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On some functional equations with conic sections as solutions

Dedicated to Professor Zenon Moszner with best wishes on his 60-th birthday

Introduction.

N. Ciorǎnescu [3] considered the functional equations

$$\frac{f(x) - f(y)}{x - y} = \frac{1}{2}[f'(x) + f'(y)], \quad x \neq y, \tag{1}$$

and

$$\left[\frac{f(x) - f(y)}{x - y}\right]^2 = f'(x)\, f'(y), \quad x \neq y. \tag{2}$$

Relations (1) and (2) reflect the following property of the curve ℓ described by the equation $z = f(t)$: For arbitrary points $A \neq B$ on ℓ the slope of the chord AB is the arithmetic or geometric mean, respectively, of the slopes of the tangents to ℓ at A and at B. It turned out that property (1) is characteristic of parabolas with vertical axis, while (2) characterizes hyperbolas with the asymptotes parallel to the axes of the coordinate system.

J. Sándor [6] (cf. also [4]) generalized equations (1) and (2) to

$$\frac{f(x) - f(y)}{x - y} = \varphi(x) + \varphi(y), \ x \neq y, \tag{3}$$

and

$$\frac{f(x) - f(y)}{x - y} = \varphi(x)\ \varphi(y), \ x \neq y \tag{4}$$

and supplemented equations (3) and (4) by the third one

$$\frac{f(x) - f(y)}{x - y} = \frac{1}{\varphi(x) + \varphi(y)}, \ x \neq y \tag{5}$$

He solved equations (3) and (5) in the class of all functions f, $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ (without any regularity assumptions). However, not all solutions found by him are defined in the whole of $\mathbb{R}$.

Equations (3), (4), (5) have the general form

$$\frac{f(x) - f(y)}{x - y} = \eta[\varphi(x), \varphi(y)], \ x \neq y, \tag{6}$$

η being the sum, the product or the reciprocal of the sum of its arguments, respectively. (These equations are essentially equivalent to equations of similar form, but with η being the arithmetic, geometric or harmonic mean, respectively). Equation (6) has a counterpart in

$$\frac{f(x) - f(y)}{x - y} = \varphi[\eta(x,y)], \quad x \neq y \tag{7}$$

Equation (7), in turn, is closely related to

$$\frac{xf(y) - yf(x)}{x - y} = \varphi[\zeta(x,y)], \; x \neq y \tag{8}$$

Equations (7) and (8), with η and ζ being the arithmetic, geometric and harmonic mean, were solved in [2] in the class of functions f, $\varphi : I \longrightarrow \mathbb{R}$ mapping a proper real interval I into the reals. In [1] and [5] equations (7) and (8), with some η and ζ, where dealt with on (subsets of) commutative fields.

As equation (7) is a counterpart of (6), equation (8) is a counterpart of

$$\frac{xf(y) - yf(x)}{x - y} = \zeta[\varphi(x), \; \varphi(y)], \; x \neq y \tag{9}$$

In the present paper we consider equations (6) and (9) on subsets of $\mathbb{R}$ or of an arbitrary commutative field. Here η and ζ may be the sum, or the product, or reciprocal of the sum of its arguments. The general idea of the proofs is similar to that in [6] and [2], but the details are different.

In the sequel $\mathbb{F}$ denotes a commutative field, $A \subset \mathbb{F}$ is a subset of $\mathbb{F}$ containing at least two elements and

$A_0 = A \setminus \{0\}$.

1. The sum.

We consider equation (3) in the equivalent form

$$f(x) - f(y) = (x-y) \, [\varphi(x) + \varphi(y)], \quad x,y \in A, \tag{10}$$

which is valid also for $x = y$.

THEOREM 1. *The general solution* $f, \varphi : A \longrightarrow \mathbb{F}$ *of equation* (10) *is given by*

$$f(x) = \alpha x^2 + 2\beta x + \gamma, \quad \varphi(x) = \alpha x + \beta, \quad x \in A, \tag{11}$$

where α, β, γ *are arbitrary constants from* $\mathbb{F}$.

