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Convex functions and some set classes

Dedicated to Professor Zenon Moszner with best wishes on his 60-th birthday

Let X be a real linear space and let $D \subset X$ be a nonempty convex set. A function $f:D \longrightarrow [-\infty,\infty)$ is called J-convex iff the inequality

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x) + f(y)}{2} \tag{1}$$

holds for every $x,y \in D$. It is a well known reult that every J-convex function $f:D \longrightarrow [-\infty,\infty)$ satisfies the inequality

$$f\big(\lambda x + (1 - \lambda)y\big) \le \lambda\, f(x) + (1 - \lambda)f(y) \tag{2}$$

for all $x,y \in D$ and every rational $\lambda \in (0,1)$. A function $f:D \longrightarrow [-\infty,\infty)$ is convex iff (2) holds for all ($x,y \in D$ and) $\lambda \in (0,1)$. Let us assume that X is endowed with a topology $\mathcal{T}$ such that the function $\varphi : \mathbb{R} \times X \times X \longrightarrow X$ given by the formula $\varphi(\lambda,x,y) = \lambda\, x + y$ is separately continuous with respect to each variable. Such a topology will be called semilinear. Some basic properties of semilinear topologies may be found in [3], [5] and [6]. Note, that any linear topology in a real linear space is semilinear. We recall that a real linear space X endowed with a semilinear topology is a

Baire space whenever every nonempty open subset of X is of the second category.

Let $(X,\mathcal{T})$ be a real linear space endowed with a semilinear topology $\mathcal{T}$. In [3] (cf. also [4]) we introduced the following class of sets;

$\zeta(X) = \{T \subset X;$ if $D \subset X$ is a convex and open set such that $T \subset D$ and $f:D \longrightarrow [-\infty,\infty)$ is a J-convex function which is lower semicontinuous at every point of T, then f is continuous in D$\}$.

We agree that f is lower semicontinuous at every point $x \in D$ at which $f(x) = -\infty$, and if $f(x) = -\infty$, then f is continuous at x iff $f = -\infty$ in a neighbourhood of x. Also, following [3] (cf. also [1] and [4]), by $\mathcal{A}_C(X)$ we denote the family of all sets $T \subset X$ with the property that if $D \subset X$ is a convex open set such that $T \subset D$ and $f:D \longrightarrow [-\infty,\infty)$ is a J-convex function such that the restriction $f|_T$ is continuous, then f is continuous in D.

The main result concerning these classes of sets presented in [3] (if X is a real linear space endowed with a semilinear Baire topology) states that

$$\mathcal{A}_C(x) \subset \zeta(X). \tag{3}$$

In this paper we shall use some facts presented in [3], [4] and [6] and therefore we shall give their full form. First we recall the definition of the lower hull of a function f defined on a subset D of a topological space $(X, \mathcal{T})$ with values in $[-\infty,\infty)$;

$$m_f(x) = \sup_{U \in \mathcal{T}_x} \inf_{t \in U \cap D} f(t), \; x \in D,$$

where $\mathcal{T}_x$ denotes the family of all open subsets of X containing the point x.

LEMMA 1 ([3, Theorem 4.4], also [4, Lemma 5]). *Let* X *be a topological space, let* $D \subset X$ *be an open set and let* $f:D \longrightarrow [-\infty,\infty)$ *be a function. The function* f *is lower semicontinuous at a point* $x \in D$ *iff* $f(x) = m_f(x)$.

LEMMA 2. ([3, Theorem 4.2], also [6, Proposition 2]). *Let* X *be a real linear space endowed with a semilinear topology* $\mathcal{T}$, *let* $D \subset X$ *be an open and convex set, and let* $f:D \longrightarrow [-\infty,\infty)$ *be a J-convex function. The lower hull* m_f *of* f *is convex in* D. *If* $(X,\mathcal{T})$ *is a Baire space then* f *is continuous.*

Theorem 1 give some sufficient conditions for a set to belong to the class $\mathcal{A}_C(X)$.

THEOREM 1. *Let* X *be a real linear space endowed with a semilinear topology and let* $T \subset X$ *be a subset. If there exist a point* $z \in T$ *and a sequence* $(U_n,\ n \in \mathbb{N})$ ($\mathbb{N}$ *denotes the set of all positive integers) of neighbourhoods of* z *such that*

$$\bigcap_{n=1}^{\infty} U_n = \{z\} \text{ and } \frac{1}{2}\,[(T \cap U_n) + (T \cap U_n)] \supset U_n \tag{4}$$

for every positive integer n, *then* T *belongs to* $\mathcal{A}_C(X)$.

