AB_2}, \tag{4.10}$$

Now we define the function $\varphi:\lambda \longrightarrow M^{\pm}$ as follows

$$\varphi(X) \overset{df}{=} \begin{cases} AX, \text{ if } X \in \overrightarrow{AB_1} \\ -AX, \text{ if } X \notin \overrightarrow{AB_1} \end{cases} \tag{4.11}$$

From (4.10) and (4.11) conditions (C0) and (C1) result by Th.2.3 and Lem.4.3.

To prove (C2) we must examinate the two cases:

a) If X_1, X_2 belong both to the same half-line $\overrightarrow{AB_k}$, then $|\varphi(X_1)-\varphi(X_2)| = |AX_1 - AX_2|$, and hence the thesis results directly from Th.2.6.

b) If $X_i \in \overrightarrow{AB_1}$ and $X_k \in \overrightarrow{AB_2}$ (where $i \neq k$), then $|\varphi(X_1)-\varphi(X_2)| = \varphi(X_i) - \varphi(X_k) = AX_1 + AX_2$, and if this value belongs to M, then (by Lem.4.2) we obtain $AX_1 + AX_2 = X_1X_2$, i.e. condition (C2) holds too.■

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