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On solutions of the translation equation commuting with some function

Dedicated to Professor Zenon Moszner with best wishes on his 60-th birthday

ABSTRACT. In this paper the general construction of solutions of the translation equation (1) such that $F\big(f(\alpha),\chi\big) = f\big(F(\alpha,\chi)\big)$, when $f \in \Gamma^{\Gamma}$ and $f^3 = f$, is presented.

Let Γ be an arbitrary set and let $(G,\cdot,e)$ be an arbitrary group. By $F:\Gamma\times G \longrightarrow \Gamma$ we denote an arbitrary solution of the translation equation

$$F\Big(F(\alpha,\chi),y\Big) = F(\alpha,\chi\cdot y) \tag{1}$$

The general solution of the translation equation has been given by Z. Moszner [2] in the following Construction C1:

1° *The function* $q:\Gamma \longrightarrow \Gamma$ *is such that* $q \circ q = q$.

2° $q(\Gamma) = \bigcup_{k\in K} \Gamma_k$ *is a disjoint union of non-empty sets (fibres)* Γ_k *such that for every* $k \in K$ *there exists a subgroup* $G_k \leq G$ *and a bijection* $g_k:G/G_k \longrightarrow \Gamma_k$, *where* G/G_k *is the set of right cosets of the group* G *with respect to subgroup* G_k.

3° *Then* $F(\alpha,\chi) = g_k\left(g_k^{-1}(q(\alpha)) \cdot \chi\right)$ *when* $q(\alpha) \in \Gamma_k$.

In this paper we will formulate and solve the problem which is essential in considerations of solutions of the translation equation on 3-adic groups (see [1]). By the results of this paper we can also describe all solutions of certain functional equation (see Remark 5).

The function $f:\Gamma \longrightarrow \Gamma$ satisfying equation

$$f^3 = f \tag{2}$$

is fixed in the sequal.

Let $F:\Gamma \times G \longrightarrow \Gamma$ be a solution of the translation equation (1) such that

$$F(\alpha,e) = f^2(\alpha). \tag{3}$$

$$F\left(f(\alpha),\chi\right) = f\left(F(\alpha,\chi)\right), \text{ for } \alpha \in \Gamma \text{ and } \chi \in G. \tag{4}$$

The condtion (4) means that every function $F(\cdot,\chi)$, for $\chi \in G$, commuts with the fixed function f.

The aim of this paper is the characterization of all solutions $F:\Gamma \times G \longrightarrow \Gamma$ of the translation equation (1) satisfying (3) and (4).

Let us denote:

$$A_f := \{\alpha \in \Gamma : f(\alpha) = f^2(\alpha)\},\ B_f := \Gamma \setminus A_f = \{\alpha \in \Gamma : f(\alpha) \neq f^2(\alpha)\}.$$

LEMMA 1. *The following inclusions are true*:

(i) $F(A_f \times G) \subseteq A_f$,

(ii) $F(B_f \times G) \subseteq B_f$.

Proof. (i). Let $\alpha \in A_f$, $\chi \in G$ and $F(\alpha,\chi) =: \beta$. We have $f(\beta) = f\big(F(\alpha,\chi)\big) = F\big(f(\alpha),\chi\big) = F\big(f^2(\alpha),\chi\big) = F(\alpha,\chi) = \beta$, and hence $f^2(\beta) = f(\beta)$, therefore $\beta \in A_f$.

(ii). Let $\beta \in B_f, \chi \in G$ and $F(\beta,\chi) =: \alpha$. It is not possible that $\alpha \in A_f$ because the equality $f(\alpha) = f^2(\alpha)$ implies $F(\alpha,\chi^{-1}) = F\big(F(\beta,\chi),\chi^{-1}\big) = F(\beta,e) = f^2(\beta)$

and

$$F(\alpha,\chi^{-1}) = F\big(f^2(\alpha),\chi^{-1}\big) = F\big(f(\alpha),\chi^{-1}\big) = f\big(F(\alpha,\chi^{-1})\big)$$

$$= f\big(F(F(\beta,\chi),\chi^{-1})\big) = f\big(F(\beta,e)\big) = f\big(f^2(\beta)\big)$$

$$= f^3(\beta) = f(\beta),$$

and this contradiction.