Proof. It is the matter of a straightforward verification to check that for arbitrary $\alpha, \beta, \gamma \in \mathbb{F}$ functions (11) actually satisfy equation (10). Conversely, suppose that functions $f, \varphi : A \longrightarrow \mathbb{F}$ satisfy equation (10). Fix a $\bar{y} \in A$ and write

$$b := f(\bar{y}), \quad c := \varphi(\bar{y}). \tag{12}$$

Putting $y = \bar{y}$ into (10) we obtain

$$f(x) = b + (x-\bar{y})\,[\varphi(x) + c], \quad x \in A, \tag{13}$$

which inserted back into (10) yields

$$(x-\bar{y})\,\big(\varphi(x) + c\big) - (y-\bar{y})\,\big(\varphi(y) + c\big)$$

$$= [(x-\bar{y}) - (y-\bar{y})]\,[\varphi(x) + \varphi(y)], \quad x,y \in A,$$

that is,

$$(y-\bar{y})\,[\varphi(x) - c] = (x-\bar{y})\,[\varphi(y) - c], \quad x,y \in A. \tag{14}$$

We fix again a $\bar{\bar{y}} \in A$, $\bar{\bar{y}} \neq \bar{y}$, we set $y = \bar{\bar{y}}$ into (14) and we divide (14) by $(x-\bar{y})(\bar{\bar{y}}-\bar{y})$ to get

$$\frac{\varphi(x) - c}{x - \bar{y}} = \frac{\varphi(\bar{\bar{y}}) - c}{\bar{\bar{y}} - \bar{y}} =: \alpha, \quad x \in A, \; x \neq \bar{y}.$$

Hence

$$\varphi(x) = c + \alpha\,(x-\bar{y}), \quad x \in A. \tag{15}$$

(For $x = \bar{y}$ formula (15) is valid in view of (12)). Relations (13) and (15) imply

$$f(x) = b + 2c\,(x-\bar{y}) + \alpha\,(x-\bar{y})^2, \quad x \in A \tag{16}$$

From (15) and (16) we obtain (11) with $\beta := c - \alpha\,\bar{y}$, $\gamma := b - 2c\bar{y} + \alpha\bar{y}^2$.

Now we pass to the related equation

$$xf(y) - yf(x) = (x-y)\,[\varphi(x) + \varphi(y)], \quad x,y \in A. \tag{17}$$

At first we consider the case where $0 \notin A$. Write $\hat{A} := A^{-1} = \{t = x^{-1} \mid x \in A\}$ and define functions $\hat{f}, \hat{\varphi} : \hat{A} \longrightarrow \mathbb{F}$ by

$$\hat{f}(t) = tf(1/t), \; \hat{\varphi}(t) = \varphi(1/t), \quad t \in \hat{A}. \tag{18}$$

For arbitrary $t,s \in \hat{A}$ equation (17) with $x = 1/t$, $y = 1/s$ goes over into

$$\hat{f}(t) - \hat{f}(s) = (t-s)\,[(\hat{\varphi}(t) + \hat{\varphi}(s)], \quad t,s \in \hat{A}. \tag{19}$$

By Theorem 1 the functions

$$\hat{f}(t) = \alpha t^2 + 2\beta t + \gamma, \quad \hat{\varphi}(t) = \alpha t + \beta, \quad t \in \hat{A},$$

yield the general solution of equation (19), whence it follows according to (18) that the functions

$$f(x) = \alpha x^{-1} + 2\beta + \gamma x, \quad \varphi(x) = \alpha x^{-1} + \beta, \quad x \in A, \tag{20}$$

(with arbitrary constants $\alpha, \beta, \gamma \in \mathbb{F}$) yield the general solution $f, \varphi : A \longrightarrow \mathbb{F}$ of equation (17).

