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The stability of the functional equation $f_o(x) + h(y) + \sum_{i=1}^{n+1} f_i(\varphi_i(x) + \psi_i(y)) = 0$

Dedicated to Professor Zenon Moszner with best wishes on his 60-th birthday

L. Szekelyhidi has proved in [6] the stability of the functional equation

$$f(x) + \sum_{i=1}^{n+1} c_i \, f(\varphi_i(x) + \psi_i(y)) = 0,$$

where f maps an Abelian group $(G, +)$ into the field $\mathbb{C}$ of all complex numbers and $\varphi_i, \psi_i, : G \longrightarrow G$ are homomorphisms for $i = 1, 2, \ldots, n + 1$.

Let Q be a field of rational numbers. Let H be a linear space over Q.

In the present paper we are going to prove the stability of the functional equation

$$f_o(x) + h(y) + \sum_{i=1}^{n+1} f_i(\varphi_i(x) + \psi_i(y)) = 0,$$

where φ_i, ψ_i are isomorphisms of H onto itself

and

$$f_i : G \longrightarrow U, \qquad i = 0, 1, \ldots, n + 1,$$

$$h : G \longrightarrow U$$

are unkown functions and $(U, \|\cdot\|)$ is a Banach space.

Theorem below may, in particular, be applied to the following functional equations:

$$f_o(x) + h(y) = f_1(x + y) \qquad \text{(of Pexider)},$$

$$f_o(x) + h(y) = f\left(\frac{x + y}{2}\right) \qquad \text{(of Jensen-Pexider)}$$

$$f_o(x) + h(y) = f_1(x + y) + f_2(x - y)$$

and

$$f_o(x) + h(y) = f(a\,x + b\,y + c) \qquad \text{(of quadratic functionals "pexiderized")}$$

Therefore, among others, we obtain a joint generalization of several ealier stability results. Let us mention here some of them. The stability of the equation of quadratic functionals:

$$2f(x) + 2f(y) = f(x + y) + f(x - y)$$

has been proved by P. Cholewa in [2].
The stability of the Pexider functional equation

$$f_o(x) + h(y) = f_1(x + y)$$

was established by K. Nikodem in [5], whereas D.H. Hyers [3] showed the stability of the Jensen functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}.$$

For any function f mapping a group (G, +) into a certain linear space we put;

$$\Delta_t^1 f(x) := f(x+t) - f(x), \quad x, t \in G,$$

and

$$\Delta_t^n f(x) := \Delta_t^{n-1} f(x+t) - \Delta_t^{n-1} f(x), \quad x,t \in G.$$

DEFINITION 1. *Let* (G,+) *be a group and let* (U, ‖·‖) *be a normed linear space. A function* f: G ⟶ U *is called polynomial of degree at most* n *iff*

$$\Delta_t^{n+1} f(x) = 0 \qquad \textit{for all } x, t \in G.$$

In the whole of the paper we assume that ε is a given nonnegative real number.

DEFINITION 2. *Let* (G, +) *be an Abelian group, let* (U, ‖·‖) *be a real normed linear space and let* n *be a nonnegative integer. A function* f: G ⟶ U *is called* ε *- approximately of degree at most* n *iff there exist functions* f_i: G ⟶ U, i = 1, ..., n+1, *a function* g: G x G ⟶ U *such that* ‖g (x, y)‖ ≤ ε *and homomorphisms* φ_i, ψ_i: G ⟶ G *such that*

$$\varphi_i(G) \subset \psi_i(G) \qquad \textit{for } i = 1, 2, \ldots\ldots, n+1$$

and the equation

$$g(x,y) + f(x) + \sum_{i=1}^{n+1} f_i(\varphi_i(x) + \psi_i(y) = 0$$

holds for all x, y e G.

DEFINITION 3. *Let* (G,+) *be an Abelian group, let* $(U,\|\cdot\|)$ *be a normed linear space, and let n be a nonnegative integer. A function* $f\colon G \longrightarrow U$ *is called* ε *- approximately polynomial of degree at most* n *iff there exist a function* $g : G \times G \longrightarrow U$ *such that* $\|g(x,y)\| \le \varepsilon$ *and the equation*

$$g(x,y) + \Delta_y^{n+1} f(x) = 0$$

holds for all $x, y \in G$.