REMARK 1. It is visible that if $A_f = \Gamma$ then every F of the translation equation (1) such that $F(\alpha,e) = f^2(\alpha)$ satisfies (4). Indeed: $F\big(f(\alpha),\chi\big) = F\big(f^2(\alpha),\chi\big) = F(\alpha,\chi) = f^2\big(F(\alpha,\chi)\big) = f\big(F(\alpha,\chi)\big)$.

REMARK 2. It is easy to prove that $f(B_f) \subseteq B_f$ and $f^2(B_f) = f(B_f)$.

By Remark 1 it follows that in the case $A_f = \Gamma$ the condition (4) is not essential. By virtue of Lemma 1 and Remark 1 we can consider the case $B_f = \Gamma$ only.

Let us denote:

$$E_\alpha(\beta) := \{\chi \in G : F(\alpha,\chi) = \beta\}, \text{ for } \alpha,\beta \in \Gamma$$

and

$$G_\alpha := E_\alpha(\alpha).$$

Lemma 2. *For every* $\alpha \in f^2(\Gamma)$ *the set* G_α *is a subgroup of the group* G *and the equalities*

$$G_\alpha = G_{f(\alpha)}, \tag{5}$$

$$E_{f(\alpha)}(\alpha) = E_\alpha\big(f(\alpha)\big) = G_\alpha \cdot \chi_0, \text{ where } \chi_0 \in E_\alpha\big(f(\alpha)\big) \tag{6}$$

are satisfied.

Proof. Because $\alpha \in f^2(\Gamma)$ then $\alpha = f^2(\beta)$, for certain $\beta \in \Gamma$. Therefore: $F(\alpha,e) = F\big(f^2(\beta),e\big) = F\big(F(\beta,e),e\big) = F(\beta,e) = f^2(\beta) = \alpha$.

By construction of solutions of the translation equation in [2] there results that G_α is a subgroup of group G. The equality (5) can be verified as follows:

$$\chi \in G_\alpha \Rightarrow F(\alpha,\chi) = \alpha \Rightarrow F\big(f(\alpha),\chi\big) = f\big(F(\alpha,\chi)\big) = f(\alpha) \Rightarrow \chi \in G_{f(\alpha)},$$

$$\chi \in G_{f(\alpha)} \Rightarrow F\big(f(\alpha),\chi\big) = f(\alpha) \Rightarrow f\big(F(\alpha,\chi)\big) = F\big(f(\alpha),\chi\big) =$$

$$f(\alpha) \Rightarrow f^2\big(F(\alpha,\chi)\big) = f^2(\alpha) \Rightarrow F(\alpha,\chi) = \alpha \Rightarrow \chi \in G_\alpha.$$

We will prove (6). Let $\chi \in E_{f(\alpha)}(\alpha)$. We have
$F\big(f(\alpha),\chi\big) = \alpha \Rightarrow f\big(F(\alpha,\chi)\big) = \alpha \Rightarrow f^2\big(F(\alpha,\chi)\big) = f(\alpha) \Rightarrow F(\alpha,\chi) = f(\alpha)$, then $\chi \in E_\alpha\big(f(\alpha)\big)$. Hence $E_{f(\alpha)}(\alpha) \subseteq E_\alpha\big(f(\alpha)\big)$.
Similarly. $E_\alpha\big(f(\alpha)\big) \subseteq E_{f(\alpha)}(\alpha)$.

Now let $\chi,\chi_0 \in E_\alpha\big(f(\alpha)\big)$. We have $\chi = \chi \cdot \chi_0^{-1} \cdot \chi_0$ and because $F(\alpha,\chi\cdot\chi_0^{-1}) = F\big(F(\alpha,\chi),\chi_0^{-1}\big) = F\big(f(\alpha),\chi_0^{-1})\big) = F\big(F(\alpha,\chi_0),\chi_0^{-1}\big) = F(\alpha,e) = \alpha$, then $\chi \cdot \chi_0^{-1} \in G_\alpha$, whence $\chi \in G_\alpha \cdot \chi_0$. Therefore, $E_\alpha\big(f(\alpha)\big) \subseteq G_\alpha \cdot \chi_0$. Since for every $\chi \in G$ and $\chi_0 \in E_\alpha\big(f(\alpha)\big)$ we have

$\chi \in G_\alpha \cdot \chi_0 \Rightarrow \chi = a \cdot \chi_0$, where $a \in G_\alpha \Rightarrow F(\alpha,\chi) = F(\alpha,a\cdot\chi_0) = F\big(F(\alpha,a),\chi_0\big) = F(\alpha,\chi_0) = f(\alpha) \Rightarrow \chi \in E_\alpha\big(f(\alpha)\big)$, then $G_\alpha \cdot \chi_0 \subseteq E_\alpha\big(f(\alpha)\big)$.

