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Quasi–isometries

Dedicated to Professor Zenon Moszner with best with best wishes on his 60-th birthday

1. Preliminaries

Let (X, ρ_1), (Y, ρ_2) be metric spaces and let $\varepsilon \geq 0$. D.H. Hyers and S.M. Ulam [3] consider the following inequality[1)]

$$\left|\rho_2\left(f(p),f(q)\right) - \rho_1(p,q)\right| \leq \varepsilon \qquad \text{for} \quad p,q \in X, \tag{1}$$

where $f\colon X \longrightarrow Y$. A mapping $f\colon X \longrightarrow Y$ satisfying this condition is called an ε -isometry [3]. Having in mind possible applications it seems interesting to consider specified below modification of the above inequality. (1) means that replacing distance between p and q by the distance of their images we are making small absolute errors. Instead of (1) we shall assume that $f\colon X \longrightarrow Y$ is such a

1) [3] dealt with the Hilbert spaces only.

mapping that replacing distance between p and q by the distance of their images or conversely we are making small relative errors. More precisely, for a given $0 \leq \varepsilon < 1$, we shall be considering a conjunction of the following conditions:

$$|\rho_2\big(f(p),\ f(q) - \rho_1(p,q)| \leq \varepsilon\rho_1(p,q)\big) \tag{2}$$

for $p,q \in X$

and

$$|\rho_2\big(f(p),f(q) - \rho_1(p,q)| \leq \varepsilon\rho_2\big(f(p),f(q)\big) \tag{3}$$

for $p,q \in X$.

This conjunction can be written as

$$|\rho_2\big(f(p),f(q) - \rho_1(p,q)| \tag{4}$$

$$\leq \varepsilon \min\Big\{\rho_1(p,q),\ \rho_2\big(f(p),f(q)\big)\Big\} \qquad \textit{for } p,q \in X.$$

PROPOSITION 1. *Conditions* (2), (3), (4) *are equivalent to the following ones, respectively*

$$(1-\varepsilon)\rho_1(p,q) \leq \rho_2\big(f(p),f(q)\big) \leq (1+\varepsilon)\ \rho_1(p,q) \tag{5}$$

for $p,q \in X$;

$$\frac{1}{1+\varepsilon}\ \rho_1(p,q) \leq \rho_2\big(f(p),f(q)\big) \leq \frac{1}{1-\varepsilon}\ \rho_1(p,q) \tag{6}$$

for $p,q \in X$;

$$\frac{1}{1+\varepsilon}\ \rho_1(p,q) \leq \rho_2\big(f(p),f(g)\big) \leq (1+\varepsilon)\ \rho_1(p,q) \tag{7}$$

for $p,q \in X$.

Proof. Equivalence of (2) and (5) is evident. If (3) holds then

$$(1-\varepsilon)\rho_2\big(f(p),f(q)\big) \le \rho_1(p,q)$$

and

$$\rho_1(p,q) \le (1+\varepsilon)\ \rho_2\big(f(p),f(q)\big)$$

whence we get

$$\frac{1}{1+\varepsilon}\ \rho_1(p,q) \le \rho_2\big(f(p),f(q)\big) \le \frac{1}{1-\varepsilon}\ \rho_1(p,q).$$

Conversely if (6) holds then

$$- \varepsilon\rho_2\big(f(p),f(q)\big) \le \rho_2\big(f(p),f(q)\big) - \rho_1(p,q)$$
$$\le \varepsilon\rho_2\big(f(p),f(q)\big)$$

i.e. (3) is valid.

Condition (4) is equivalent to the conjunction of (2) and (3) hence to the conjunction of (5) and (6).

Since $1 - \varepsilon \le \frac{1}{1+\varepsilon}$ and $1 + \varepsilon \le \frac{1}{1-\varepsilon}$ (4) is equivalent to (7). Condition (7) implies directly that f is Lipschitzian and injective.

PROPOSITION 2. *Condition* (7) *is equivalent to the following one*

$$\frac{1}{1+\varepsilon}\ \rho\ (p,q) \le \rho\ \big(f^{-1}(p),f^{-1}(q)\big) \le (1+\varepsilon)\ \rho\ (p,q) \qquad (8)$$

for $p,q \in f(X)$.

