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Translations and translative partitions of quasigroups

Dedicated to Professor Zenon Moszner with best wishes on his 60-th birthday

INTRODUCTION.

Let X be a nonempty set, $(Q,\cdot)$ a groupoid. In [3] Z. Moszner proved the following theorem: If $(Q,\cdot)$ is a group then a function $F\colon X \times Q \longrightarrow X$ satisfies the functional equation

$$F(F(x,b),a) = F(x,\ b\cdot a) \qquad \textit{for } x \in X, \quad a,b \in Q \tag{i}$$

iff there exists a partition χ of the set X such that for every $Y \in \chi$ there exist a subroup H_Y of the group $(Q,\cdot)$, a mapping $f_Y\colon Y \longrightarrow Y$ and a bijection g_Y of the set $f_Y(Y)$ onto the collection $\{H_Y a\colon a \in Q\}$ such that

$$f_Y(f_Y(x)) = f_Y(x) \qquad \textit{for } x \in Y,$$

$$F(x,a) = g_Y^{-1}\left[g_Y\left(f_Y(x)\right)a\right] \qquad \textit{for } x \in Y, \quad a \in Q. \tag{ii}$$

In the case where $(Q,\cdot)$ is a group one can analogously characterize the solutions of the following functional equation

$$F(F(x,b),\ a) = F(x,\ a\cdot b) \qquad \textit{for } x \in X, \quad a \in Q. \tag{iii}$$

It is enough to replace in Moszner's theorem the collection $\{H_Y a:\ a \in Q\}$ by the collection $\{aH_Y:\ a \in Q\}$ and the condition (ii) by the following condition

$$F(x,a) = g_Y^{-1}[a\ g_Y(f_Y(x))] \qquad \textit{for } x \in Y, \quad a \in Q.$$

In this note we shall show that the similar characterization of the solutions of (iii) is correct in the case, where $(Q,\cdot)$ is a quasigroup. To obtain this result we define left and right translative partitions of the quasigroup $(Q,\cdot)$, left and right translative subquasigroups of the quasigroup $(Q,\cdot)$ and consider the properties of such objects. Finally, using this notions we characterize the solutions of (iii).

Basic definitions and theorems

DEFINITION 1. *A pair* $(Q,\cdot)$ *is called a groupoid, if* Q *is a nonempty set and* $\cdot$ *is a mapping of* $Q \times Q$ *into* Q.

If $(Q,\ \cdot)$ is a groupoid, $A,B \subset Q$ and $a \in Q$ then we shall use the following notations

$U_l := \{f \in Q:\ \ \exists_{a \in Q}\ (fa = a)\}$,

$U_r := \{e \in Q:\ \ \exists_{a \in Q}\ (ea = a)\}$,

$U := U_l \cup U_r$,

$AB := \{a\cdot b:\ \ a \in A,\ \ b \in B\}$,

$aA := \{a\}A, \qquad Aa := A\{a\}$,

$LA := \{aA:\ \ a \in Q\}$,

$RA := \{Aa:\ \ a \in Q\}$.

DEFINITION 2. *A family* M *of subsets of the set* Q *is called the partition of* Q *if it has the following properties*

$A \neq \emptyset \qquad$ *for* $A \in M$,

$Q = \bigcup M$,

$A \cap B = \emptyset \qquad$ *for* $\qquad A, B \in M,\ A \neq B$.

DEFINITION 3. *A family* M *of subsets of the set* Q *is called the left translative partition of the groupoid* $(Q, \cdot)$ *if the following conditions are satisfied*

M *is a partition of* Q, (1)

$$\underset{a \in Q}{\forall}\ \underset{A \in M}{\forall}\ \underset{B \in M}{\exists}(aA \subset B), \tag{2}$$

$$\underset{a,b \in Q}{\forall}\ \underset{A,B \in M}{\forall}((ab)A \subset B \Longrightarrow a(bA) \subset B). \tag{3}$$

DEFINITION 4. *A family* M *of subsets of the set* Q *is called the right translative partition of the groupoid* $(Q, \cdot)$ *if condition* (1) *is satisfied and*

$$\underset{a \in Q}{\forall}\ \underset{A \in M}{\forall}\ \underset{B \in M}{\exists}(Aa \subset B), \tag{2'}$$

$$\underset{a,b \in Q}{\forall}\ \underset{A,B \in M}{\forall}(A(ba) \subset B \Longrightarrow (Ab)a \subset B). \tag{3'}$$

Conditions (3) and (3') are obviously satisfied if $(Q, \cdot)$ is a group. In this case it has been proved (cf. [2]) that M is a partition of Q satisfying (2') if there exists a subgroup H of the group $(Q, \cdot)$ such that $M = RH$.

DEFINITION 5. *A groupoid* $(Q,\cdot)$ *is called the quasigroup if for each* $a,b \in Q$ *every of equations* $ax = b$, $xa = b$ *has exactly one solution.*

A quasigroup possesing the unity element is called a loop.

If $(Q, \cdot)$ is a quasigroup and $a \in Q$, then f_a, e_a, ${}^{-1}a$, a^{-1} denote elements of Q satisfying the conditions

$$f_a \cdot a = a, \qquad a \cdot e_a \qquad {}^{-1}a \cdot a = e_a, \qquad a \cdot {}^{-1}a = f_a.$$

DEFINITION 6. *Let* $(Q,\cdot)$ *be a quasigroup (loop). A nonempty subset* H *of* Q *is called the subquasigroup (subloop) of* $(Q,\cdot)$ *if for each* $a,b \in H$ *the solutions of equations* $ax = b$, $xa = b$ *belong to* H.

DEFINITION 7. *A subquasigroup* H *of the quasigroup* $(Q,\cdot)$ *is called left translative if*

$$a(b(cH)) = (ab)(cH) \qquad \textit{for} \quad a,b,c \in Q \tag{4}$$

and right translative if

$$((Hc)b)a = (Hc)(ba) \qquad \textit{for} \quad a,b,c \in Q. \tag{5}$$

DEFINITION 8. *Let* X *be a nonempty set,* $(Q,\cdot)$ *a groupoid. A function* $F\colon X \times Q \longrightarrow X$ *is called the solution of the translation equation if*

$$F(F(x,b),a) = F(x, a\cdot b) \qquad \textit{for} \quad x \in X, \quad a,b \in Q. \tag{6}$$

The functional equation (6) is called the translation equation.

DEFINITION 9. *A function* $F\colon X \times Q \longrightarrow X$ *is called almost transitive if*

$$\underset{y \in F(X,Q)}{\forall} \; \underset{x \in X}{\forall} \; \underset{a \in Q}{\exists} (F(x,a) = y) \tag{7}$$

and transitive if

$$\underset{x,y \in X}{\forall} \; \underset{a \in Q}{\forall} (F(x,a) = y). \tag{8}$$

If a function $F\colon X \times Q \longrightarrow X$ is transitive, then it is almost transitive. From (8) it follows, that $F(X,Q) = X$. If $\mathbb{R}_o$ denotes the set of all real numbers without zero and $\mathbb{R}_+$ denotes the multiplicative group of all positive real numbers then the function $F\colon \mathbb{R}_o \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ defined as follows

$$F(x,a) = |ax| \qquad \textit{for} \;\; x \in \mathbb{R}_o,\; a \in \mathbb{R}_+$$

is the almost transitive solution of the translation equation and it is not transitive, since $F(\mathbb{R}_o, \mathbb{R}_+) = \mathbb{R}_+ \neq \mathbb{R}_o$.