LEMMA 1. *Let* (G, +) *be a linear space over* Q, *let* $(U, \|\cdot\|)$ *be a normed linear space, and let* n *be a nonnegative integer. If a function* $f\colon G \longrightarrow U$ *is* ε *- approximately of degree at most* n, *then* f *is* $2^{n+1}\varepsilon$ *- approximately polynomial of degree at most* n. *Conversely, if a function* $f\colon G \longrightarrow U$ *is* ε *- approximately polynomial of degree at most* n, *then* f *is* ε *- approximately of degree at most* n.

Proof. Let f be ε - approximately polynomial of degree at most n. Hence

$$g(x, y) + \Delta_y^{n+1} f(x) = g(x,y)$$

$$+ \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} f(x + (n+1-i)y) = 0$$

and

$$g(x,y) + \sum_{i=0}^{n} (-1)^{i} \binom{n+1}{i} f(x + (n+1-i)y)$$

$$+ (-1)^{n+1} f(x) = 0,$$

whence

$$f(x) + \sum_{i=0}^{n} (-1)^{n+1+i} \binom{n+1}{i} f(x +(n+1-i)y)$$

$$+ (-1)^{n+1} g(x, y) = 0.$$

Conversely, let f be ε - approximately of degree at most n. We shall first show that for any ε - approximately of degree at most n function f, the function $\Delta_t f$ is 2ε - approximately of degree at most n - 1. Since φ_i and ψ_i are homomorphisms and $Rg\ \varphi_i \subset Rg\psi_i$, for every $t \in G$ we can find an $s \in G$ such that

$$\varphi_{n+1}(t) + \psi_{n+1}(s) = 0.$$

Setting x + t instead of x and y + s instead of y we obtain that

$$g(x + t, y + s) + f(x + t)$$

$$+ \sum_{i=1}^{n+1} f_i(\varphi_i(x+t) + \psi_i(y+s)) = 0,$$

whence

$$f(x+t) + \sum_{l=1}^{n} f_l(\varphi_l(x) + \psi_l(y) + \varphi_l(t) + \psi_l(s))$$

$$+ f_{n+1}(\varphi_{n+1}(x) + \psi_{n+1}(y)) + g(x+t,\ y+s) = 0.$$

We add and subtract here the term $g(x,y) + \sum_{l=1}^{n} f_l(\varphi_l(x) + \psi_l(y))$ getting

$$f(x+t) + \sum_{l=1}^{n} [f_l(\varphi_l(x) + \psi_l(y) + \varphi_l(t) + \psi_l(s))$$

$$- f_l(\varphi_l(x) + \psi_l(y))] + g(x+t,\ y+s) - g(x,y)$$

$$+ \sum_{l=1}^{n+1} f_l\ (\varphi_l(x) + \psi_l(y)) + g(x+y) = 0.$$

Now, it follows that the function $\Delta_t f(x)$ is 2ε - approximately of degree at most n-1; in fact,

$$\Delta_t\ f(x) + g(x+t,\ y+s) - g(x,y)$$

$$+ \sum_{l=1}^{n} [f(\varphi_l(x) + \psi_l(y)$$

$$+ \varphi_l(t) + \psi_l(s)) - f_l(\varphi_l(x) + \psi_l(y))] = 0$$

for all $x,\ t,\ y \in G$.

By induction, after n+1 steps, we get

$$\|\Delta_t^{n+1} f(x)\| \le 2^{n+1}\varepsilon,$$

which completes the proof.

LEMMA 2. *Let* (G,+) *be a linear space over* Q, *let* $(U, \|\cdot\|)$ *be a normed linear space and let* n *be a nonnegative integer. Let further*

$$\varphi_i, \psi_i : G \longrightarrow G, \ i = 1, \dots, n+1,$$

be isomorphisms such that

$$Rg\left(\psi_j - \varphi_j \circ \varphi_i^{-1} \circ \psi_i\right) = G$$

for all $j \neq i$; $i, j = 1, \dots, n+1$.
If functions $f_i : G \longrightarrow U$, $i = 0,1, \dots n+1$ *satisfy the inequality*

$$\|f_o(x) + \sum_{i=1}^{n+1} f_i(\varphi_i(x) + \psi_i(y))\| \le \varepsilon \tag{1}$$

for every x, y, *then*

$$\|\Delta_t^{n+1} f_i(x)\| \le 2^{n+1}\varepsilon$$

for all $x, t \in G$ *and* $i = 0,1, \dots, n+1$.