Proof. Assume (7) and consider $p,q \in f(X)$. Then we have

$$\frac{1}{1+\varepsilon} \ \rho_1\left(f^{-2}(p),f^{-1}(q)\right) \le \rho_2(\rho,q) \le (1+\varepsilon) \ \rho_1\left(f^{-1}(p),f^{-1}(q)\right),$$

whence (8), follows directly, Converse implication can be proved similarly.

2. Quasi-isometries in normed spaces

From now on we assume that X and Y are normed spaces.

THEOREM 1. *Let* $f\colon X \longrightarrow Y$ *be a differentiable solution of* (4). *Then*

$$\frac{1}{1+\varepsilon} \| h \| \le \| f'(x)h \| \le (1+\varepsilon)\| h \| \tag{9}$$

for $x,h \in X$.

Proof. Fix an $x \in X$. From (7) we get

$$\frac{1}{1+\varepsilon} \| h \| \le \| f(x+h)-f(x) \| \le (1 + \varepsilon)\| h \| \tag{10}$$

for $h \in X$.

From differentiability of f we have

$$f(x+h)-f(x) = f'(x)h + r(x,h) \tag{11}$$

for $h \in X$,

where

$$\lim_{h\to 0} \frac{\| r(x,h) \|}{\| h \|} = 0. \tag{12}$$

Making use of (10) and (11) we obtain

$$\frac{1}{1+\varepsilon} \| h \| \le \| f'(x)h+r(x,h) \| \le (1+\varepsilon)\| h \|$$

$$\text{for } h \in X$$

and so

$$\frac{1}{1+\varepsilon} \le \| f'(x) \frac{h}{\| h \|} + \frac{r(x,h)}{\| h \|} \| \le 1 + \epsilon$$

$$\text{for } h \in X,\ h \ne 0.$$

Letting $h \longrightarrow 0$ and applying (12) we get

$$\frac{1}{1+\varepsilon} \le \| f'\ (x)u \| \le 1 + \varepsilon \text{ for } u \varepsilon X,\ \| u \| = 1$$

whence (9) follows immediately.

THEOREM 2. *Let* X, Y *be algebraically isomorphic normed spaces, let* f *be differentiable bijection satisfying* (9) *such that* $f'(x)$ *is invertible for* a ll $x \in X$, *and* f^{-1} *is continuous. Then* f *satisfies* (4).

Proof. By the mean-value theorem [1] we have

$$\| f(x) - f(y) \| \le (1 + \epsilon) \| x-y \|. \tag{13}$$

Under our assumptions f^{-1} is differentiable and [1]

$$(f^{-1})'\big(f'(x)\big) = \big(f'(x)\big)^{-1} \text{ for } x \in X. \tag{14}$$

From the left hand side inequality of (9) we get

$$\|(f'(x))^{-1}\| \le 1 + \varepsilon \qquad \textit{for } x \in X$$

and hence in view of (14)

$$\|(f^{-1})'(f(x))\| \le 1 + \varepsilon \qquad \textit{for } x \in X,$$

i.e. (since f is onto) $\|(f^{-1})'(y)\| \le 1 + \varepsilon$ *for* $y \in Y$.
Making again use of the mean-value theorem we obtain

$$\|f^{-1}(x') - f^{-1}(y')\| \le (1 + \varepsilon)\|x' - y'\| \qquad \textit{for } x',y' \in Y.$$

Putting $x' = f(x)$, $y' = f(y)$ we get

$$\|x - y\| \le (1 + \varepsilon)\|f(x) - f(y)\| \qquad \textit{for } x,y \in X$$

i.e.

$$\|f(x) - f(y)\| \le \frac{1}{1+\varepsilon}\|x - y\| \qquad \textit{for } x,y \in X.$$

Joining this inequality and (13) we have

$$\frac{1}{1+\varepsilon}\|x - y\| \le \|f(x) - f(y)\| \le (1+\varepsilon)\|x - y\| \qquad \textit{for } x,y \in X$$ [2].

Theorem 2 gives sufficient condition for condition (4). For C^1 mappings of Banach spaces we can weaken its assumptions using the following Theorem of J.T. Schwartz [4].

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[2] cf. Proposition 1.