THEOREM 1. *Let* $(Q,\cdot)$ *be a groupoid,* X *a set,* M *a left translative partition of* $(Q,\cdot)$, k *a function mapping* X *into* X *such that*

$$k(k(x)) = k(x) \qquad \textit{for } x \in X. \tag{10}$$

Let g *be a bijection of* $k(X)$ *into* M *and let* h *be a relation defined as follows*

$$h(aA) = g^{-1}(B) \Leftrightarrow aA \subset B \qquad \textit{for } a \in Q,\; A,B \in M.$$

Then h *is a function defined on the family* $\{aA\colon a \in Q,\ A \in M\}$ *and the function* F *defined as follows*

$$F(x,a) = h\big(ag(k(x))\big) \qquad \textit{for} \quad x \in X,\ a \in Q$$

is the solution of the translation equation.

Proof. From (1), (2) and the definition of h we conclude, that h is a function defined on the family $\{aA\colon a \in Q,\ A \in M\}$. Let $x \in X$, $a,b \in Q$ be arbitrary and let $A,B,D \in M$ be such that

$$g(k(x)) = A, \qquad (ab)A \subset D, \qquad bA \subset B.$$

Then $a(bA) \subset aB$ and, by (3), $a(bA) \subset D$. Hence, by (2) and (1) the inclusion $aB \subset D$ holds. By (10) and $g^{-1}(B) \in k(X)$ we have $k(g^{-1}(B)) = g^{-1}(B)$. Using the above results we get

$$\begin{aligned} F(F(x,b),a) &= F\Big(h\big(bg(k(x))\big),a\Big) = F(h(bA),\ a) \\ &= F(g^{-1}(B),\ a) = h\Big(ag\big(k(g^{-1}(B))\big)\Big) = h\big(ag(g^{-1}(B))\big) \\ &= h(aB) = h((ab)A) = h\big((ab)g(k(x))\big) = F(x,ab), \end{aligned}$$

which completes the proof.

By theorem 1 we are able to characterize some solutions of the translation equation using the left translative partitions of $(Q,\cdot)$. We shall show that, in the case where $(Q,\cdot)$ is a quasigroup, then each solution of the translation equation can be characterized with the help of left translative partitions of $(Q,\cdot)$.

LEMMA 1. *Let* M *be a left translative partition of the groupoid* $(Q,\cdot)$ *and let* $a,b,c \in Q$, $A,B \in M$ *be arbitrary. Then the following conditions are equivalent:*

$$[(ab)c]\, A \subset B, \tag{11}$$

$$(ab)(cA) \subset B, \tag{12}$$

$$a(b(cA)) \subset B, \tag{13}$$

$$a[(bc)A] \subset B, \tag{14}$$

$$[a(bc)]A \subset B. \tag{15}$$

Proof. Let $D,E \in M$ be such that

$$cA \subset D, \qquad bD \subset E. \tag{16}$$

Then we have

$$b(cA) \subset bD \subset E, \tag{17}$$

$$a(b(cA)) \subset a(bD) \subset aE. \tag{18}$$

By (2), (3), (1) we deduce immediately, that the following condition

$$a(bA) \subset B \Longrightarrow (ab)A \subset B \qquad \text{for } a,b \in Q, \qquad A,B \in M \tag{19}$$

holds. By (17) and (19) we obtain

$$(bc)A \subset E, \tag{20}$$

whence

$$a[(bc)A] \subset AE. \tag{21}$$

Moreover by (16) we get

$$(ab)(cA) \subset (ab)D. \tag{22}$$

Now we shall show the equivalency of conditions (11) - (15). By (3) and (19) conditions (11) and (12) are equivalent. Now suppose that condition (12) holds. From (22), (2), (1) we obtain $(ab)D \subset B$, which together with (3) gives $a(bD) \subset B$. Hence and from (18) we get $a(b(cA)) \subset B$, i.e. (13) holds. From (13), (18), (2), (1) we get $aE \subset B$, which together with (21) gives (14). From (19) we conclude, that (14) implies (15). Let now condition (15) be fulfilled. Then by (3) condition (14) holds. Hence and from (21), (2), (1) we get $aE \subset B$, which together with (18), (19), (22) gives (12). This completes the proof, since condition (12) is equivalent to (11).

Left translative partitions of quasigroups

LEMMA 2. *If* M *is the left translative partition of the quasigroup* $(Q, \cdot)$ *then the following conditions are satisfied*

$$\forall_{a \in Q} \forall_{A \in M} \exists_{B \in M} (aB \subset A), \tag{23}$$

$$fA \subset A \quad \textit{for } f \in U_l, \quad A \in M, \tag{24}$$

$$aA \subset B \Longrightarrow (fa)A \subset B \quad \textit{for } a \in Q,\ f \in U_l,\ A,B \in M, \tag{25}$$

$$eA \subset A \quad \textit{for } e \in U_r,\ A \in M, \tag{26}$$

$$aA \subset B \Longrightarrow {}^{-1}aB \subset A \quad \textit{for } a \in Q,\ A,B \in M, \tag{27}$$

$$[aA \subset B,\quad aD \subset B] \Longrightarrow A = D \quad \textit{for } a \in Q,\ A,B,D \in M. \tag{28}$$

Proof. Let $a \in Q$, $A \in M$, $b \in A$ be arbitrary. Let $c \in Q$, $B \in M$ be such that $ac = b$ and $c \in B$. Then $b \in aB \cap A$ which means (by (1) and (2)) that $aB \subset A$. Thus condition (23) is fulfilled. Now let

$A \in M$, $f \in U_l$ be arbitrary and let $a \in Q$ be such that $fa = a$. According to (23) there exists $B \in M$ such that $aB \subset A$, whence

$$f(aB) \subset fA, \qquad (fa)B \subset A.$$

Hence and from (3), (2), (1) we get $fA \subset A$ and conclude that condition (24) is satisfied. Let $f \in U_l$, $a \in Q$, $A, B \in M$ be arbitrary and let $aA \subset B$. Using (24) we get $f(aA) \subset fB \subset B$, which together with (19) gives $(fa)A \subset B$. Thus condition (25) is fulfilled. Let $e \in U_r$, $A \in M$ be arbitrary and let $B \in M$, $a \in Q$ be such that

$$eA \subset B, \quad ae = a. \tag{29}$$

Let $f \in U_l$, $c \in Q$ be such that $ca = f$. From (29) and (25) we get $(fe)A \subset B$, whence $[(ca)e]A \subset B$. Hence, according to lemma 1, we obtain $[c(ae)]A \subset B$, which together with (29) gives $(ca)A \subset B$, i.e. $fA \subset B$. Using (24) and (1) we get $A = B$, which together with (29) gives (26). Assume that $a \in Q$, $A,B \in M$ are such that $aA \subset B$. Then

$$^{-1}a(aA) \subset {}^{-1}aB. \tag{30}$$

By (26) we have $({}^{-1}a\ a)A = e_a A \subset A$, whence by (3) we get ${}^{-1}a(aA) \subset A$. Hence and from (30), (2), (1) we obtain ${}^{-1}aB \subset A$. Thus condition (27) is satisfied. If $aA \subset B$, $aD \subset B$ then by (27) we have ${}^{-1}aB \subset A$, ${}^{-1}aB \subset D$, which together with (1) gives $A = D$ and completes the proof.

