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Superstability is not natural

Dedicated to Professor Zenon Moszner with best wishes on his 60-th birthday

Let $(S,+)$ be an arbitrary semigroup and let f map S into the field $\mathbb{C}$ of all complex numbers. Assume that there exists a nonnegative number ε such that

$$|f(x+y) - f(x)\, f(y)| \le \varepsilon \qquad \text{for} \qquad x, y \in S. \tag{1}$$

Then f is either bounded or exponential (see J.A. Baker, J. Lawrence and F. Zorzitto [3], J.A. Baker [2] and M. Kuczma [8]). Such a behaviour of approximate homomorphisms (superstability phenomenon) may seem somewhat shocking all the more so as in the additive case inequality

$$|f(x+y) - f(x) - (y)| \le \varepsilon, \qquad x,y \in S, \tag{2}$$

admits pretty rich family of unbounded nonhomommorphic solutions; indeed, any function of the form $f = a + r$ where $a\colon S \longrightarrow \mathbb{C}$ is a nontrivial additive function and $r\colon S \longrightarrow \left\{z \in \mathbb{C} : |z| \le \frac{1}{3}\varepsilon\right\}$ $(\varepsilon > 0)$ is quite arbitrary, yields a solution of (2).

Our aim is to show that the superstability phenomenon is caused by the fact that problem (1) is ill-posed in a sense. Such an observation was made by M. Taylor (1987, oral communication; see also R. Ger [6]). Namely, contrary to (2), posing problem (1) one disregards the natural group structure in $\mathbb{C}$ which in this case should obviously be the multiplicative group $(\mathbb{C}^*, \cdot)$ of all nonzero complex numbers. Since we are looking for functions which are near to exponential mappings after the example of (2) we should rather pose the problem in the following way:

$$\left|\frac{f(x+y)}{f(x)f(y)} - 1\right| \le \varepsilon \quad , \quad x,y \in S, \tag{3}$$

for functions $f\colon S \longrightarrow \mathbb{C}^*$. This apparently diminishes the class of functions considered since we have eliminated their possible zero values. However such a restriction is inessential indeed as the following proposition shows.

PROPOSITION 1. *Let* $(S,+)$ *be a semigroup and let* $f\colon S \longrightarrow \mathbb{C}$ *be an unbounded solution of inequality* (1) *such that* $0 \in f(S)$. *Then there exists a proper subsemigroup* $(S_0, \cdot)$ *of the semigroup* $(S,+)$ and *a function* $g\colon S \longrightarrow \mathbb{C}^*$ *such that*

$$\big(S + (S\setminus S_0)\big) \cup \big((S\setminus S_0) + S\big) \subset S\setminus S_0, \tag{4}$$

$$|g(x+y) - g(x)\, g(y)| \le \varepsilon \, , \qquad x,y \in S_0,$$

and

$$f(x) = \begin{cases} g(x) & \text{for } x \in S_0 \\ 0 & \text{for } x \in S\setminus S_0 \end{cases} \tag{5}$$

Proof. Put $Z := \{x \in S: f(x) = 0\}$ and $S_0 := S\setminus Z$. Since $0 \in f(S)$ and f is unbounded we infer that both Z and S_0 are nonempty. Fix a $z \in Z$ and take an $s \in S$; then $\varepsilon \geq |f(s+z) - f(s)\, f(z)| = |f(s+z)|$ whence $|f|_{S+z}| \leq \varepsilon$. Consequently, for any $x \in S$ and $z \in Z$ one has

$$\varepsilon \geq |f(x+y+z) - f(x)\, f(y+z)| \geq |f(x)|\cdot|f(y+z)| - |f(x+y+z)|$$

and

$$|f(x)|\cdot|f(y+z)| \leq \varepsilon + |f(x+y+z)|$$

for all $x,y \in S$ and $z \in Z$. But f is unbounded; therefore $f(y+z) = 0$ for all $y \in S$ and $z \in Z$. Thus $S + (S\setminus S_0) = S + Z \subset Z = S\setminus S_0$. Analogously we show that $(S\setminus S_0) + S \subset S\setminus S_0$.