The proof of Lemma 2 is finished.

According to the point 2° of Construction C1 let us denote the family of fibres of the solution F by $\{\Gamma_k\}_{k\in K}$.

LEMMA 3. *For every fibre* Γ_k *one of the following equalities is true:*

$$f(\Gamma_k) = \Gamma_k \tag{6}$$

or

$$\exists\ l \in K : l \neq k \text{ and } f(\Gamma_k) = \Gamma_l. \tag{7}$$

Proof. Let us choose $\alpha_0 \in \Gamma_k$ and let us consider two possibilities:

a) $f(\alpha_0) \in \Gamma_k$,

b) $f(\alpha_0) \in \Gamma_l$ and $l \neq k$.

Ad a). We will prove (6). Let $\alpha \in \Gamma_k$. Hence $F(\alpha_0,\chi) = \alpha$, for a certain $\chi \in G$. We have $F\big(f(\alpha_0),\chi\big) = f\big(F(\alpha_0,\chi)\big) = f(\alpha)$ and by the condition a): $f(\alpha) \in \Gamma_k$. Therefore, $f(\Gamma_k) \subseteq \Gamma_k$. Since

$$\forall\ \alpha \in \Gamma_k : f^2(\alpha) = F(\alpha,e) = \alpha$$

then

$$\Gamma_k = f^2(\Gamma_k) = f\big(f(\Gamma_k)\big) \subset f(\Gamma_k) \subset \Gamma_k$$

whence $f(\Gamma_k) = \Gamma_k$.

Ad b). We will prove (7). For evry $\alpha \in \Gamma_k$ there exists $\chi \in G$ such that $F(\alpha_0,\chi) = \alpha$, hence $F\big(f(\alpha_0),\chi\big) = f\big(F(\alpha_0,\chi)\big) = f(\alpha)$, whence by b) : $f(\alpha) \in \Gamma_1$. Therefore, $f(\Gamma_k) \subseteq \Gamma_1$. Using the equality $f\big(f(\alpha_0)\big) = \alpha_0$ we obtain similarly the converse inclusion.

LEMMA 4. *If the fibre* Γ_k *satisfies* (6) *and* $\alpha \in \Gamma_k$ *and* $F(\alpha,\chi_0) = f(\alpha)$ *then* $G^o_\alpha := G_\alpha \cup G_\alpha \cdot \chi_0$ *is a subgroup of* G.

Proff. Since $G_\alpha \subseteq G^o_\alpha$ then $e \in G^o_\alpha$. Let $\chi,\psi \in G^o_\alpha$. We will show that:

(i) $\chi,\psi \in G_\alpha \Rightarrow \chi \cdot \psi^{-1} \in G_\alpha \subset G^o_\alpha$,

(ii) $\chi \in G_\alpha, \psi \in G_\alpha \cdot \chi_0 \Rightarrow \chi \cdot \psi^{-1} \in G_\alpha \cdot \chi_0 \subset G^o_\alpha$,

(iii) $\chi \in G_\alpha \cdot \chi_0, \psi \in G_\alpha \Rightarrow \chi \cdot \psi^{-1} \in G_\alpha \cdot \chi_0 \subset G^o_\alpha$,

(iv) $\chi,\psi \in G_\alpha \cdot \chi_0 \Rightarrow \chi \cdot \psi^{-1} \in G_\alpha \subset G^o_\alpha$.

Ad (i). It is evident by Lemma 2.

Ad (ii). Let $\psi = a \cdot \chi_0$, where $a \in G_\alpha$. We have

$$F(f(\alpha),\chi \cdot \psi^{-1}) = F\big(F(f(\alpha),\chi),\psi^{-1}\big) = F\big(f(\alpha),\psi^{-1}\big)$$

$$= F\big(f(\alpha),\chi_0^{-1} \cdot a^{-1}\big) = F\big(F(\alpha,\chi_0),\ \chi_0^{-1} \cdot a^{-1}\big) = F(\alpha,a^{-1}) = \alpha$$

whence by Lemma 2

$$\chi \cdot \psi^{-1} \in E_{f(\alpha)}(\alpha) = G_\alpha \cdot \chi_0.$$

Ad (iii). Similarly, if $\chi = a \cdot \chi_0$, where $a \in G_\alpha$ then:

$$F(\alpha,\chi \cdot \psi^{-1}) = F\big(F(\alpha,a),\chi_0 \cdot \psi^{-1}\big) = F(\alpha,\chi_0 \cdot \psi^{-1})$$
$$= F\big(F(\alpha,\chi_0),\psi^{-1}\big) = F(f(\alpha),y^{-1}) = f(\alpha),$$

whence $\chi \cdot \psi^{-1} \in E_\alpha\big(f(\alpha)\big) = G_\alpha \cdot \chi_0$.