Proof. Concerning f_o we use Lemma 1. To prove the assertion for the remaining $f_i (i = 1,2,\dots n+1)$ we set $x = \varphi_i^{-1}(u) - \varphi_i^{-1}\circ\psi_i(y)$ in (1).
Then we have

$$\|f_i(u) + f_o(\varphi_i^{-1}(u) - \varphi_i^{-1} \circ\psi_i(y))$$

$$+\sum_{\substack{j=1\\ j\neq i}}^{n+1} f_j\left[\varphi_i\, o\varphi_i^{-1}(u) + \left(\psi_j - \varphi_j\, o\varphi_i^{-1}\, o\psi_i\right)(y)\right]\| \le \varepsilon$$

which means that f_i is ε - approximately of degree at most n. Hence f_i is 2^{n+1} ε polynomial of degree at most n, $i = 1, 2, \ldots, n+1$, and the proof is completed.

LEMMA 3. *Let* $(G, +)$ *be a linear space over* Q, *let* $(U, \|\cdot\|)$ *be a normed linear space and let* n *be a nonnegative integer. Let* φ_i, $\psi_i : G \longrightarrow G$, $i=1, \ldots, n+1$, be automorphisms such that

$$Rg\ (\psi_j - \varphi_j\, o\varphi_i^{-1}\, o\psi_i) = G,$$

$$i \neq j;\ i,\ j = 1, \ldots, n+1;$$

assume that there exists a $c \in Q$ *such that* $c\varphi_i + \psi_i$ *is an automorphism of* G *for* $i = 1, \ldots n+1$.
If functions $h, f_i : G \longrightarrow U$ $(i = 0,1, \ldots n+1)$ *satisfy the inequality*

$$\|h(y) + f_o(x) + \sum_{i=1}^{n+1} f_i(\varphi_i(x) + \psi_i(y))\| \le \varepsilon \qquad (2)$$

for every $x, y \in G$, *then*

$$\|\Delta_t^{n+1} f_i(x)\| \le 2^{n+2}\varepsilon,\ i = 0,1, \ldots n+1,$$

and

$$\|\Delta_t^{n+1} h(x)\| \le 2^{n+2}\varepsilon.$$

Proof. Let $c \in Q$ and $c\varphi_i + \psi_i$ be isomorphisms of G for $i = 1,2, \dots, n+1$. Substituting $u+cy$ instead of x in (2), we obtain the following inequality

$$\|h(y) + f_o(u + cy) + \sum_{i=1}^{n+1} f_i(\varphi_i(u) + (c\varphi_i + \psi_i)(y))\| \le \varepsilon.$$

We shall show, that the assumptions of Lemma 2 are satisfied in this case. In fact, for $i \neq j$ and $i,j = 1,\dots, n+1$ we have

$$Rg\left[c\varphi_j + \psi_j - \varphi_j \circ \varphi_i^{-1} \circ (c\varphi_i + \psi_i)\right]$$

$$= Rg\left[\psi_j - \varphi_j \circ \varphi_i^{-1} \circ \psi_i + c\varphi_j - \varphi_j \circ \varphi_i^{-1} \circ (c\varphi_i)\right]$$

$$= Rg(\psi_j - \varphi_j \circ \varphi_i^{-1} \circ \psi_i) = G,$$

whereas for $j = 0$ and $i = 1, \dots, n+1$ we have

$$Rg(\varphi_i c + \psi_i - \varphi_i \circ c\, .id) = Rg\psi_i = G.$$

COROLLARY 1. *Let* $(G,+)$ *be a linear space over* Q, *let* $(U, \|\cdot\|)$ *be a normed linear space, and let* n *be a nonnegative integer. Let* $p_i, q_i \in Q - \{0\}$, $i = 1, \dots, n+1$, *be such that* $p_i \cdot q_j \neq p_j \cdot q_i$ *for* $i \neq j$ *and* $i,j = 1, \dots, n+1$.
If functions $h, f_i : G \longrightarrow U$; $i = 1, \dots, n+1$ *satisfy the inequality*