To show that $S_0 + S_0 \subset S_0$ assume the contrary, i.e. that there exist $x,y \in S_0$ such that $x + y \in S\setminus S_0 = Z$. Then, in particular, $f(x+y+u) = 0$ for all $u \in S$ and

$$\varepsilon \geq |f(x+y+u) - f(x)\, f(y+u)| = |f(x)|\cdot|f(y+u)|$$

for all $u \in S$. Hence

$$|f(u)| = \frac{|f(y)\, f(u)|}{|f(y)|} \leq \frac{|f(y)\, f(u) - f(y+u)| + |f(y+u)|}{|f(y)|} \leq \frac{\varepsilon + \frac{1}{|f(x)|}\varepsilon}{|f(y)|}$$

for all $u \in S$, i.e. f is bounded, a contradiction.

To finish the proof it suffices to put $g := f|_{S_0}$.

REMARK 1. If $(S, +)$ is a group then every solution $f: S \longrightarrow \mathbb{C}$ of inequality (1) such that $0 \in f(S)$ has to be bounded. In fact, let $s_0 \in S$ be such that $f(s_0) = 0$. Then $|f(x+s_0)| \le \varepsilon$ for all $x \in S$ and consequently, replacing here x by $x - s_0$, we infer that $|f(x)| \le \varepsilon$ for all $x \in S$.

REMARK 2. Since $\varepsilon = 0$ is admissible in (1) Proposition 1 allows to determine all complex solutions f on a semigroup $(S,+)$ of the equation

$$f(x+y) = f(x)\, f(y)\,, \qquad x,y \in S, \tag{6}$$

in terms of homomorphisms $g: S \longrightarrow \mathbb{C}^*$. Namely, we have the following

PROPOSITION 2. *Let $(S, +)$ be a semigroup and let $f: S \longrightarrow \mathbb{C}$ be a solution of (6). Then either $f = 0$ or f is a homomorphism from S into $\mathbb{C}^*$ or $0 \in f(S) \neq \{0\}$ and there exists a subsemigroup $(S_0,+)$ of $(S, +)$ and a homomorphism $g: S_0 \longrightarrow \mathbb{C}^*$ such that inclusion (4) holds true and f is given by (5). Conversely, any function $f: S \longrightarrow \mathbb{C}$ of such form yields a solution to (6).*

REMARK 3. (M. Sablik, oral communication). Let $(\mathbb{N}, \cdot)$ be the semigroup of all positive integers with the usual multiplication and let $S_0 := \{1,2,4,8,\ldots\}$. Then $(S_0, \cdot)$ is a subsemigroup of $(\mathbb{N}, \cdot)$ such that (4) is fulfilled with $S = \mathbb{N}$ and the function $f: \mathbb{N} \longrightarrow \mathbb{C}$ given by the formula

$$f(n) := \begin{cases} n & \text{for} \quad n \in S_0 \\ 0 & \text{for} \quad n \in \mathbb{N}\setminus S_0 \end{cases}$$

yields an unbounded solution of (6) with infinite number of zeros.

From now on we are going to deal with the stability question formulated like in (3). We shall see that in such a case superstability phenomenon disappears and the stability behaviour of homomorphisms with values in the group $(\mathbb{C}^*, \cdot)$ is much the same like in the additive case.