Ad (iv). Let $\chi = a_1 \cdot \chi_0$, $\psi = a_2 \cdot \chi_0$, where $a_1, a_2 \in G_\alpha$.

We have:

$$\chi \cdot \psi^{-1} = a_1 \cdot \chi_0 \, \chi_0^{-1} \cdot a_2^{-1} = a_1 \cdot a_2^{-1} \in G_\alpha^o \subset G_\alpha.$$

We have shown Lemma 4.

The subgroup G_α is evidently invariant subgroup of G_α^o, as the subgroup of index 2. Therefore,

if $W = G_\alpha \cdot a \in G/G_\alpha$ and $\chi_0 \in G_\alpha^o \setminus G_\alpha$ then $\chi_0 \cdot W = \chi_0 \cdot G_\alpha \cdot a = G_\alpha \cdot \chi_0 \cdot a \in G/G_\alpha$.

REMARK 3. *If the fibre* Γ_k *satisfies* (6) *and* $\alpha \in \Gamma_k$ *then the function* $h_\alpha : G/G_\alpha \longrightarrow \Gamma_k$ defined as follows

$$h_\alpha(W) := F(\alpha, W)$$

satisfies equality

$$h_\alpha(\chi_0 \cdot W) = f\big(h_\alpha(W)\big), \textit{ for every } W \in G/G_\alpha,$$

where χ_0 *is a fixed element of the set* $G_\alpha^o \setminus G_\alpha$.

Proof. For arbitrary $W \in G/G_\alpha$ we have:

$$f\big(h_\alpha(W)\big) = f\big(F(\alpha, W)\big) = F\big(f(\alpha), W\big) = F\big(F(\alpha, \chi_0), W\big)$$

$$= F(\alpha, \chi_0 \cdot W) = h_\alpha(\chi_0 \cdot W).$$

REMARK 4. *If the fibre* Γ_k *satisfies* (7) *and* $\alpha \in \Gamma_k$, $f(\alpha) \in \Gamma_l$ *and the functions* $h_\alpha : G/G_\alpha \longrightarrow \Gamma_k$, $h_{f(\alpha)} : G/G_\alpha \longrightarrow \Gamma$ *are defined as follows* $h_\alpha(W) = F(\alpha_0, W)$ *and* $h_{f(\alpha)}(W) = F(f(\alpha), W)$ *then*

$$h_{f(\alpha)}(W) = f\left(h_\alpha(W)\right), \qquad \textit{for every } W \in G/G_\alpha.$$

Indeed:

$$h_{f(\alpha)}(W) = F\left(f(\alpha),W\right) = f\left(F(\alpha,W)\right) = f\left(h_\alpha(W)\right).$$

DEFINITION. *The solution* $F\ \Gamma\times G \longrightarrow \Gamma$ *of the translation equation* (1) *satisfying* (3), (4) *is called* f*-compatible (respectively not* f*-compatible) if all its fibres satisfy condition* (6) *(respectively* (7)*).*

First we will characterize solutions which are f-compatible.

THEOREM 1. *Let* $(G,\cdot,e)$ *be a group,* Γ *an arbitrary set and a function* $f \in \Gamma^\Gamma$ *satisfies equalities:* $f^3 = f$ and $A_f = \emptyset$. *All* f*-compatible solutions* $F : \Gamma\times G \longrightarrow \Gamma$ *of the translation equation* (1) *satisfying* (3), (4) *and only such we obtain by the presented below.*

CONSTRUCTION C2

1° *Let us denote* $B_f^\circ := \left\{\langle f(\alpha),\ f^2(\alpha)\rangle : \alpha \in \Gamma\right\}$. *We decompose* $B_f^\circ = \bigcup\limits_{k\in K} \Gamma_k^\circ$ *in a disjoint union of non-empty sets such that*

a) $\forall\ k \in K\ \exists\ G_k^\circ \le G : \text{card } \Gamma_k^\circ = \text{card } G/G_k^\circ.$

b) $\forall\ k \in K\ \exists\ G_k < G : \left((G_k^\circ : G_k) = 2\right).$

2° *For every* k *belonging to* K *let us denote by* Γ_k^1 *a selection of the set* Γ_k° *and let us denote by* S_k *a selection of the set* $\left\{\langle W,\chi_k \cdot W\rangle : W \in G/G_k\right\}$, *where* χ_k *is a fixed element of the set* $G_k^\circ\setminus G_k$.