Now suppose that $0 \in A$ and that the set A_0 contains at least two elements, say $u, v \in A_0$, $u \neq v$. If functions $f, \varphi : A \longrightarrow \mathbb{F}$ satisfy equation (17), then, by what we have already proved, there exist $\alpha, \beta, \gamma \in \mathbb{F}$ such that (20) holds for $x \in A_0$.Thus putting into (17) first $x = u$, $y = 0$, and next $x = v$, $y = 0$, we obtain

$$\alpha u^{-1} + \beta + \varphi(0) = f(0) = \alpha v^{-1} + \beta + \varphi(0), \tag{21}$$

whence $\alpha = 0$ and consequently

$$f(x) = \gamma x + 2\beta, \quad \varphi(v) = \beta, \quad x \in A_0. \tag{22}$$

Moreover, (21) with $\alpha = 0$ implies that $f(0) = \varphi(0) + \beta$, whence with $\delta := \varphi(0)$

$$f(0) = \beta + \delta, \quad \varphi(0) = \delta. \tag{23}$$

On the other hand, it is clear that the functions $f, \varphi : A \longrightarrow \mathbb{F}$ given by (22) and (23) (with arbitrary $\beta, \gamma, \delta \in \mathbb{F}$) actually satisfy equation (17). Thus we have obtained the following consequence of Theorem 1.

THEOREM 2. *Suppose that the set* A_0 *contains at least two elements. The general solution* $f, \varphi : A \longrightarrow \mathbb{F}$ *of equation* (17) *is given by* (20) *when* $0 \notin A$, *and by* (22), (23) *when* $0 \in A$, *where* α, β, γ, δ *are arbitrary constants from* $\mathbb{F}$.

2. The product.

In this section we are going to deal with the equation

$$f(x) - f(y) = (x-y)\,\varphi(x)\,\varphi(y), \quad x,y \in A, \tag{24}$$

which is an equivalent form of (4).

THEOREM 3. *The general solution* $f, \varphi : A \longrightarrow \mathbb{F}$ *of equation* (24) *is given by the formulas*

$$\begin{cases} f(x) = \lambda, \ x \in A, \\ \varphi(x) = 0, \ x \in A\setminus\{y_0\}, \ \varphi(y_0) = \mu \end{cases} \tag{25}$$

or

$$f(x) = \frac{\alpha x + \beta}{\gamma x + \delta}, \ \varphi(x) = \frac{c}{\gamma x + \delta}, \quad x\in A, \tag{26}$$

where y_0 *is an arbitrary point of* A, *whereas* $\alpha, \beta, \gamma, \delta, \lambda, \mu$ *are arbitrary constants from* $\mathbb{F}$ *such that* $-\delta/\gamma \notin A$ *whenever* $\gamma = 0$, $\Delta := \alpha\delta - \beta\gamma \neq 0$ *and the quadratic equation*

$$c^2 = \Delta \tag{27}$$

has solutions $c \in \mathbb{F}$; *finally,* $c \in \mathbb{F}$ *is an arbitrary root of equation* (27).

Proof. Again it is clear that functions (25) as well as functions (26), with the parameters as specified, satisfy equation (24). In order to prove the converse assume that functions $f, \varphi : A \longrightarrow \mathbb{F}$ satisfy equation (24). We distinguish two cases.

Case I. There exists a $\tilde{y} \in A$ such that $\varphi(\tilde{y}) = 0$. Then we get by (24)

$$f(x) = f(\tilde{y}) =: \lambda, \; x \in A. \tag{28}$$

that is, the first part of (25). Relations (24) and (28) imply now that

$$\varphi(x)\, \varphi(y) = 0 \text{ for all } x,y \in A, \; x \neq y. \tag{29}$$

If there exists a $y_0 \in A$ such that $\varphi(y_0) \neq 0$, then by (29) φ is zero on $A \setminus \{y_0\}$ and we obtain the second part of (25) with $\mu := \varphi(y_0)$. If $\varphi(x) = 0$ for all $x \in A$, then the second part of (25) holds with $\mu = 0$ and an arbitrary $y_0 \in A$.