THEOREM 3. *Let* X *and* Y *be Banach spaces and* $f\colon X \longrightarrow Y$ *a* C^1 *mapping and suppose* $f'(x)$ *is invertible at every* $x \in X$, *and moreover that* $\| (f'(x))^{-1} \| \le K < \infty$ *uniformly in* X. *Then* $f\colon X \longrightarrow Y$ *is a homeomorphism of* X *into* Y.

Proof. [4] p. 16.

COROLLARY 1. *If* X *and* Y *are finitedimensional and isomorphic then for* C^1 *- mappings conditions* (9) *and* (4) *are equivalent.*

Proof. From Theorem 1 we have (4) $\Longrightarrow$ (9).
If (9) holds, since X and Y are finite dimensional, from the left hand side inequality follows that $f'(x)$ is invertible. By Theorem 3 $f\colon X \longrightarrow Y$ is a homeomorphism. Finally Theorem 2 gives condition (4).

REMARK 1. For the Hilbert space l^2 of 2-summable sequences the shift operator

$$S : (\xi_0, \xi_1, \dots) \qquad\qquad (0, \xi_0, \xi_1, \dots)$$

gives an example that condition (9) is not sufficient for the invertibility of differential.

3. Quasi-isometries in Euclidean space

We consider now the case $X = Y = R^n$, where $\| \cdot \|$ is given by $\| (x_1,\dots,x_n)\| = (x_1^2 + \dots + x_n^2)^{\frac{1}{2}}$. We begin with $n = 1$.

REMARK 2. *A function* $f\colon R \longrightarrow R$ *satisfies* (4) *if and only if either*

$$\frac{1}{1+\varepsilon} \le \frac{f(x) - f(y)}{x - y} \le 1+\varepsilon \qquad \textit{for } x,y \in R,\ x \ne y \tag{15}$$

or

$$\frac{1}{1+\varepsilon} \le - \frac{f(x) - f(y)}{x - y} \le 1 + \varepsilon \qquad \textit{for } x,y \in R,\ x \neq y. \tag{16}$$

Proof. From (15) and (16) we get immediately

$$\frac{1}{1+\varepsilon} |x - y| \le | f(x) - f(y)| \le (1 + \varepsilon)|x - y| \tag{17}$$

$$\textit{for } x,y \in R.$$

Conversely it follows from (17) that f is continuous and injective and hence strictly monotonic.

Monotonicity of f and (17) implies (15) or (16).

REMARK 3.

Let now $X = Y = R^n$. Write $f\colon R^n \longrightarrow R^n$ in the form $f = (f_1,\dots,f_n)$ where $f_i\colon R^n \longrightarrow R$. Assume that f is C^1, then

$$\| f'(x)h \|^{\frac{1}{2}} = \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(x)h_j \right)^2 \right]^{\frac{1}{2}}$$

$$\textit{for } h = (h_1,\dots,h_n).$$

We have

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} h_j \right)^2 = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} h_j \right) \left(\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} h_j \right)$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} h_j \right) \left(\sum_{k=1}^{n} \frac{\partial f_i}{\partial x_k} h_k \right)$$

$$= \sum_{i=1}^{n} \left(\sum_{j,k=1}^{n} \frac{\partial f_i}{\partial x_j} \cdot \frac{\partial f_i}{\partial x_k} h_j h_k \right)$$

$$= \sum_{j,k=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j} \cdot \frac{\partial f_i}{\partial x_k} \right) h_j h_k .$$

Thus $\| f'(x)h \|^2 = \sum_{j,k}^{n} \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j}(x) \cdot \frac{\partial f_i}{\partial x_k}(x) \right) h_j h_k$ and consequently (9) becomes

$$\frac{1}{(1+\varepsilon)^2} \left(\sum_{i=1}^{n} h_i^2 \right) \le \sum_{j,k=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j} \cdot \frac{\partial f_i}{\partial x_k} \right) h_j h_k \tag{18}$$

$$\le (1+\varepsilon)^2 \left(\sum_{i=1}^{k} h_i^2 \right) \qquad \textit{for } h = (h_1, \ldots, h_n).$$

Denoting by I the Jacobi matrix of f (18) can be in matrix form, written as

$$\frac{1}{(1+\varepsilon)^2} h\, h^t \le (I^t h)(I h) \le (1+\varepsilon)^2 h\, h^t$$

where $(\quad)^t$ denotes matrix transposition.

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