THEOREM 1. *Let* $(S, +)$ *be an amenable semigroup and let* $\varepsilon \in [0,1)$ *be a given number. Assume that* $f\colon S \longrightarrow \mathbb{C}^*$ *is such that relation* (3) *holds true. Then there exists exactly one pair of functions* $m\colon S \longrightarrow (0,\infty)$ *and* $q\colon S \longrightarrow \left\{z \in \mathbb{C}\colon 1 - \varepsilon \le |z| \le \frac{1}{1-\varepsilon}\right\}$ *such that* $m(x+y) = m(x)\, m(y)$, $x,y \in S$, *and* $f(x) = q(x)\, m(x)$, $x \in S$. *In particular,*

$$\left|\frac{m(x)}{f(x)} - 1\right| \le \frac{2-\varepsilon}{1-\varepsilon} \quad \text{and} \quad \left|\frac{f(x)}{m(x)} - 1\right| \le \frac{2-\varepsilon}{1-\varepsilon}$$

for all $x \in S$.

Proof. Put $\varphi(x) := |f(x)|$, $x \in S$. Then

$$\varepsilon \ge \left| \frac{f(x+y)}{f(x)\cdot f(y)} - 1 \right| \ge \left| \frac{\varphi(x+y)}{\varphi(x)\varphi(y)} - 1\right|,$$

i.e.

$$1 - \varepsilon \le \frac{\varphi(x+y)}{\varphi(x)\varphi(y)} \le 1 + \varepsilon$$

for all $x,y \in S$. Consequently

$$|\ln \varphi(x+y) - \ln \varphi(x) - \ln\varphi(y)| \le \ln \frac{1}{1-\varepsilon}, \quad x,y \in S,$$

and in view of L. Székelyhidi's result [10], there exists a homomorphism a from (S,+) into the additive group $(\mathbb{R},+)$ of all real numbers such that

$$|a(x) - \ln\varphi(x)| \le \ln \frac{1}{1-\varepsilon}, \qquad x \in S.$$

It remains to put $m := \exp a$ and $q := f\cdot\exp(-a)$.

To prove the uniqueness, assume that we are given two pairs (m_1,q_1) and (m_2,q_2) of suitable mappings such that such that $q_1m_1 = f = q_2m_2$. Then $m := \frac{m_1}{m_2}$ is a positive exponential function bounded away from 0 and ∞. Thus $m = 1$ whence $m_1 = m_2$ and, consequently, $q_1 = q_2$. This completes the proof.

REMARK 4. Professor Zenon Moszner ([9], Exemple 3) has proved the above result (existence) in the case where $(S,+) = (\mathbb{R},+)$ and $f(\mathbb{R}) \subset \mathbb{R}\setminus\{0\}$. He also proposes three essentially different definitions of stability of homomorphism. His definition 3 is the closest (but not equivalent) to this what we mean by Hyers-Ulam stability. G.L. Forti [4] introduces yet another notion of stability. In what follows we do not apply any formal definition of stability confining ourselves to the statement that all the facts established here yield some stability properties.

REMARK 5. At first glance the restriction $\varepsilon < 1$ in Theorem 1 looks a bit artificial. The following example shows however that without this assumption the result is no longer valid. Take $(S,+) = (\mathbb{R},+)$ and $f: \mathbb{R} \longrightarrow (0,\infty)$ defined by the formula $f(x) := \exp\sqrt{|x|}$, $x \in \mathbb{R}$. Then

$$\left|\frac{f(x+y)}{f(x)\ f(y)} - 1\right| = \left|\exp\left(\sqrt{|x+y|} - \sqrt{|x|} - \sqrt{|y|}\right) - 1\right|$$

$$= 1 - \exp\left(\sqrt{|x+y|} - \sqrt{|x|} - \sqrt{|y|}\right) \le 1 =: \varepsilon,$$

for all $x \in \mathbb{R}$, in view of the fact that both: absolute value and the square root are subadditive functions. Suppose that there exists an exponential mapping $m: \mathbb{R} \longrightarrow (0,\infty)$ such that $\frac{f(x)}{m(x)} \le c < \infty$, $x \in \mathbb{R}$. Then $m = \exp a$ where $a: \mathbb{R} \longrightarrow \mathbb{R}$ is additive and $\exp\left(\sqrt{|x|} - a(x)\right) \le c$, $x \in \mathbb{R}$. Consequently, a possesses a measurable (even continuous) minorant on $\mathbb{R}$ and hence is continuous (see J. Aczél [1] or M. Kuczma [8], for instance). Thus $a(x) = \alpha\, x$, $x \in \mathbb{R}$, for some $\alpha \in \mathbb{R}$, and $\sqrt{|x|} \le \alpha\, x + \ln c$, $x \in \mathbb{R}$, a contradiction.