Case II. We have

$$\varphi(x) \neq 0, \; x \in A. \tag{30}$$

In this case the proof is similar to that of Theorem 1. We fix $\bar{y}$, $\bar{\bar{y}} \in A$, $\bar{y} \neq \bar{\bar{y}}$, and with notation (12) we obtain by (24) in turn

$$f(x) = b + c(x-\bar{y})\, \varphi(x), \; x \in A, \tag{31}$$

$$c(x-\bar{y})\ \varphi(x) - c(y-\bar{y})\ \varphi(y) = \tag{32}$$

$$= [(x-\bar{y}) - (y-\bar{y})]\ \varphi(x)\ \varphi(y),\ x,y \in A$$

and (cf. (30))

$$\frac{c - \varphi(x)}{(x-\bar{y})\ \varphi(x)} = \frac{m - \varphi(\bar{\bar{y}})}{(\bar{\bar{y}}-\bar{y})\ \varphi(\bar{\bar{y}})} =: \gamma,\ x \in A,\ x \neq y. \tag{33}$$

Suppose that $\gamma \neq 0$. Relation (33) can be written in the form (valid also for $x = \bar{y}$)

$$c = [\gamma x + 1 - \gamma\bar{y}]\ \varphi(x) = (\gamma x+\delta)\varphi(x),\ x \in A, \tag{34}$$

where we have put

$$\delta := 1 - \gamma\bar{y} \text{ so that } \bar{y} = (1-\delta)\gamma^{-1}. \tag{35}$$

Relation (34) gives the second part of (26), which inserted into (31) yields

$$f(x) = \frac{(b\gamma+c^2)x + (b\delta-c^2\bar{y})}{\gamma x + \delta},\ x \in A,$$

so that with (cf. (35))

$$\alpha := b\gamma + c^2,\ \beta := b\delta - c^2\bar{y} = b\delta - c^2\gamma^{-1}(1 - \delta), \tag{36}$$

we obtain the first part of (26).

Since by (3) we have

$$c \neq 0, \tag{37}$$

relation (34) implies in particular that $\gamma x + \delta \neq 0$ for all $x \in A$, which means that $-\delta/\gamma \notin A$. Moreover, by (36) and (37),

$$\Delta = \alpha\delta - \beta\gamma = \left(b\gamma\delta + c^2\delta\right) - \left(b\delta\gamma - c^2 + c^2\delta\right) = c^2 \neq 0$$

and thus c is a root of equation (27).

Now suppose that γ given by (33) is zero. Then

$$\varphi(x) = c \text{ for all } x \in A, \tag{38}$$

which may be written in form (26) with $\gamma = 0$ and $\delta = 1$. Inserting (38) into (31) we obtain

$$f(x) = b + c^2 (x-\bar{y}), \ x \in A,$$

which again may be written in form (26)

$$\alpha := c^2, \ \beta := b - c^2\bar{y}, \ \gamma = 0, \ \delta = 1. \tag{39}$$

We have by (39) and (37)

$$\Delta = \alpha\delta - \beta\gamma = \alpha = c^2 \neq 0$$

and thus c is a roct of equation (27).

Now we pass to the related equation

$$xf(y) - yf(x) = (x-y)\,\varphi(x)\,\varphi(y), \quad x,y \in A. \tag{40}$$

THEOREM 4. *Suppose that the set* A_0 *contains at least two elements. The general solution* f, φ : A $\longrightarrow$ $\mathbb{F}$ *of equation* (40) *is given by the formulas*

$$\begin{cases} f(x) = \lambda x, \quad x \in A, \\ \varphi(x) = 0, \quad x \in A \setminus \{y_0\}, \quad \varphi(y_0) = \mu, \end{cases} \tag{41}$$

and

$$f(x) = \frac{\alpha x + \beta x^2}{\gamma + \delta x}, \quad \varphi(x) = \frac{cx}{\gamma + \delta x}, \quad x \in A, \tag{42}$$

and, if $0 \in A$, *also by*

$$\begin{cases} f(x) = \alpha + \beta x, \quad \varphi(x) = c, \qquad x \in A_0 \\ f(0) = \mu c, \qquad (0) = \mu. \end{cases} \tag{43}$$

In (41)-(43) y_0 *denotes an arbitrary point of* A, *the parameters* α, β, γ, δ, λ, μ *are arbitrary constants from* $\mathbb{F}$ *such that* $-\gamma/\delta \notin A$ *whenever* $\delta \neq 0$; $\Delta \neq 0$ *and equation* (27) *has solutions* $c \in \mathbb{F}$; and c $\in$ F *is an arbitrary root of equation* (27). *Here* $\Delta = \alpha\delta - \beta\gamma$ *in case of formula* (42), *while* $\Delta = \alpha$ *in case of formula* (43).