THEOREM 2. *A family* M *of subsets of the set* Q *is the left translative partition of the quasigroup* $(Q,\cdot)$ *if condition* (1) *is satisfied and*

$$aA \in M \qquad \textit{for} \quad a \in Q,\ A \in M, \tag{31}$$

$$(ab)A = a(bA) \qquad \textit{for} \quad a,b \in Q,\ A \in M. \tag{32}$$

Proof. Let M be a family of subsets of the set Q. In virtue of definition 3 it is obvious that M is the left translative partition of $(Q,\cdot)$ if conditions (1), (31), (32) are satisfied. Now assume that conditions (1), (2), (3) are satisfied. Take arbitrary $a,b \in Q$, $A \in M$. By (2) we have

$$aA \subset B \tag{33}$$

for some B belonging to M. Let $d \in B$ be arbitrary and let $c \in Q$, $D \in M$ be such that

$$d = ac, \qquad c \in D. \tag{34}$$

Then by (1), (2) we have $aD \subset B$, whence by (33) and (28) we get $D = A$. Hence and from (34) we obtain $d \in aA$, which gives $B \subset aA$ and consequently, by (33), $B = aA$. Thus condition (31) is fulfilled. By (31) we conclude that $(ab)A = B$ for some $B \in M$. Hence and from (3), (31) we get $a(bA) = B$. Thus $a(bA) = (ab)A$, which completes the proof.

From lemma 1 and theorem 2 we obtain the following

LEMMA 3. *If* M *is a left translative partition of the quasigroup* $(Q,\cdot)$ *then*

$$[(ab)c]A = (ab)(cA) = a(b(cA)) = [a(bc)]A = a[(bc)A] \tag{35}$$

for $a,b,c \in Q$, $A \in M$.

In virtue of (35) and (32) we will often do not note the brackets in the products of the form abA, $abcA$.

LEMMA 4. *If* M *is a left translative partition of the quasigroup* $(Q,\cdot)$ *then the following conditions are satisfied*

$$fA = A \qquad \textit{for} \quad f \in U_l, \quad A \in M, \tag{36}$$

$$faA = aA \qquad \textit{for} \quad f \in U_l, \quad a \in Q, \quad A \in M, \tag{37}$$

$$eA = A \qquad \textit{for} \quad e \in U_r, \quad A \in M, \tag{38}$$

$$aA \subset B \Longrightarrow {}^{-1}aB = A \qquad \textit{for} \quad a \in Q, \quad A,B \in M, \tag{39}$$

$$abA = afbA \qquad \textit{for} \ a,b \in Q, \quad f \in U, \quad A \in M, \tag{40}$$

$$ca = f = ad \Longrightarrow abA = dbA \quad \textit{for} \ a,b,c,d \in Q, \quad A \in M, \tag{41}$$

$${}^{-1}aA = a^{-1}A \qquad \textit{for} \ a \in Q, \quad A \in M, \tag{42}$$

$${}^{-1}(ab)A = b^{-1}\, a^{-1}A \qquad \textit{for} \ a,b \in Q, \quad A \in M. \tag{43}$$

Proof. Conditions (36) - (39) we obtain immediately from lemma 2 and theorem 2. From (31), (36), (38) we obtain (40). Let $f \in U$, $a,c,d \in Q$ be such that $ca = f = ad$ and let $b \in Q$ be arbitrary. By (40), (31), (38), (32) and (35) we have

$$cbA = cfbA = c(ad)ba = [c(ad)](bA) = [(ca)d](bA) = (fd)bA$$

$$= f(dbA) = dbA,$$

thus condition (41) is fulfilled. Take arbitrary $a,b \in Q$, $A \in M$. By (36), (35), (31) and (38) we have

$${}^{-1}aA = {}^{-1}af_aA = {}^{-1}aaa^{-1}A = e_aa^{-1}A = a^{-1}A,$$

$${}^{-1}(ab)A = {}^{-1}(ab)f_aA = {}^{-1}(ab)(aa^{-1}A) = {}^{-1}(ab)(af_ba^{-1}A)$$

$$= {}^{-1}(ab)(abb^{-1}a^{-1}A) = {}^{-1}(ab)(ab)(b^{-1}a^{-1}A)$$

$$= e_{ab}(b^{-1}a^{-1}A) = b^{-1}a^{-1}A.$$

Thus conditions (42) and (43) are satisfied, which completes the proof.

LEMMA 5. *If* M *is a left translative partition of the quasigroup* $(Q,\cdot)$ *then there exists an unique* $H \in M$ *such that the following conditions are satisfied*

$$U \subset H, \tag{44}$$

$$a \in A \Longleftrightarrow aH = A \qquad \textit{for} \qquad a \in Q, \quad A \in M, \tag{45}$$

$$H \textit{ is a subquasigroup of } Q. \tag{46}$$

Proof. Let $f \in U$ be arbitrarily fixed and let $H \in M$ be such that $f \in H$. Let $g \in U$ be arbitrary and let $A \in M$, $h \in Q$ be such that $g \in A$ and $f = hg$. In virtue of theorem 2 and (36), (38) we have

$$H = hA = h(gA) = (hg)A = fA = A,$$

whence $g \in H$. Thus condition (44) is fulfilled.
Now take arbitrary $a \in Q$, $A \in M$. By (44) we conclude that the equality $aH = A$ implies $a \in A$. Assume that $a \in A$. Then by (44) we have $a = ae_a \in aH$, which together with $a \in A$ and (31) gives $aH = A$. Thus condition (45) holds. Now assume that $h,g \in H$ are arbitrary. By (45) we get $hH = H$, whence we conclude that $HH = H$ and that the equality $hk = g$ holds for some $k \in H$. Let $s \in Q$ be such that $sh = g$. Then $sH = H$, which together with (45) gives $s \in H$. From the

above considerations we deduce that H is a subquasigroup of the quasigroup $(Q,\cdot)$. Moreover, by (44) we conclude that H is the only subquasigroup of $(Q,\cdot)$ belonging to M, which completes the proof.

LEMMA 6. *Let* H *be a subquasigroup of the quasigroup* $(Q,\cdot)$. *If* H *has the property*

$$(ah)H = aH \qquad \textit{for} \qquad a \in Q, \quad h \in H \tag{47}$$

then LH *is the partition of* Q. *Moreover, if* H *has the property*

$$a(bH) = (ab)H \qquad \textit{for} \qquad a,b \in Q \tag{48}$$

then (47) *holds.*

Proof. The family LH covers the set Q and the elements of LH are nonempty. Suppose that (47) holds and that $aH \cap bH \neq \emptyset$ for some $a,b \in Q$. Then $ah = bg$ for some $h,g \in H$. Hence and by (47) we get $aH = (ah)H = (bg)H = bH$. Thus LH is the partition of Q. If ir (48) b belongs to H, then $bH = H$ since H is a subquasigroup of $(Q,\cdot)$. Therefore (48) implies (47), which completes the proof.

THEOREM 3. *Let* $(Q,\cdot)$ *be a quasigroup. Then a family* M *of subsets of the set* Q *is the left translative partition of* $(Q,\cdot)$ *iff there exists a left translative subquasigroup* H *of* $(Q,\cdot)$ *such that* $M = LH$.