Note that the following three facts were essential in the proof of Theorem 1:

(i) with $\varepsilon \in (0,1)$ the neighbourhood $U = \left(1-\varepsilon, \frac{1}{1-\varepsilon}\right)$ of the neutral element 1 of the multiplicative group $((0,\infty),\cdot)$ is bounded;

(ii) the absolute value function $|\cdot|$ establishes a continuous homomorphism of the group $(\mathbb{C}^*,\cdot)$ onto its subgroup $((0,\infty),\cdot)$;

(iii) $|\cdot|$ is a projection, i.e. $|\cdot| \circ |\cdot| = |\cdot|$.

These observations lead to the following

THEOREM 2. *Let* $(S,+)$ *be an amenable semigroup and let* $(H,\ \cdot)$ *be a Hausdorff topological group with the neutral element* e. *Assume that there exists a uniquely two-divisible locally compact Abelian subgroup* $(H_0,\ \cdot)$ *of the group* $(H,\ \cdot)$ *with the following properties:*

(a) *the dual group* $(H_o^*,+)$ *of all continuous real characters on* H_o *separates the points of* H_o;

(b) *there exists a continuous homomorphism* h *of* H *onto* H_o *such that* $h \circ h = h$.

Let further U_e *be a neighbourhood of* e *whose projection* $W_e := h(U_e)$ *onto* H_o *is symmetric and such that*

(c) $W_e \cdot W_e \subset W_e^2$;

(d) W_e *is bounded, i.e. for any neighbourhood* $W \subset H_o$ *of the neutral element* e *there exists an* $n \in \mathbb{N}$ *such that* $W_e \subset W^{2^n}$.

Then for every function $f: S \longrightarrow H$ *such that*

$$f(x+y) \in f(x) \cdot f(y) \cdot U_e, \qquad x,y \in S, \tag{7}$$

there exists exactly one homomorphism $m: S \longrightarrow H_o$ *such that*

$$f(x) \in m(x) \text{ cl } \big((\text{ker. } h) \cdot W_e\big) , \qquad x \in S.$$

Proof. Fix arbitrarily a map $f: S \longrightarrow H$ fulfilling condition (7) and put $\varphi := h \circ f$. Then we obviously have

$$\varphi(x+y) \in \varphi(x) \cdot \varphi(y) \cdot W_e, \qquad x,y \in S, \tag{8}$$

and consequently, since $(H_o, \cdot)$ is commutative, an easy induction shows that

$$\varphi(2^n x) \in \varphi(x)^{2^n} \cdot W_e \cdot W_e^2 \cdot \ldots \cdot W^{2^{n-1}}$$

holds true for any $x \in S$ and $n \in \mathbb{N}$. Hence by means of (c)

$$\varphi(2^n x) \in \varphi(x)^{2^n} \cdot W_e^{2^n}, \qquad x \in S, \qquad n \in \mathbb{N},$$

i.e.

$$\varphi_n(x) := \varphi(2^n x)^{\frac{1}{2^n}} \in \varphi(x) \cdot W_e, \qquad x \in S, \quad n \in \mathbb{N}, \tag{9}$$

because of the unique two-divisibility of the group $(H_o, \cdot)$. We shall show that for any $x \in S$ the sequence $\left(\varphi_n(x)\right)_{n\in\mathbb{N}}$ is fundamental. Indeed, in view of (9)

$$\varphi_{n+m}(x) \cdot \varphi_n(x)^{-1} = \varphi\left(2^{2+m}x\right)^{\frac{1}{2^{n+m}}} \cdot \varphi(2^n x)^{-\frac{1}{2^n}}$$