Proof. Let D be an open and convex subset of X and assume that $T \subset D$. Let $f:D \longrightarrow [-\infty,\infty)$ be an arbitrary J-convex function such that $f|_T$ is continuous. Hence and by (4) we get $f(t) \le M$ for all $t \in U_n \cap T$, where M is a constant, and n is a positive integer. It follows from the J-convexity of f and (4) that f is bounded above on a nonempty open set U_n. By a theorem of Bernstein-Doetsch ([3], [5]) f is continuous and, therefore, T belongs to the class $A_C(X)$.

COROLLARY 1. *Let* X *be a real linear space endowed with a semilinear topology. Every set* T *containing a second category subset wit the Baire property belongs to the class* $\mathcal{A}_C(X)$.

Proof. Let $S = (G \setminus P) \cup R$ be a subset of T such that G is a nonempty open set and P and R are of the first category. Let us fix a $z \in G \setminus P$ and take an arbitrary sequence of neighbourhoods $(U_n; n \in \mathbb{N})$ of z such $\bigcap_{n=1}^{\infty} U_n = \{z\}$ and $U_n \subset G_n$ for every positive integer n. As in Lemma 3 [5], we can prove the second part of (4). The rest of the proof follows from Theorem 1.

In the case of a real linear topological space, Corollary 1 may be found in [4]. It is still an open problem whether the inclusion in (3) is strict. However, we have the following.

THEOREM 2. *Let* X *be a real linear space endowed with a semilinear topology, let* D *be an open and convex subset of* X. *If* $T \subset D$ *is a J-convex subest (i.e.* $\frac{T+T}{2} \subset T$*) belonging to the class* $\zeta(X)$ *and* T *is dense in* D, *then* T *belongs to the class* $\mathcal{A}_c(X)$.

Proof. It is enough to show that every J-convex function $f: D \longrightarrow [-\infty, \infty)$ such that $f|_T$ is continuous, is continuous in D. So, let us consider such a function f. We define a function $F: D \longrightarrow [-\infty, \infty)$ by the formula

$$F(x) = \sup_{U \in \mathcal{T}_x,\, U \subset D} \inf_{t \in U \cap T} f(t), \quad x \in D. \tag{5}$$

It is easily seen that $F(x) \le f(x)$ for every $x \in T$. We shall show that

$$F(x) = f(x) \text{ for every } x \in T \tag{6}$$

Suppose that $F(x) < f(x)$ for an $x \in T$. It follows from (5) that there exists an $\varepsilon > 0$ with the property

$$\inf_{t \in U \cap T} f(t) < f(x) - \varepsilon$$

for all neighbourhoods $U \in \mathcal{T}_x$. By the density of T in D there exists a sequence $(t_n, n \in \mathbb{N}) \subset T$ convergent to x and such that

$$f(t_n) < f(x) - \varepsilon \tag{7}$$

for each positive integer n. Since $f|_T$ is continuous and $x \in T$, the inequality (7) is false. This proves (6).

Let us fix an arbitrary $x \in D$ and $\varepsilon > 0$. According to (5), there exists a neighbourhood $U \in \mathcal{T}_x$, $U \subset D$, such that

$$F(x) - \varepsilon \le \inf_{t \in U \cap T} f(t). \tag{8}$$

Let us take an arbitrary $u \in U$. Then $U \in \mathcal{T}_u$ and hence also (cf. (5)).

$$\inf_{t \in U \cap T} f(t) \le F(u). \tag{9}$$

Inequalities (8) and (9) imply that

$$F(x) - \varepsilon \le F(u) \text{ for every } u \in U,$$

which shows that F is lower semicontinuous at x. Due to arbitrariness of x F is lower semicontinuous at every point x of D.

Let x,y be arbitrary points of D. Since T is dense in D there exist sequences $(x_n, n \in \mathbb{N})$ and $(y_n, n \in \mathbb{N})$ of elements of T convergent to x and y, respectively. We may assume (see (5)) that

$$f(x_n) \leq F(x) \text{ and } f(y_n) \leq F(y) \tag{10}$$

for any positive integer n. Evidently $z_n = \frac{x_n + y_n}{2}$ is convergent to $\frac{x+y}{2}$ and, moreover, by our assumption on T, z_n belongs to the set T. By the lower semicontinuity of F at every point of D, given $\varepsilon > 0$, there exist a positive integer m such that

$$F\left(\frac{x+y}{2}\right) - \varepsilon \leq F\left(\frac{x_n + y_n}{2}\right) = F(z_n) \tag{11}$$

for each positive integer $n \geq m$. On account of (6), by the J-convexity of f and by (10) we get

$$F(z_n) = f(z_n) = f\left(\frac{x_n + y_n}{2}\right) \leq \frac{f(x_n) + f(y_n)}{2} \tag{12}$$

$$\leq \frac{F(x) + F(y)}{2}.$$

Inequalities (11) and (12) imply

$$F\left(\frac{x+y}{2}\right) - \varepsilon \leq \frac{F(x) + F(y)}{2},$$

and consequently F is J-convex function in D. Since F is lower semicontinuous at every point of T and $T \in \zeta(X)$, F is continuous in D.