Proof. Let H be a left translative partition of $(Q,\cdot)$. In virtue of lemma 5 there exists a subquasigroup H of $(Q,\cdot)$ such that $M = LH$. Hence and from lemma 3 we get $a(b(cH)) = (ab)(cH)$ for $a,b,c \in Q$, which means, that H is a left translative subquasigroup of $(Q,\cdot)$. Conversely, suppose that H is a left translative subquasigroup of the quasigroup $(Q,\cdot)$. Putting $h \in H$ in place of c in (5) we get $a(bH) = (ab)H$. Concequently, by lemma 6, LH is a partition of Q satisfying (31). Moreover, it follows immediately from (5) that condition (32) is satisfied for LH. Thus, in virtue of theorem 2, LH is a left translative partition of $(Q, \cdot)$, which completes the proof.

For the relation $\rho \subset Q \times Q$ we shall use the following notation

$$[x] := \{y \in Q: \quad y \rho x\} \qquad \textit{for} \qquad x \in Q,$$

$$Q/\rho := \{[x]: \quad x \in Q\}.$$

THEOREM 4. *Let* $(Q,\cdot)$ *be a quasigroup,* $M \subset 2^Q$. *Then* M *is the left translative partition of* $(Q,\cdot)$ *iff the relation* $\rho \subset Q \times Q$ *defined as follows*

$$x \rho y \Longleftrightarrow \underset{A \in M}{\exists}(x,y \in A) \tag{49}$$

is an equivalency such that the following conditions are satisfied

$$M = Q/\rho, \tag{50}$$

$$x \rho ex \qquad \textit{for} \qquad x \in X, \quad e \in U_r, \tag{51}$$

$$x \rho y \Longrightarrow (ab)x \rho a(by) \qquad \textit{for} \qquad x,y,a,b \in Q. \tag{52}$$

Proof. Suppose that $M \subset 2^Q$ and that the relation ρ defined by (49) is an equivalence relation such that conditions (50) - (52) are satisfied. Then M is a partition of Q. From $x \rho x$, using (52), (51) and the transitivity of ρ we obtain sequently

$$({}^{-1}aa)x \rho {}^{-1}a(ax),$$

$$e_a x \rho {}^{-1}a(ax),$$

$$x \rho {}^{-1}a(ax) \qquad \textit{for } x,a \in Q \tag{53}$$

and

$$(ae_a)x \ \rho \ a(e_a x),$$

$$ax \ \rho \ a(e_a x) \qquad \textit{for} \ x,a \in Q. \tag{54}$$

Let us observe now that the following condition

$$x \ \rho \ y \Longrightarrow ax \ \rho \ ay \qquad \textit{for} \ x,y,a \in Q \tag{55}$$

holds true. Take arbitrary $x,y,a \in Q$. If $x\rho y$ then from (52) we obtain $(ae_a)x \ \rho \ a(e_a y)$, which together with (54) and the transitivity of ρ gives $ax \ \rho \ ay$. If $ax\rho ay$ then from the above we get ${}^{-1}a(ax) \ \rho \ {}^{-1}a(ay)$, which together with (53) and the transitivity of ρ gives $x \ \rho \ y$. Thus condition (55) is fulfilled.

Let $x,a \in Q$ be arbitrary. If $z \in [x]$, then $z \ \rho \ x$ and from [55] we have $az \ \rho \ ax$, whence we conclude that $a[x] \subset [ax]$. If $y \in [ax]$ and $z \in Q$ is such that $y = az$, then $az \ \rho \ ax$. Hence, by (55), we get $z \ \rho \ x$, i.e. $z \in [x]$ and consequently $[ax] \subset a[x]$. Thus we have shown that the following condition is satisfied

$$a[x] = [ax] \qquad \textit{for} \ a,x \in Q. \tag{56}$$

This means, that for M condition (31) if fulfilled. By (52) and the reflexivity of ρ we get

$$[(ab)x] = [a(bx)] \qquad \textit{for} \ a,b,x \in Q.$$

Hence and from (56) we get

$$(ab)[x] = a(b[x]) \qquad \textit{for} \ a,b,x \in Q,$$

thus for M condition (32) is satisfied. From the above considerations and theorem 2 we conclude that M is the left translative partition of $(Q,\cdot)$, which completes the first part of the proof. Now assume that M is a left translative partition of $(Q,\cdot)$. Then ρ is an equivalence relation and $M=Q/\rho$. Hence and from (31), (32), (38) we get

$$a[x] = [ax], \qquad (ab)[x] = a(b[x]), \qquad e[x] = [x]$$

for $a,b,x \in Q$, $e \in U_r$, whence we conclude that conditions (51), (52) are satisfied, which completes the proof.

It is easy to see that for the right translative partitions and right translative subquasigroups of the quasigroup $(Q,\cdot)$ lemmas corresponding to lemmas 1 - 6 and theorems corresponding to theorems 2 - 4 are true. In particular theorem 5 and lemma 7 given below are true.

THEOREM 5. *A family* M *of subsets of the set* Q *is a right translative partition of the quasigroup* $(Q,\cdot)$ *iff there exists a right translative subquasigroup* H *of the quasigroup* $(Q,\cdot)$ *such that* $M = RH$.

LEMMA 7. *If* H *is a right translative partition of the quasigroup* $(Q,\cdot)$ *then the following conditions are fulfilled*

$$H[a(bc)] = (Ha)(bc) = ((Ha)b)c = H[(ab)c] = [H(ab)]c \tag{57}$$

for $a,b,c \in Q$,

$$(Ha)b^{-1} = (Ha)({}^{-1}b) \qquad \textit{for } a,b \in Q \tag{58}$$

$$(Ha)(b^{-1}c^{-1}) = (Ha)(cb)^{-1} \qquad \textit{for } a,b,c \in Q, \tag{59}$$

$$(Ha)u = Ha \qquad \textit{for } a \in Q, \quad u \in U, \tag{60}$$

$$U \subset H. \tag{61}$$

Now we shall present some examples of quasigroups which will illustrate objects defined so far.

EXAMPLE 1. $Q := \{1,2,3,4,5\}$ and $\cdot$ is defined as follows

$\cdot$	1	2	3	4	5
1	1	2	3	4	5
2	2	1	4	5	3
3	3	4	5	2	1
4	4	5	1	3	2
5	5	3	2	1	4

$(Q,\cdot)$ is the loop, $H := \{1,2\}$ is the subloop of $(Q,\cdot)$ and neither of families LH, RH is the partition of Q.

EXAMPLE 2. $Q := \{1,2,3,4,5,6,7,8\}$ and $\cdot$ is defined as follows

$\cdot$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	5	6	7	8	3	4
3	3	4	2	1	8	7	5	6
4	4	3	7	8	1	2	6	5
5	5	6	4	3	2	1	8	7
6	6	5	8	7	3	4	1	2
7	7	8	6	5	4	3	2	1
8	8	7	1	2	6	5	4	3

$(Q,\cdot)$ is the loop with the unit 1, $H := \{1,2\}$ is the subloop of $(Q,\cdot)$. LH is the partition of Q, RH is not the partition of Q. Condition (32) is not satisfied for the family LH since $\{3,4\} \in LH$ and $(3\cdot3)\{3\cdot4\} = \{5,6\} \neq \{3,4\} = 3(3\{3,4\})$.