$$= \left(\varphi\left(2^m y\right)^{\frac{1}{2^m}} \cdot \varphi(y)^{-1}\right)^{\frac{1}{2^n}} \in W_e^{\frac{1}{2^n}}$$

for all $x \in S$ and every $n,m \in \mathbb{N}$ (here $y:= 2^n x$). Therefore, taking an arbitrary neighbourhood $W \subset H_o$ of the element e and choosing an $n_o \in \mathbb{N}$ such that $W_e \subset W^{2^{n_o}}$ (see (d)) we infer that

$$\varphi_{n+m}(x) \cdot \varphi_n(x)^{-1} \in W_e^{\frac{1}{2^{n_o}}} \subset W, \quad x \in S,$$

provided that $n,m \in \mathbb{N}$, $n \geq n_o$, because (c) implies that $W_e \subset W_e^{2^k}$ for all $k \in \mathbb{N}$.

Being a projection h is an open mapping; thus $W_e = h(U_e)$ is a neighbourhood of e in H_o and on account of (d) the family

$\left\{W_e^{\frac{1}{2^n}} : n \in \mathbb{N}\right\}$ forms a countable base of neighbourhoods of e in H_o.

This means that the topology of $(H_o,\cdot)$ satisfies the first countability axiom (actually $(H_o,\cdot)$ being Hausdorff is then metrizable). Consequently the local compactness of $(H_o,\cdot)$ implies its sequential completeness which enables us to define a map m: $S \longrightarrow H_o$ by the formula

$$m(x) := \lim_{n\to\infty} \varphi_n(x), \qquad x \in S. \tag{10}$$

To prove that m establishes a homomorphism on (S,+) we shall adapt an idea applied by Z. Gajda [5] (cf. also G.L. Forti [4]) in case of sequentially complete locally convex linear topological spaces. To this aim fix arbitrarily a continuous real character $\chi \in H_o^*$ and put $\psi := \chi\circ\varphi$. Then, for any $x,y \in S$, one has

$$\psi(x+y) - \psi(x) - \psi(y) = \chi\left(\varphi(x+y)\varphi(x)^{-1}\varphi(y)^{-1}\right) \subset \chi(W_e).$$

Observe that the set $\chi(W_e)$ is bounded in $\mathbb{R}$; in fact, let $W \subset H_o$ be a neighbourhood of e such that $cl_{H_o} W$ is compact. In view of the boundedness of W_e there exists an $n \in \mathbb{N}$ such that $W_e \subset W^{2^n}$. Hence

$$\chi(W_e) \subset \chi\left(W^{2^n}\right) = 2^n\,\chi(W) \subset 2^n\,\chi(cl\ W)$$

and the latter set is compact in $\mathbb{R}$ as the image of a compact set under the continuous map $2^n\chi$. Consequently, there exists an $M > 0$ such that

$$\left|\psi(x+y) - \psi(x) - \psi(y)\right| \le M \qquad \text{for all } x,y \in S.$$

Now, Székelyhidi's theorem [10] states that exists a homomorphism $a_\chi : S \longrightarrow \mathbb{R}$ such that

$$|\psi(x) - a_\chi(x)| \le M, \qquad x \in S.$$

It is known (see Forti's paper [4]) that a_χ has to have the form $a_\chi(x) = \lim_{n\to\infty} \frac{1}{2^n} \psi(2^n x)$, $x \in S$. On the other hand the continuity of χ gives

$$\chi\bigl(m(x)\bigr) = \lim_{n\to\infty} \chi\bigl(\varphi_n(x)\bigr) = \lim_{n\to\infty} \frac{1}{2^n} \chi\bigl(\varphi(2^n x)\bigr)$$

$$= \lim_{n\to\infty} \frac{1}{2^n} \psi(2^n x) = a_\chi(x)$$

for all $x \in S$, i.e. $\chi \circ m = a_\chi$.