Proof. The functions f, φ given by (41) or (42) or (43) actually satisfy equation (40) as may be verified by the straightforward substitution. To prove the converse assume that functions f, $\varphi : A \longrightarrow \mathbb{F}$ satisfy equation (40). If $0 \notin A$, then we apply transformation (18) and by virtue of Theorem 3 we get either (41) or (42). Now suppose that $0 \in A$ and put $y = 0$ into equation (4) to obtain

$$f(0) = \varphi(x)\,\varphi(0),\ x \in A_0. \tag{44}$$

We distinguish three cases.

Case I. We have

$$f(0) = \varphi(0) = 0. \tag{45}$$

Then (44) yields no further information and, as we have just seen, on A_0 the functions f, φ are given either by (41) (with a $y_0 \in A_0$) or by (42). In the former case (45) shows that formula (41) is valid in the whole of A. Also in the latter case (42) is valid in the whole of A provided that $\gamma \neq 0$. Now we consider the case where f, φ are given by (42) on A_0 and by (45) at 0 and $\gamma = 0$. Moreover, we have by Theorem 3 (cf., in particular, relation (27))

$$\alpha\delta = \alpha\delta - \beta\gamma = c^2 \neq 0. \tag{46}$$

Formula (42) now can be written as

$$f(x) = \alpha' + \beta' x,\ \varphi(x) = c',\ x \in A_0, \tag{47}$$

where $\alpha' := \alpha/\delta$, $\beta' := \beta/\delta$, $c' := c/\delta$. (We must have $\delta \neq 0$ because of (46)). Relation (46) implies that $\alpha' \neq 0$ and $(c')^2 = c^2/\delta^2 = \alpha/\delta = \alpha'$. Suppressing the primes in (47) and taking (45) into account we obtain formula (43) with $\mu = 0$.

Case II. We have

$$f(0) = 0 \neq \varphi(0) =: \mu$$

Then (41) holds with $y_o = 0$ and $\lambda := f(\bar{y})/\bar{y}$, with a fixed $\bar{y} \in A_0$.

Case III. We have

$$f(0) \neq 0, \ \mu := \varphi(0) \neq 0$$

Then we obtain (43) with $c := f(0)/\varphi(0)$, $\alpha := c^2 (\neq 0)$, $\beta := (f(\bar{y}) - c^2)/\bar{y}$, with a fixed $\bar{y} \in A_0$. Equation (27) is a consequence of the definition of α.

3. The reciprocal of the sum.

In this section we take $\mathbb{F} = \mathbb{R}$, the field of real numbers. The extension of the results obtained to the case of arbitrary fields of characteristic different from 2 presents no serious difficulties. We leave this task to the reader.

Suppose that functions $f, \varphi : A \longrightarrow \mathbb{F}$ (where $A \subset \mathbb{R}$ contains at least two elements) satisfy equation (5) in A. The right-hand side of (5) is never zero, which means that

$$f(x) \neq f(y) \text{ for } x \neq y, \ x,y \in A.$$

In other words, the function f is invertible in A. Let $g : f(A) \longrightarrow A$ be its inverse and write $\psi := \varphi \circ g$. Putting into (5)

$x = g(u)$, $y = g(v)$, $(u,v \in f(A))$, taking the reciprocals of both sides and multiplying by $u - v$ we obtain

$$g(u) - g(v) = (u - v)[\psi(u) + \psi(v)], \quad u,v \in f(A), \tag{48}$$

originally for $u \neq v$, but for $u = v$ (48) is trivial. Up to the notation, equation (48) is identical with (10). By virtue of Theorem 1 there exist real constants α, β, γ such that

$$g(u) = \alpha u^2 + 2\beta u + \gamma, \quad \psi(u) = \alpha u + \beta, \quad u \in f(A). \tag{49}$$