It is easy to observe that condition' (31) is satisfied. Hence and from $H \in LH$, $1 \in H$ we deduce that $x(yH) \in LH$, $xy \in (xy)H$,

$xy \in x(yH)$ for $x,y \in Q$, whence, because LH is the partition of Q we get $(xy)H = x(yH)$ for $x,y \in Q$.

EXAMPLE 3. $Q := \{1,2,3,4,5,6\}$ and $\bullet$ is defined as follows

	1	2	3	4	5	6
1	1	2	5	6	3	4
2	2	1	6	5	4	3
3	5	6	3	4	1	2
4	6	5	4	3	2	1
5	3	4	1	2	5	6
6	4	3	2	1	6	5

$H_1 := \{1,2\}$, $H_2 := \{3,4\}$, $H_3 := \{5,6\}$ are subquasigroups of the quasigroup $(Q,\bullet)$, $3 \notin 3H_1$, thus condition

$$a \in aH_1 \qquad \textit{for } a \in Q$$

is not satisfied. Moreover $LH_1 = LH_2 = LH_3 = \{H_1, H_2, H_3\}$ whence, by lemma 5 we conclude that LH_i $(i = 1,2,3)$ is not the left translative partition of $(Q,\bullet)$.

EXAMPLE 4. $Q := \{1,2,3,4,5,6\}$, $\bullet$ is defined as follows

	1	2	3	4	5	6
1	1	2	4	5	3	6
2	2	1	3	6	4	5
3	3	4	6	2	5	1
4	4	3	5	1	6	2
5	6	5	1	3	2	4
6	5	6	2	4	1	3

$H := \{1,2\}$ is a subquasigroup of the quasigroup $(Q,\bullet)$, LH is the partition of Q and H is the only subquasigroup belonging to LH. Moreover $3(4H) = \{2,6\} \notin LH$, thus condition (31) is not satisfied for LH. It is easy to observe, that $U_l = \{1,2,3,6\}$, $U_r = \{1,2\}$.

EXAMPLE 5. $Q:=\{1,2,3,4,5,6,a,b,c,d,e,f\}$ and $\cdot$ is defined as follows

$\cdot$	1	a	5	e	2	b	4	d	3	c	6	f
1	1	a	5	e	2	b	4	d	3	c	6	f
a	a	1	e	5	b	2	d	4	c	3	f	6
5	5	e	1	a	f	6	3	c	d	4	2	b
e	e	5	a	1	6	f	c	3	4	d	b	2
2	2	b	d	4	3	c	f	6	a	1	5	e
b	b	2	4	d	c	3	6	f	1	a	e	5
4	4	d	b	2	e	5	a	1	f	6	c	3
d	d	4	2	b	5	e	1	a	6	f	3	c
3	3	c	f	6	a	1	e	5	2	b	4	d
c	c	3	6	f	1	a	5	e	b	2	d	4
6	6	f	3	c	4	d	2	b	5	e	1	a
f	f	6	c	3	d	4	b	2	e	5	a	1

$H:=\{1,a,5,e\}$ is the left translative subquasigroup of the quasigroup $(Q,\cdot)$. The proof of this fact is very cumbersome and therefore we omit it. Since $(d6)e = 6 \neq f = d(6f)$ thus $(Q,\cdot)$ is not the group.

The translation equation.

In this part X denotes an arbitrary nonempty set, $(Q,\cdot)$ an arbitrary quasigroup.

THEOREM 6. *If a function* $F\colon X \times Q \longrightarrow X$ *is the solution of the translation equation and* $f \in Q$ *then the relation* $\tau \subset X \times X$ *defined as follows*

$$x \,\tau\, y \Longleftrightarrow \underset{a \in Q}{\exists} (F(y,a) = F(x,f)) \qquad \textit{for } x,y \in X \tag{62}$$

is an equivalence relation and the following conditions are satisfied

$$F(z,a)\,\tau\, x \qquad \textit{for } x \in X, \qquad z \in [x], \qquad a \in Q, \tag{63}$$

$F|_{[x]\times Q}$ *is the almost transitive solution of the translation equation,* (64)

$$F = \bigcup_{x\in X} F|_{[x]\times Q} \tag{65}$$

Proof. Let $F\colon X \times Q \longrightarrow X$ be a solution of the translation equation, let $f \in Q$ be arbitrarily fixed and let $\tau \subset X \times X$ be the relation defined by (62). Condition (65) is satisfied. By the equality $F(x,f) = F(x,f)$ we obtain the reflexivity of the relation τ. Let $F(y,a) = F(x,f)$ and let $b \in Q$ be such that $ba = f$. Then by (7) we have

$$F(y,f) = F(y,ba) = F(F(y,a),b) = F(F(x,f),b) = F(x,bf),$$

whence we obtain the symmetricity of the relation τ. If $F(y,a) = F(x,f)$, $F(z,b) = F(y,f)$ and $c \in Q$ is such that $cf = a$ then by (7) we have

$$F(x,f) = F(y,a) = F(y,cf) = F(F(y,f),c) = F(F(z,b),c) \\ = F(z,cb),$$

whence we obtain the transitivity of the relation τ. Thus we have shown that τ is an equivalence relation. Now let $x \in X$, $z \in [x]$, $a \in Q$ be arbitrary and let $b,c \in Q$ be such that $F(z,b) = F(x,f)$ and $ca = b$. We have

$$F(x,f) = F(z,b) = F(z,ca) = F(F(z,a),c),$$

thus condition (63) is satisfied. From (63) we get $F([x],Q) \subset [x]$ for $x \in X$. Hence and from (7) we conclude that $F|_{[x]\times Q}$ is the solution of the translation equation. Let $x \in X$, $y \in F([x],Q)$, $z \in [x]$ be arbitrary. Then $y = F(t,a)$ for some $t \in [x]$, $a \in Q$ and $F(z,b) = F(x,f) = F(t,c)$ for some $b,c \in Q$. Let $d \in Q$ be such that $a = dc$. We have

$$y = F(t,a) = F(t,dc) = F(F(t,c),d) = F(F(z,b),d) = F(z,kb).$$

Hence and by (63) we deduce that $F|_{[x]xQ}$ is an almost transitive function, which completes the proof.

LEMMA 8. *Let* $F\colon X \times Q \longrightarrow X$ *be a solution of the translation equation. Then the following conditions are satisfied*

$$F(x,(ab)c) = F(x,a(bc)) \qquad \textit{for } x \in X,\ a,b \in Q \tag{66}$$

$$F(X,Q) = F(X,\{a\}) \qquad \textit{for } a \in Q, \tag{67}$$

$$F(y,f) = y \qquad \textit{for } y \in F(X,Q),\ f \in U_1, \tag{68}$$

$$F(x,fa) = F(x,a) \qquad \textit{for } x \in X,\ a \in Q,\ f \in U_1, \tag{69}$$

$$F(x,a(fb)) = F(x,ab) \qquad \textit{for } x \in X,\ a,b \in Q,\ f \in U_1, \tag{70}$$

$$ca = f = ad \Longrightarrow F(x,cb) = F(x,db)$$

$$\textit{for } x \in X,\ a,b,c,d \in Q, \tag{71}$$

$$F(x,af) = F(x,a) \qquad \textit{for } x \in X,\ a \in Q,\ f \in U_1, \tag{72}$$

$$F(x,f) = F(x,e) \qquad \textit{for } x \in X,\ e,f \in U, \tag{73}$$

$$F(x,{}^{-1}a) = F(x,a^{-1}) \qquad \textit{for } x \in X,\ a \in Q, \tag{74}$$

$$F(x,{}^{-1}(ab)) = F(x,b^{-1}a^{-1}) \qquad \textit{for } x \in X,\ a,b \in Q. \tag{75}$$