Fix arbitrarily points $x,y \in S$; then

$$\chi\bigl(m(x+y) \circ m(x)^{-1} \circ m(y)^{-1}\bigr) = (\chi\circ m)(x+y) - (\chi\circ m)(x) - (\chi\circ m)(y)$$

$$= a_\chi(x+y) - a_\chi(x) - a_\chi(y) = 0.$$

In virtue of the arbitrarness of the character χ and the fact that $(H_o^*, \cdot)$ separates points (see (a)) we get the equality

$$m(x+y)\cdot m(x)^{-1}\cdot m(y)^{-1} = e, \qquad x,y \in S,$$

which states that m is exponential.

Finally, relation (9) and definition (10) imply that

$$m(x) \in \varphi(x) \cdot \mathrm{cl}_{H_o} W_e, \qquad x \in S,$$

or, equivalently, since W_e is symmetric

$$h(f(x)) = \varphi(x) \in m(x)\cdot cl_{H_o} W_e \subset m(x)\cdot cl\ W_e, \qquad x \in S.$$

But $h|_{H_o}$ is the identity mapping and $m(x) \in H_o$, $x \in S$, whence

$$h\left(f(x)\cdot m(x)^{-1}\right) \in cl\ W_e, \qquad x \in S. \tag{11}$$

Note that for any $z \in H$ there exists a $t(z) \in \ker h$ such that $z = t(z)\cdot h(z)$; in fact, taking $t(z) := z\cdot h(z)^{-1}$ one has $h(t(z)) = h(z)\cdot h(h(z)^{-1}) = h(z)\cdot h(h(z))^{-1} = h(z)\cdot h(z)^{-1} = e$. Therefore, by (11), for any $x \in S$, we obtain

$$\begin{aligned} z := f(x)\cdot m(x)^{-1} = t(z)\cdot h(z) &\in t(z)\cdot cl\ W_e \\ &= cl(t(z)\cdot W_e) \\ &\subset cl[(\ker h)\cdot W_e] =: Z \end{aligned}$$

whence

$$f(x) \in m(x)\cdot Z, \qquad x \in S, \tag{12}$$

as claimed.

It remains to prove the uniqueness of the exponential mapping $m\colon S \longrightarrow H_o$. To this aim, assume that (12) holds true and simultaneously $f(x) \in \tilde{m}(x)\cdot Z$, $x \in S$, with exponential mappings m, $\tilde{m} : S \longrightarrow H_o$. Then

$$\begin{aligned} m_o(x) &:= m(x)^{-1} \cdot \tilde{m}(x) \\ &= m(x)^{-1} \cdot f(x) \cdot f(x)^{-1} \cdot \tilde{m}(x) \in Z \cdot Z^{-1} \end{aligned}$$

for all $x \in S$. Note that m_o is exponential, too, and

$$m_o(x) = h\big(m_o(x)\big) \in h(Z \cdot Z^{-1}) = h(Z) \cdot h(Z^{-1}), \quad x \in S.$$

Now, in view of the continuity of h,

$$h(Z) = h\big(\mathrm{cl}[(\ker h) \cdot W_e]\big) \subset \mathrm{cl}\, h\big((\ker h) \cdot W_e\big)$$

$$\subset \mathrm{cl}[h(\ker h) \cdot h(W_e)] = \mathrm{cl}\, W_e.$$

and by the symmetry of W_e

$$h(Z)^{-1} \subset (\mathrm{cl}\, W_e)^{-1} = \mathrm{cl}\, W_e^{-1} = \mathrm{cl}\, W_e.$$