3° *Let* $\Gamma_k := \Gamma_k^1 \cup f(\Gamma_k^1)$ *and let us define the bijection* $g_k : G/G_k \longrightarrow \Gamma_k$ *in the following way*

$$g_k(W) := \begin{cases} g_k^*(W) : W \in S_k, \\ f(g_k^*(\chi_k \cdot W)) : W \notin S_k, \end{cases} \tag{8}$$

where $g_k^* : S_k \longrightarrow \Gamma_k^1$ *is an arbitrary bijection.*

4° *The function* $F : \Gamma \times G \longrightarrow \Gamma$ *is defined as follows:*

$$F(\alpha,\chi) := g_k\left(g_k^{-1}(f^2(\alpha)) \cdot \chi\right), \text{ when } f^2(\alpha) \in \Gamma_k, \chi \in G. \tag{9}$$

Proof. It is visible by (2) and by Construction C1 that every function $F : \Gamma \times G \longrightarrow \Gamma$ obtained by Construction C2 satisfies the translation equation and the condition (3). Because for every k from K we have

$$f(\Gamma_k) = f(\Gamma_k^1) \cup f^2(\Gamma_k^1) = f(\Gamma_k^1) \cup \Gamma_k^1 = \Gamma_k$$

then the solution F is f-compatible.

We will show the condition (4). Let $\alpha \in \Gamma, \chi \in G, f(\alpha), f^2(\alpha) \in \Gamma_k$. We have:

$$F\left(f(\alpha),\chi\right) = g_k\left(g_k^{-1}\left(f^3(\alpha)\right) \cdot \chi\right) = g_k\left(g_k^{-1}\left(f(\alpha)\right) \cdot \chi\right)$$

and

$$f\left(F(\alpha,\chi)\right) = f\left(g_k(g_k^{-1}(f^2(\alpha)) \cdot \chi)\right).$$

In virtue of (8) : if $W \in G/G_k$ and $g_k(W) = f(\alpha)$ then $g_k(\chi_k \cdot W) = f^2(\alpha)$. Indeed: if $W \in S_k$ then $\chi_k \cdot W \notin S_k$ and $g_k(\chi_k \cdot W) = f\left(g_k^*(\chi_k^2 \cdot W)\right) = f\left(g_k^*(W)\right) = f^2(\alpha)$.
(Similarly in the case : $W \notin S_k$).
From the above

$$F\left(f(\alpha),\chi\right) = g_k(W \cdot \chi)$$

and

$$f\left(F(\alpha,\chi)\right) = f\left(g_k(\chi_k \cdot W \cdot \chi)\right)$$

If $W \cdot \chi \in S_k$ then $\chi_k \cdot W \cdot \chi \notin S_k$ and we have

$$F\left(f(\alpha),\chi\right) = g_k(W \cdot \chi) = g_k^*(W \cdot \chi)$$

and

$$f\left(F(\alpha,\chi)\right) = f\left(g_k(\chi_k \cdot W \cdot \chi)\right) = f\left(f(g_k^*(\chi_k^2 \cdot W \cdot \chi))\right)$$

$$= f^2\left(g_k^*(W \cdot \chi)\right) = g_k^*(W \cdot \chi),$$

therefore

$$F\left(f(\alpha),\chi\right) = f\left(F(\alpha,\chi)\right).$$

The second case $W \cdot \chi \notin S_k$ implies $\chi_k \cdot W \cdot \chi \in S_k$ and

$$F\left(f(\alpha),\chi)\right) = g_k(W \cdot \chi) = f\left(g_k^*(\chi_k \cdot W \cdot \chi)\right)$$

and

$$f\left(F(\alpha,\chi)\right) = f\left(g_k(\chi_k \cdot W \cdot \chi)\right) = f\left(g_k^*(\chi_k \cdot W \cdot \chi)\right),$$

therefore

$$F\left(f(\alpha),\chi\right) = f\left(F(\alpha,\chi)\right).$$

This ends one part of the proof of Theorem 1.