Let us put

$$g(x) = \max \left(f(x), F(x)\right), \quad x \in D. \tag{13}$$

Of course, $g:D \longrightarrow [-\infty,\infty)$ is a J-convex function. Moreover, by the definitions (13) and of the lower hull

$$m_g(x) \geq m_F(x) \text{ for each } x \in D.$$

On account of Lemma 1, $m_F(x) = F(x)$ for every $x \in D$. Hence

$$m_g(x) \geq F(x), \ x \in D.$$

Suppose that for some $x \in D$ we have

$$m_g(x) > F(x).$$

In virtue of the lower semicontinuity of m_g (cf. Lemma 2) and of the continuity of F in D we infer that there exist an $\varepsilon > 0$ and a neighbourhood $U \in \mathcal{T}_x$ such that

$$F(u) + \varepsilon < m_g(x) - \varepsilon \leq m_g(u)$$

for every $u \in U$. Thus, for every $u \in U$, we have

$$F(u) + \varepsilon < g(u), \tag{14}$$

in virtue of the inequality $m_g \leq g$ (cf. the definition of the lower hull). By (13) and (14) we obtain

$$g(u) = f(u) \text{ for every } u \in U.$$

Thus

$$f(u) > F(u) + \varepsilon, \; u \in U. \tag{15}$$

It follows from the continuity of F at x that there exists a neighbourhood $V \in \mathcal{T}_x$, $V \subset U$, such that

$$F(x) < F(v) + \frac{\varepsilon}{2}$$

for all $v \in V$, which together with (15) implies

$$f(v) > F(x) + \frac{\varepsilon}{2}, \; v \in V.$$

The last inequality is false (see (5)) and, therefore,

$$m_\varepsilon(x) = F(x) \text{ for every } x \in D. \tag{16}$$

Relations (6), (13) and (16) imply that

$$m_\varepsilon(x) = g(x) \text{ for every } x \in T.$$

According to Lemma 1 g is lower semicontinuous at every point of T. Since $T \in \zeta(X)$, g is continuous in D. Hence g is bounded above on a nonempty open subset W of D and since $f \le g$, f is bounded above on W. Thus f is continuous, in virtue of a theorem of Bernstein-Doetsch ([5], [6]). This finishes the proof of Theorem 2.

As we see, the sets of the class $\mathcal{A}_c(X)$ cannot be "too small". In [2] we have proved that there exists a subset T of real line $\mathbb{R}$ which does not belong to $\mathcal{A}_c(\mathbb{R})$ and such that $T + T = \mathbb{R}$. Now we shall show that there exists a "large" family of sets T which does not belong to $\zeta(\mathbb{R})$ (and hence also to $\mathcal{A}_c(\mathbb{R})$) and such that $T + T = \mathbb{R}$.

Let $\mathcal{F}$ be the family of all non-monotonic convex functions defined on the space $\mathbb{R}$. Take a discontinuous additive function $a:\mathbb{R} \longrightarrow \mathbb{R}$ such that $a(Q) = \{0\}$ (Q denotes here the set of all rationals). For an arbitrary $f \in \mathcal{F}$ we define

$$T = \{x \in \mathbb{R};\ a(x) \le f(x)\}. \tag{17}$$

First we shall show that $T + T = \mathbb{R}$. Consider an $x \in R$. There exists a rational r such that the points $x + r$ and $x - r$ belong to T (since otherwise f must be monotonic). Therefore

$x = \frac{x+r+x-r}{2} \in (T + T)\,\frac{1}{2}$. Hence $T + T = \mathbb{R}$

Let us put

$$F(x) = \max\big(a(x),\ f(x)\big),\ x \in \mathbb{R}. \tag{18}$$

Note that F is a discontinuous J-convex function. It is not hard to check that

$$m_F(x) = f(x),\ x \in \mathbb{R} \tag{19}$$

(cf. also Example 1 in [4]). If $x \in T$ then $a(x) \le f(x)$ and consequently $F(x) = f(x)$, $x \in T$. Hence and by (19)

$m_F(x) = F(x)$ for evry $x \in T$.

This means (by Lemma 1) that F is lower semicontinuous at every point of the set T. Since F is discontinuous T does not belong to $\zeta(\mathbb{R})$.

REMARK. If we put $f(x) = 0$ for all $x \in \mathbb{R}$ in (17) we get an example of a dense set T with the properties: $\frac{T + T}{2} \subset T$ and $T \notin \mathcal{A}_c(\mathbb{R})$ (the function F defined by (18) is J-convex and the restriction of $F_{|T}$ (= 0) is continuous). This shows that the assumption that $T \in \zeta(X)$ is essential in Theorem 2.

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