Proof. Take arbitrary $x \in X$, $y \in F(X,Q)$, $a,b,c,d \in Q$, $f \in U_l$, $e \in U_r$. Using (7) we get

$$F(x,(ab)c) = F(F(x,c),ab) = F\big(F(F(x,c),b),a\big) = F(F(x,bc),a)$$
$$= F(x,a(bc)),$$

thus condition (66) is satisfied. In virtue of (66) we shall often write $F(x,abc)$ instead of $F(x,a(bc))$ and $F(x,(ab)c)$ in the case where F is the solution of the translation equation. Suppose now that the equality $y = F(z,k)$ holds true for some $z \in X$, $k \in Q$. Let $p \in Q$ be such that $ap = d$. By (7) we have

$$y = F(z,k) = F(x,ap) = F(F(x,p),a),$$

whence $y \in F(X,\langle a\rangle)$. Hence and from the obvious inclusion $F(X,\langle a\rangle) \subset F(X,Q) \subset F(X,Q)$ we obtain (67). Let $k \in Q$ be such that $fk = k$. By (67) there exists $z \in X$ such that $y = F(z,k)$ and we get

$$y = F(z,k) = F(z,fk) = F(F(z,k),f) = F(y,f).$$

Thus condition (68) is satisfied. Hence and from (7) we have

$$F(x,fa) = F(F(x,a),f) = F(x,a),$$

whence we obtain (69). From (69) we immediately obtain (70). Now assume that the equalities $ca = f = ad$ hold. Using (70), (66), (69) we get

$$F(x,cb) = F(x,cfb) = F(x,c(ad)b) = F(x,(ca)db) = F(x,fdb)$$
$$=F(x,db),$$

thus condition (71) holds. Let $k,p \in Q$ be such that $ka=f=ap$. From (71) we get

$$F(x,km) = F(x,pm) \qquad \textit{for } m \in Q,$$

whence

$$F(x,ka) = F(x,pa).$$

Hence and from (66), (7), (69) we obtain

$$F(x,af) = F(x,aka) = F(F(x,ka),a) = F(F(x,pa),a)$$

$$= F(x,apa) = F(x,fa) = F(x,a),$$

thus (72) holds. Let $k,p \in Q$ be such that $pe = p$, $kp = f$. Then by (69) we obtain

$$F(x,e) = F(x,fe) = F(x,kpe) = F(x,kp) = F(x,f).$$

Thus we have shown that the following condition is satisfied

$$F(x,f) = F(x,e) \qquad \textit{for} \qquad f \in U_l,\ e \in U_r.$$

Hence, since the sets U_l, U_r are nonempty we get (73). By (72), (66), (73), (68) and in virtue of the definitions of the symbols ${}^{-1}a$, a^{-1}, f_a, e_a we get

$$F(x, {}^{-1}a) = F(x, {}^{-1}af_a) = F(x, {}^{-1}aaa^{-1}) = F(x, e_a a^{-1})$$

$$= F(F(x, a^{-1}), e_a) = F(F(x,a^{-1}), f_a) = F(x,a^{-1}),$$

thus condition (74) holds.

By (66), (70), (73) we obtain

$$F(x,(ab)(b^{-1}a^{-1})) = f(x,f) \qquad \textit{for } x \in X,\ a,b \in Q,\ f \in U.$$

Hence and from (72), (7), (66), (73), (68) we get

$$F(x, {}^{-1}(ab)) = F(x, {}^{-1}(ab)f) = F(F(x,f), {}^{-1}(ab))$$

$$= F(F(x, (ab)(b^{-1}a^{-1})), {}^{-1}(ab))$$

$$= F(x, {}^{-1}(ab)(ab)(b^{-1}a^{-1})) = F(x, e_{ab} b^{-1}a^{-1})$$

$$= F(F(x, b^{-1}a^{-1}), e_{ab}) = F(F(x, b^{-1}a^{-1}), f)$$

$$= F(x, b^{-1}a^{-1}),$$

which completes the proof.

For the function $F:X \times Q \longrightarrow X$, $f \in U$ we shall use the following notation

$$Q_{xy} := \{a \in Q:\ F(y,a) = F(x,f)\} \qquad \textit{for} \qquad x,y \in X. \tag{76}$$

Let us note that for the transitive solution of the translation equation (74) holds and $F(X,Q) = X$. Thus in this case

$$Q_{xy} = \{a \in Q: F(y,a) = x\} \qquad \textit{for} \qquad x,y \in X.$$

LEMMA 9. *If* $F\colon X \times Q \longrightarrow X$ *is a solution of the translation equation then the following conditions are satisfied*

$$aQ_{xy} = Q_{F(x,a)\,y}, \qquad Q_{yx}a = Q_{yF(x,a^{-1})} \qquad \textit{for } x,y \in X,\ a \in Q, \tag{77}$$

$$aQ_{yy} = Q_{xy} = Q_{xx}a \qquad \textit{for } a \in Q_{xy}, \quad x,y \in X, \tag{78}$$

Q_{xx} *is a left and right translative subquasigroup of* (79)
$(Q,\cdot)$ *for every* $x \in X$.

Proof. Let us note, that conditions

$$F(x,a) = F(y,b), \tag{80}$$

$$F(x,ca) = F(y,cb) \tag{81}$$

are equivalent for $x \in X$, $a,b,c \in Q$. Indeed, from (80) we immediately obtain (81). If we assume that (81) holds true then by (68), (73) and 7 we get

$$F(x,a) = F(x,f_c a) = F(F(x,a),f_c) = F(F(x,a),\ e_c) = F(x,e_c,a)$$

$$= F(x,\ {}^{-1}c\,ca) = F(F(x,ca),\ {}^{-1}c) = F(F(y,cb),\ {}^{-1}c)$$

$$= F(y,\ {}^{-1}c\,cb) = F(y,e_c b) = F(F(y,b),e_c)$$

$$= F(F(y,b),\ f_c) = F(y,b).$$

Thus we have shown that conditions (81), (80) are equivalent.