If $\alpha = 0$, then necessarily $\beta \neq 0$ and we get from (49)

$$f(x) = (x-\gamma)/2\beta, \quad \varphi(x) = \beta, \quad x \in A. \tag{50}$$

If $\alpha \neq 0$, then there appear some conditions binding together α, β, γ and the set A. Relation (49) yields (for $x \in A$)

$$f(x) = \frac{-\beta + \varepsilon(x)\sqrt{\beta^2 - \alpha(\gamma-x)}}{\alpha}, \quad \varphi(x) = \varepsilon(x)\sqrt{\beta^2 - \alpha(\gamma-x)}, \tag{51}$$

where $\varepsilon : A \longrightarrow \{-1,1\}$ is an arbitrary function on A with values ± 1 only. Moreover, in order for (51) to make sense we must have

$$\alpha \neq 0 \text{ and } \beta^2 - \alpha(\gamma-x) \geq 0, \quad x \in A, \tag{52}$$

or, equivalently, either

$$\alpha > 0 \text{ and } A \subset [(\alpha\gamma-\beta^2)/\alpha,\infty[, \tag{53}$$

or

$$\alpha < 0 \text{ and } A \subset]-\infty, (\alpha\gamma- \beta^2)/\alpha]. \tag{54}$$

Thus we have established that functions f, $\varphi : A \longrightarrow \mathbb{R}$ satisfying equation (5) on A must be given either by (50) or by (51) (with the parameters fulfilling the corresponding conditions). Since the converse is evident, we have proved, in fact, the following result.

THEOREM 5. *The general solution* f, $\varphi : A \longrightarrow \mathbb{R}$ ($A \subset \mathbb{R}$ *contains at least two elements) of equation* (5) *in* A *is given by formulas* (50) *and* (51), *where* $\varepsilon : A \longrightarrow \{-1,1\}$ *is an arbitrary function on* A *with values* ±1 *only and* α, β, γ *are arbitrary real constants fulfilling the condition* $\beta \neq 0$ *in case of formula* (50) *and condition* (52) *(or, what ammounts to the same,* (53) *or* (54)) *in case of formula* (51).

REMARK. Relations (53) and (54) show that if $\inf A = -\infty$ and $\sup A = \infty$ (in particular, if $A = \mathbb{R}$), then the general solution f, $\varphi : A \longrightarrow \mathbb{R}$ of equation (5) in A is given by formula (50) with arbitrary real constants $\beta \neq 0$ and γ.

We state yet (without proof) the result concerning the related equation

$$\frac{xf(y) - yf(x)}{x - y} = \frac{1}{\varphi(x) + \varphi(y)}, \quad x,y \in A,\ x \neq y. \tag{55}$$

The proof of Theorem 6 below is similar to that of Theorems 2 and 4 and is left to the reader.

THEOREM 6. *Let* $A \subset \mathbb{R}$ *and suppose that the set* $A_0 := A \setminus \{0\}$ *contains at least two elements. If* $0 \notin A$, *then the general solution* $f, \varphi : A \longrightarrow \mathbb{R}$ *of equation* (55) *is given by the formulas*

$$f(x) = (1-\gamma x)/2\beta, \ \varphi(x) = \beta, \ x \in A, \tag{56}$$

or

$$\left.\begin{aligned} f(x) &= \left[-\beta x + x\varepsilon(x)\sqrt{\beta^2 - \alpha(\gamma - x^{-1})}\right]/\alpha, \ x \in A \\ \varphi(x) &= \varepsilon(x)\sqrt{\beta^2 - \alpha(\gamma - x^{-1})}, \ x \in A, \end{aligned}\right\} \tag{57}$$

where $\varepsilon : A \longrightarrow \{-1,1\}$ *is an arbitrary function on* A *with values* ± 1 *only, and* α, β, γ *are arbitrary real constants fulfilling the condition* $\beta \neq 0$ *in case of formula* (56) *and the condition*

$$\alpha \neq 0 \text{ and } \beta^2 - \alpha(\gamma - x^{-1}) \geq 0, \ x \in A,$$

in case of formula (57). *If* $0 \in A$, *then the general solution* $f, \varphi : A \longrightarrow \mathbb{R}$ *of equation* (55) is given by formula (56) *for* $x \in A_0$ and by $f(0) = (\beta + \delta^{-1})$, $\varphi(0) = \delta$ *at zero. Here* β, γ, δ *are arbitrary real constants such that* $\beta \neq 0$ *and* $\beta + \delta \neq 0$.

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