The continuity of the projection h implies easily that H_o is closed in H whence, in particular, $\mathrm{cl}\, W_e = \mathrm{cl}_{H_o} W_e$. We have already remarked that $\mathrm{cl}_{H_o} W_e$ is compact. Consequently so is also the product $C := (\mathrm{cl}\, W_e) \cdot (\mathrm{cl}\, W_e)$ as the continuous image of the compact set $(\mathrm{cl}\, W_e) \times (\mathrm{cl}\, W_e) \subset H_o \times H_o$. Thus the range $m_o(H)$ of H under m_o is contained in the compact set $C \subset H_o$. Recalling that the group $(H_o, \cdot)$ is uniquely two-divisible this forces the boundedness of $m_o(H)$ in the sense that for any neighbourhood W of e the set $m_o(H)$ is contained in W^{2^n} for some $n \in \mathbb{N}$. Therefore the exponential mapping m_o has to be constant: $m_o(x) = e$, $x \in S$, which means that $\tilde{m} = m$. This completes the proof.

REMARK 6. The assumptions of Theorem 2 imply that H can be split into a direct product $H_o \odot H_1$ (with $H_1 := \ker h$) of two closed subgroups $(H_o, \cdot)$ and $(H_1, \cdot)$ of the group $(H, \cdot)$. The hypothesis of such a factorization might alternatively be assumed instead of the existence of a suitable projection h.

REMARK 7. Separations of points in the dual group $(H_o^*,+)$ of continuous real characters is equivalent to any of the following two statements:

(*) $(H_o^*,+)$ is connected;

(**) the group $(H_o,\cdot)$ itself is topologically isomorphic to a product of the form $\mathbb{R}^n \times F$ where $n \in \mathbb{N}$ and $(F,\cdot)$ yields a discrete torsion free Abelian group (see Hewitt-Ross [7]).

REMARK 8. The target group $(H,\cdot)$ of the map f need not be neither Abelian nor divisible. In the simplest case $(H,\cdot) = (\mathbb{C}^*,\cdot)$ (cf. Theorem 1) is divisible indeed but the division is not uniquely performable.

REMARK 9. Replacing the amenability assumption of the domain semigroup $(S,+)$ by the stronger requirement of commutativity of $(S,+)$ allows one to weaken the hypotheses regarding the group $(H_o,\cdot)$ and to simplify the proof considerably. Namely, assumption (a) (or their equivalent forms mentioned in Remark 7) is not longer needed in such a case. The proof that m is an exponential mapping may then be carried out in the usual way, i.e. by setting $2^n x$ and $2^n y$ in (7) instead of x and y, respectively and applying (9).

REFERENCES

[1] Aczél J., *Lectures on functional equations and their applications,* Academic Press, New York and London, 1966.

[2] Baker J.A., *The stability of the cosine equation,* Proc. Amer. Math. Soc. 80 (1980), 411-416.

[3] Baker J.A., Lawrence J. and Zorzitto F., *The stability of the equation $f(x+y) = f(x)\, f(y)$,* Proc. Amer. Math. Soc. 74 (1979), 242-246.

[4] Forti G.L., *The stability of homomorphisms and amenability with applications to functional equations,* Abh. Math. Sem. Univ. Hamburg 57 (1987), 215-226.

[5] Gajda Z., *On the stability of the Cauchy equation on semi-groups*, Aequationes Math. 36 (1988), 76-79.

[6] Ger R., *Abstract of the talk delivered at the 26-th International Symposium on Functional Equations* (April 24 - May 3, 1988, Sant Feliu de Guixols, Catalonia, Spain), Aequationes Math. 37 (1989), 68.

[7] Hewitt E. and Ross K.A., *Abstract harmonic analysis*, vol. I, Springer Verlag, Berlin-Göttingen-Heidelberg, 1963.

[8] Kuczma M., *An introduction to the theory of functional equations and inequalities*, PWN, Warszawa-Kraków-Katowice, 1985.

[9] Moszner Z., *Sur la stabilité de l'equation d'homomorphisme*, Aequationes Math. 29 (1985), 290-306.

[10] Székelyhidi L., *Remark presented at the 22-nd International Symposium on Functional Equations* (December 16 - December 22, 1984, Oberwolfach, Germany), Aequationes Math. 29 (1985), 95-96.