Since conditions (80), (81) are equivalent, thus by (69) and (72), (73) it follows that conditions

$$F(y,b) = F(x,f), \qquad F(y,ab) = F(x,fa) = F(F(x,a),\ f)$$

are equivalent for $x,y \in X$, $a,b \in Q$, $f \in U$. Hence and by (76) we obtain the eqivalency of conditions

$$b \in Q_{xy}, \qquad ab \in Q_{F(x,a)y}$$

for $x,y \in X$, $a,b \in Q$. Hence

$$aQ_{xy} = Q_{F(x,a)y} \qquad \textit{for} \quad x,y \in X, \qquad a \in Q.$$

From (72) and (7) we obtain the equivalency of conditions

$$F(x,b) = F(y,f), \qquad F(F(x,\ a^{-1}),\ ba) = F(y,f)$$

for $x,y \in X$, $a,b \in Q$, which gives the equivalency of conditions

$$b \in Q_{yx}, \qquad ba \in Q_{yF(x,a^{-1})},$$

and consequently the equality

$$Q_{yx}a = Q_{yF(x,a^{-1})}$$

for $x,y \in X$, $a \in Q$. Thus we have shown that condition (77) holds. Analogously one can show the equivalency of conditions

$$F(y,a) = F(x,f), \qquad F(y,a) = F(F(x,t),f),$$

and the equivalency of conditions

$$F(y,a) = F(x,f), \qquad F(F(y,f),a) = F(x,f).$$

Hence immediately we obtain the equivalency of conditions

$$a \in Q_{xy}, \qquad a \in Q_{F(x,f)y}$$

and the equivalency of conditions

$$a \in Q_{xy}, \qquad a \in Q_{xF(y,f)}.$$

Hence and from (77) we get

$$Q_{xF(y,f)} = Q_{xy} = Q_{F(x,f)y} = fQ_{xy} \qquad \text{for } x,y \in Q,\ f \in U_1. \tag{82}$$

Let now $a \in Q_{xy}$, i.e. $F(y,a) = F(x,f)$. Thus we have

$$aQ_{yy} = Q_{F(y,a)y} = Q_{F(x,f)y} = Q_{xy}$$

and

$$Q_{xx}a = Q_{xF(x,a^{-1})} = Q_{xF(y,f)} = Q_{xy},$$

which completes the proof of (78). By (73) the set Q_{xx} is nonempty for $x \in X$. From (78) we get

$$aQ_{xx} = Q_{xx} = Q_{xx}a \qquad \textit{for } a \in Q_{xx},$$

whence we conclude that $Q_{xx}Q_{xx} = Q_{xx}$ and that for each $a,b \in Q_{xx}$ the solutions of equations $ax = b$, $ya = b$ belong to Q_{xx}. Thus Q_{xx} is a subquasigroup of $(Q,\cdot)$. By (77), (7) and (66) we obtain

$$a(b(cQ_{xy})) = a(bQ_{F(x,c)y}) = aQ_{F(x,bc)y} = Q_{F(x,a(bc))y}$$

$$= Q_{F(x,(ab)c)y} = Q_{F(F(x,c),ab)y} = (ab)Q_{F(x,c)y}$$

$$= (ab)\ (cQ_{xy}).$$

Similarly, by (77), (7), (74), (75), we get

$$((Q_{xy}c)b)a = Q_{xF(y,\ a^{-1}b^{-1}c^{-1})} = Q_{xF(F(y,c^{-1}),\ a^{-1}b^{-1})}$$

$$= Q_{xF(F(y,c^{-1}),(ba)^{-1})} = Q_{xF(y,c^{-1})}(ba)$$

$$= (Q_{xv}c)(ba).$$

From the above considerations and in virtue of definition 7 we conclude that Q_{xx} is a left and right subquasigroup of $(Q,\cdot)$, which completes the proof.

COROLLARY 1. *If* $F\colon X \times Q \longrightarrow X$ *is an almost transitive solution of the translation equation then the following conditions*

$$LQ_{yy} = \{Q_{xy}:\quad x \in F(X,Q)\} = \{Q_{xy}:\ x \in X\}, \tag{83}$$

$$RQ_{yy} = \{Q_{yx}:\quad x \in F(X,Q)\} = \{Q_{yx}:\ x \in X\}, \tag{84}$$

LQ_{yy} *is the left translative partition of* $(Q,\cdot)$, (85)

RQ_{yy} *is the right translative partition of* $(Q,\cdot)$. (86)

are satisfied for every $y \in X$.

Proof. Let $y \in X$ be arbitrary, fixed. In virtue of theorems 3, 5 and by the condition (79) of lemma 9 we immediately obtain (85) and (86). F is an almost transitive function, thus $F(x,f) = F(y,a)$ for some $a \in Q$, i.e. $Q_{xy} \neq \emptyset$. Hence and from (78) we get $\{Q_{xy}: x \in X\} \subset LQ_{yy}$. Moreover by (77) we have $LQ_{yy} \subset \{Q_{xy}: x \in F(X,Q)\} \subset \{Q_{xy}: x \in X\}$. Thus we have shown that condition (83) is satisfied. The proof of (84) is analogous.

If H is a subquasigroup of $(Q,\cdot)$ then it is obvious that

$$hH = H = Hh \qquad \textit{for } h \in H, \tag{87}$$

whence we conclude that $H \in LH$ and $H \in RH$. We will use this property in the proofs of the next two lemmas.

LEMMA 10. *If* H *is a left translative subquasigroup of* $(Q,\cdot)$ *then the function* $F\colon LH \times Q \longrightarrow LH$ *defined as follows*

$$F(aH,\ b) = (ba)H \qquad \textit{for} \qquad a,b \in Q \tag{88}$$

is a transitive solution of the translation equation and $Q_{HH} = H$.

Proof. Let $a,b,c \in Q$ be arbitrary and let $d \in Q$ be such that $da = b$. In virtue of theorem 3 and lemmas 3, 5 we get

$$F(F(cH,b),a) = F((bc)H,a) = [a(bc)]H = [(ab)c]H = F(cH,\ ab),$$

$$F(aH,d) = (da)H = bH,$$

thus F is a transitive solution of the translation equation.
H is a subquasigroup of $(Q,\cdot)$, thus

$$aH = H \Longleftrightarrow a \in H \qquad \textit{for} \qquad a \in Q. \tag{89}$$

By (36), (87), (32) we get

$$F(H,a) = F(fH,\ a) = (af)H = a(fH) = aH$$

for $a \in Q$, $f \in U$, whence $F(H,f) = fH = H$ for $f \in U$. Thus for $a \in Q$ the equality $F(H,a) = F(H,f)$ holds if and only if $aH = H$. Hence, by (89) and (76) we get $Q_{HH} = H$, which completes the proof.

LEMMA 11. *If* H *is a right translative subquasigroup of* $(Q,\cdot)$ *then the function* $F\colon RH \times Q \longrightarrow RH$ *defined as follows*

$$F(Ha,\ b) := H(ab^{-1}) \qquad \textit{for} \qquad a,b \in Q \tag{90}$$

is a transitive solution of the translation equation and $Q_{HH} = H$.

Proof. In virtue of theorem 5 the family RH is the right translative partition of $(Q,\cdot)$. By (57) and (87) we immediately obtain

$$(Ha)b = H(ab) \qquad \textit{for} \qquad a,b \in Q. \tag{91}$$

Let $a,b,c \in Q$ be arbitrary and let $d \in Q$ be such that $ad = b$. Using (90), (57), (59), (91) we have

$$F(F(Hc,b)a) = F(H(cb^{-1}),a) = H[(cb^{-1})a^{-1}] = (Hc)(b^{-1}a^{-1})$$

$$= (Hc)(ab)^{-1} = H[c(ab)^{-1}] = F(Hc,\ ab).$$

By (90), (91), (60), (57) we obtain

$$F(Ha,d^{-1}) = H[a(d^{-1})^{-1}] = (Ha)(d^{-1})^{-1} = [(Ha)f_d](d^{-1})^{-1}$$

$$= [(Ha)(dd^{-1})](d^{-1})^{-1} = < H[(ad)d^{-1}]>(d^{-1})^{-1}$$

$$= [H(ad)][d^{-1}(d^{-1})^{-1}] = [H(ad)]f_{d^{-1}} = H(ad) = Hb.$$

Thus F is a transitive solution of the translation equation. H is a subquasigroup of $(Q, \cdot)$ thus

$$Ha = H \Longleftrightarrow a \in H \qquad \textit{for} \quad a \in Q \tag{92}$$