$$\|h(y) + f_o(x) + \sum_{i=1}^{n+1} f_i(p_i x + q_i y)\| \le \varepsilon$$

for every $x, y \in G$ *then*

$$\|\Delta_t^{n+1} f_i(x)\| \le 2^{n+2}\varepsilon$$

for $i = 0,1, \ldots, n+1$ *and*

$$\|\Delta_t^{n+1} h(x)\| \le 2^{n+2}\varepsilon.$$

THEOREM. *Let* $(G,+)$ *be a linear space over* Q, *let* $(U, \|\cdot\|)$ *be a Banach space and let* n *be a nonnegative integer.*
If functions $h, f_i : G \longrightarrow U$ $i = 0,1, \ldots. n+1$ *satisfy the inequality*

$$\|h(y) + f_o(x) + \sum_{i=1}^{n+1} f_i(\varphi_i(x) + \psi_i(y))\| \le \varepsilon$$

for every $x, y \in G$, *where* $\varphi_i, \psi_i : G \longrightarrow G$ *are automorhisms of* G; $i = 1, \ldots. n+1$, *such that*

$$\text{(i) } Rg(\psi_j - \varphi_j \circ \varphi_i^{-1} \circ \psi_i) = G$$

and

(ii) *there exists a* $c \in Q$ *such that* $c\,\varphi_j + \psi_j$ *are automorphisms of* G *for* $j = 1, \ldots, n+1$,
then there exist polynomial functions v, u_i; $i = 0, 1, \ldots. n+1$ *such that*

$$\|v - h\| \le \frac{\varepsilon}{n+3} + 2^{n+4} \cdot \varepsilon / \max_m \binom{n}{m}$$

and

$$\|f_i - u_i\| \le \frac{\varepsilon}{n+3} + 2^{n+4} \cdot \varepsilon/\max_m \binom{n}{m}$$

for $i = 0,1, \ldots, n+1$.

Moreover,

$$v(y) + u_o(x) + \sum_{i=1}^{n+1} u_i(\varphi_i(x) + \psi_i(y)) = 0,$$

for all $x,y \in G$.

Proof: From the paper [1] of M.A. Albert and J.A. Baker and from Lemma 3 we know that there exist polynomial functions w_o, w_1, ..., w_{n+2} such that

$$\|f_i(x) - w_i(x)\| \le 2^{n+3} \cdot \varepsilon/\max_m \binom{n}{m}$$

$i = 0, 1, \ldots\ n+1$,

and

$$\|h(x) - w_{n+2}(x)\| \le 2^{n+3} \cdot \varepsilon/\max_m \binom{n}{m}$$

Let $\mu := 2^{n+3} \cdot \varepsilon/\max_m \binom{n}{m}$. For all $x, y \in G$ we have

$$(n+3)\mu \ge \sum_{i=1}^{n+1} \| f_i(\varphi_i(x) + \psi_i(y)) - w_i(\varphi_i(x) + \psi_i(y)) \|$$

$$+ \| h(y) - w_{n+2}(y) \| + \| f_o(x) - w_o(x)\|$$

$$\ge \| \sum_{i=1}^{n+1} \Big[f_i\big(\varphi_i(x) + \psi_i(y)\big) - w_i(\varphi_i(x) + \psi_i(y))\Big]$$

$$+ f_o(x) - w_o(x) + h(y) - w_{n+2}(y)\|$$

$$= \| h(y) + f_o(x) + \sum_{i=1}^{n+1} f_i\big(\varphi_i(x) + \psi_i(y)\big)$$

$$- \sum_{i=1}^{n+1} w_i\big(\varphi_i(x) + \psi_i(y)\big) - w_o(x) - w_{n+2}(y)\|.$$

and

$$(n+3)\mu + \varepsilon \geq \| h(y) + f_o(x) + \sum_{i=1}^{n+1} f_i\big(\varphi_i(x) + \psi_i(y)\big) \tag{3}$$

$$- \sum_{i=1}^{n+1} w_i\big(\varphi_i(x) + \psi_i(y)\big) - w_o(x) - w_{n+2}(y)\|$$

$$+ \| - h(y) - f_o(x) - \sum_{i=1}^{n+1} f_i\big(\varphi_i(x) + \psi_i(y)\big)\|$$

$$\geq \| \sum_{i=1}^{n+1} w_i\big(\varphi_i(x) + \psi_i(y)\big) + w_o(x) + w_{n+2}(y)\big)\|.$$