Using the equality $aa^{-1} = f_a$ and (61) we get

$$a \in H \Longleftrightarrow a^{-1} \in H \qquad \textit{for} \quad a \in Q \tag{93}$$

and consequently

$$f \in H \Longleftrightarrow f^{-1} \in H \qquad \textit{for} \quad f \in U. \tag{94}$$

Moreover by (92), (91), (61) we obtain

$$F(H,a) = F(Hf,a) = H(fa^{-1}) = Ha^{-1} \tag{95}$$

for $a \in Q$, $f \in U$. Hence and from (61), (94), 92) we get

$$F(H,f) = Hf^{-1} = H \qquad \textit{for} \quad f \in U. \tag{96}$$

Let $a \in Q$, $f \in U$ be arbitrary. Using (76), (95), (96), (92), (93) we obtain

$$a \in Q_{HH} \Longleftrightarrow F(H,a) = F(H,f) \Longleftrightarrow Ha^{-1} = H \Longleftrightarrow a^{-1} \in H \Longleftrightarrow a \in H,$$

whence $Q_{HH} = H$. This completes the proof.

By lemma 9 we know that if a function $F\colon X \times Q \longrightarrow X$ is the solution of the translation equation then every set Q_{xx} $(x \in X)$ is the left and right translative subquasigroup of $(Q,\cdot)$. Hence and from lemmas 10 and 12 we obtain the following.

THEOREM 7. *Let* $H \subset Q$. *Then* H *is a left translative subquasigroup of* $(Q,\cdot)$ *iff* H *is a right translative subquasigroup of* $(Q,\cdot)$.

In virtue of theorem 7 we accept the following.

DEFINITION 10. *A subquasigroup* H *of* $(Q,\cdot)$ *is called the translative subquasigroup of* $(Q,\cdot)$ *iff one of conditions* (5), (6) *is satisfied.*

THEOREM 8. *A function* $F\colon X \times Q \longrightarrow X$ *is an almost transitive solution of the translation equation iff there exist a translative subquasigroup* H *of* $(Q,\cdot)$ *and functions* $k\colon X \longrightarrow X$, $g\colon k(X) \longrightarrow LH$ such that the following conditions are satisfied

$$k(k(x)) = k(x) \qquad \textit{for} \quad x \in X, \tag{97}$$

g *is an one-to-one mapping of* $k(X)$ *onto* LH, (98)

$$F(x,a) = g^{-1}\big(ag(k(x))\big) \qquad \textit{for} \quad x \in X, \quad a \in Q. \tag{99}$$

Proof. Let H be a translative subquasigroup of $(Q,\cdot)$, let for functions k,g conditions (97), (98) be satisfied and let F be a function defined by (99). Then LH is a left translative partition of $(Q,\cdot)$ and by theorem 1 we conclude that F is a solution of the translation equation (it is easy to see that the function h from theorem 1 coincides with g^{-1}). Let $y \in F(X,Q)$, $x \in X$ be arbitrary and let $z \in X$, $a,b,c \in Q$ be such that

$$y = F(z,c), \qquad g(k(x)) = aH, \qquad g(k(z)) = bH.$$

Then by (99), (32), (38), (35) we have

$$y = F(z,c) = g^{-1}\big(cg(k(z))\big) = g^{-1}(c(bH)) = g^{-1}((cb)H)$$

$$= g^{-1}((cb)(e_a H)) = g^{-1}[(cb)\ (^{-1}aa)H]$$

$$= g^{-1} < [(cb)a^{-1}](aH) > = g^{-1}[<(cb)\ ^{-1}a > g(k(x))]$$

$$= F(x,(cb)\ ^{-1}a),$$

which means thet F is an almost transitive function. This completes the first part of the proof. Now assume that $F\colon X \times Q \longrightarrow X$ is an almost transitive solution of the translation equation and that $y \in X$, $f \in U$ are arbitrary, fixed. We put

$$H := Q_{yy},$$

$$k(x) := F(x,f) \qquad \textit{for} \quad x \in X,$$

$$g(x) := Q_{xy} \qquad \textit{for} \quad x \in k(X).$$

In virtue of lemma 9 and theorem 7 we conclude that H is a translative subquasigroup of $(Q,\cdot)$. From (67), (68) it follows that the equality $F(X,Q) = k(X)$ holds and that for the function k condition (97) is fulfilled. From (83) we obtain the equality $\{Q_{xy}\colon x \in X\} = LQ_{yy}$. Hence and by the definition of the function g it follows that g maps $k(X)$ onto LQ_{yy}. Let $x,z \in k(X)$ and let $g(x) = g(z)$. Then $Q_{xy} = Q_{zy}$ and for $a \in Q_{xy}$ we get $F(y,a) = F(x,f) = F(z,f)$. Hence and from (68) we obtain $x = z$, which means that g is an one-to-one function. Finally, by (77), (72) we get

$$g^{-1}\big(ag(k(x))\big) = g^{-1}\big(ag(F(x,f))\big) = g^{-1}(aQ_{F(x,f)y})$$

$$= g^{-1}(Q_{F(x,f)y}) = F(x,a)$$

for $x \in X$, $a \in Q$, which completes the proof.

COROLLARY 2. *A function* $F: X \times Q \longrightarrow X$ *is a transitive solution of the translation equation iff there exist a translative subquasigroup* H *of* $(Q, \cdot)$ *and a bijection* g of X *onto* LH *such that*

$$F(x,a) = g^{-1}(ag(x)) \qquad \textit{for} \qquad x \in X,\ a \in Q. \tag{100}$$

Proof. Assume that $F: X \times Q \longrightarrow X$ is a transitive solution of the translation equation. Then $F(X,Q) = X$ and by theorem 8 there exist a translative subquasigroup H of $(Q, \cdot)$, function $k: X \longrightarrow X$ and a bijection g of $k(X)$ onto LH such that F is of the form (99). Hence and from (68), (36) we get

$$x = F(x,f) = g^{-1}\left(fg(k(x))\right) = g^{-1}\left(g(k(x))\right) = k(x) \tag{101}$$

for $x \in X$. Hence we conclude that $k(X) = X$, and therefore g is a bijection of X onto LH and F is of the form (100). Now suppose that H is a translative subquasigroup of $(Q, \cdot)$, g is a bijection of X onto LH and that F is of the form (100). For $x \in X$, $f \in U$ we have

$$x = g^{-1}(g(x)) = g^{-1}(fg(x)) = F(x,f),$$

whence $X = F(X,Q)$. Thus, since in virtue of theorem 8 F is an almost transitive solution of the translation equation we conclude that F is a transitive function, which completes the proof.

REFERENCES

[1] Biełousow W.D. *Osnowy tieorii kwazigrupp i łup*, Izdatielstwo Nauka, Moskwa 1967.

[2] Midura S., Moszner Z., *Quelques remarques au sujet de la notion de l'objet et de l'objet geometrique*, Ann. Pol. Math. XVIII, (1966), 323-338.

[3] Moszner Z., *Solution generale de l'equation de translation et ses applications*, Aeq. Math. t.1,3, (1968), 291-293.