Fix a $y_o \in G$. Then the function

$$G \ni x \longrightarrow w_{n+2}(y_o) + w_o(x) + \sum_{i=1}^{n+1} w_i\big(\varphi_i(x) + \psi_i(y_o)\big) \in U$$

is a polynomial function in x.

From (3) norm of this function is bounded. There exists an $F \in U$ such that

$$F = w_{n+2}(y_o) + w_o(x) + \sum_{i=1}^{n+1} w_i(\varphi_i(x) + \psi_i(y_o))$$

for every $x \in G$, because every polynomial bounded function is a constant function (see Kuczma [4, Chapter XV], for instance; M. Kuczma considers the case of polynomial functions from $\mathbb{R}^N$ into $\mathbb{R}$ only, but this restriction is inessential).

Now, fix an $x_o \in G$. Then the function

$$G \ni y \longrightarrow w_{n+2}(y) + w_o(x_o) + \sum_{l=1}^{n+1} w_l \left(\varphi_l(x_o) + \psi_l(y)\right) \in U$$

is a polynomial function in y. From (3) norm of this function is bounded. There exists $F_1 \in U$ such that

$$F_1 = w_{n+2}(y) + w_o(x_o) + \sum_{l=1}^{n+1} w_l\left(\varphi_l(x_o) + \psi_l(y)\right)$$

for every $y \in G$.

Both of the functions defined are equal at $x = x_o$ and $y = y_o$. Therefore we have $F = F_1$. Hence, for every $x, y \in G$, we get

$$w_o(x) + w_{n+2}(y) + \sum_{l=1}^{n+1} w_l \left(\varphi_l(x) + \psi_l(y)\right) = F.$$

Obviously,

$$\left(w_o(x) - \frac{F}{n+3}\right) + \left(w_{n+2}(y) - \frac{F}{n+3}\right)$$

$$+ \sum_{l=1}^{n+1} \left[w_l\left(\varphi_l(x) + \psi_l(y)\right) - \frac{F}{n+3}\right] = 0.$$

The polynomial functions

$$w'_i(x) = w_i(x) - \frac{F}{n+3}, \qquad i = 0, 1, ..., n+1,$$

satisfy the equation

$$w'_o(x) + w'_{n+2}(y) + \sum_{i=1}^{n+1} w'_i(\varphi_i(x) + \psi_i(y)) = 0, \qquad x, y \in G,$$

and since $\|f_i(x) - w_i(x)\| \le \mu$, $x \in G$, we obtain

$$\|f_i(x) - w'_i(x)\| = \|f_i(x) - w_i(x) + \frac{F}{n+3}\|$$

$$\le \|f_i(x) - w_i(x)\| + \|\frac{F}{n+3}\|$$

$$\le \mu + \frac{(n+3)\mu + \varepsilon}{n+3} = 2\mu + \frac{\varepsilon}{n+3}$$

Thus for all $x \in G$ the theorem has been proved.

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REFERENCES

[1] Albert M.A., Baker J.A., *Functions with bounded m-th differences*, Ann. Polon. Math. 43 (1983), 93-103.

[2] Cholewa P.W., *Remarks on the stability of functional equations*, Aequationes Math. 27 (1984), 76-86.

[3] Hyers D.K, *Transformations with bounded m-th differences*, Pacific J. Math. 11 (1961), 591-602.

[4] Kuczma M., *Introduction to the Theory of Functional Equations and Inequalitis,* Polish Scientific Publishers, PWN, Warszawa, 1985.

[5] Nikodem K., *The Stability of the Pexider equation, Annales Mathematicae Silesianae.* 5 (1991), 91-93.

[6] Szekelyhidi L., *The stability of linear functional equations,* C.R. Math. Rep. Acad. Sci. Canada. 3 (#2) (1981), 63-67.