Assume now that $F:\Gamma\times G \longrightarrow \Gamma$ is a solution of the translation equation (1) satisfying (3), (4) and f-compatible. By the general Construction C1 of solutions of the translation equation we have the following parameters: the family $\{\Gamma_k\}_{k\in K}$ of fibres such that $f^2(\Gamma) = \bigcup_{k\in K} \Gamma_k$; the family of subgroups $\{G_k\}_{k\in K}$ and the family of bijections $g_k:G/G_k \longrightarrow \Gamma_k$ such that the condition (9) is satisfied. It is visible that

$$\forall\ k \in K : G_{\alpha_k} = G_k, \text{ where } \alpha_k := g_k(G_k).$$

By virtue of Lemma 4 $:G_k^o := G_{\alpha_k} \cup G_\alpha \cdot \chi_k$, where χ_k is an arbitrary element of G satisfying equality $F(\alpha_k,\chi_k) = f(\alpha_k)$, is a subgroup of G and obviously $(G_k^o : G_k) = 2$.

If we define

$$B_f^o := \bigcup_{k\in K} \Gamma_k^o, \qquad \text{where } \Gamma_k^o := \bigcup_{\alpha\in\Gamma_k} \{f(\alpha),f^2(\alpha)\}$$

then

$$\operatorname{card} \Gamma_k^o = \operatorname{card} G_k^o, \text{ for } k \in K.$$

Therefore, the point 1° of construction C2 is fulfilled.

Let S_k be a selection of the set $\left\{(W, \chi_k \cdot W) \in G/G_k\right\}$ such that $G_k \in S_k$. We define the function g_k^* as follows

$$g_k^*(W) := F(\alpha_k, W), \text{ for } W \in S_k.$$

Let $\Gamma_k^1 := \{\bar{g}_k(W) \colon W \in S_k\}$. It is visible by Remark 3 that the condition (8) is fulfilled because $g_k(W) = F(\alpha_k, W) = h_{\alpha_k}(W)$, $W \in G/G_k$.

The proof of Theorem 1 is finished.

Now, we will characterize all solutions which are not f-compatible.

THEOREM 2. *Let* $(G, \cdot, e)$ *be a group,* Γ *an arbitrary non-empty set and a function* $f \in \Gamma^\Gamma$ *satisfies equalities:* $f^3 = f$ *and* $A_f = \emptyset$. *All not f-compatible solutions* $F\colon \Gamma \times G \longrightarrow \Gamma$ *of the translation equation* (1) *satisfying* (3), (4) *and only such we obtain by presented below*

CONSTRUCTION C3

1° *Let us denote* $B_f^o := \left\{(f(\alpha), f^2(\alpha)) \colon \alpha \in \Gamma\right\}$. *We decompose the set* B_f^o *in a disjoint union* $\bigcup_{t \in T} \Gamma_t^o$ *of non-empty sets such that*

$$\forall\, t \in T\ \exists\, G_t \le G : \operatorname{card} \Gamma_t^o = \operatorname{card} G/G_t.$$

2° *For* $t \in T$ *we take a selection* Γ_t *of the set* Γ_t^o and we define $\bar{\Gamma}_t := f(\Gamma_t)$ (*By* (2) $\bar{\Gamma}_t$ *is a selection of* Γ_t^o *and* $\Gamma_t \cap \bar{\Gamma}_t = \emptyset$ *by* (7)).

3° *For* $t \in T$ *we take a bijection* $g_t : G/G_t \longrightarrow \Gamma_t$, *put* $\bar{G}_t := G_t$ *and we define the bijection* $\bar{g}_t : G/\bar{G}_t \longrightarrow \bar{\Gamma}_t$ *as follows*

$$\bar{g}_t(W) = f\left(g_t(W)\right). \tag{10}$$

4° *The family of sets* $\{\Gamma_t\}_{t\in T} \cup \{\bar{\Gamma}_t\}_{t\in T}$ *we denote by* $\{\Gamma_k\}_{k\in K}$. *The family of subgroups* $\{G_t\}_{t\in T} \cup \{\bar{G}_t\}_{t\in T}$ *we denote respectively by* $\{G_k\}_{k\in K}$. *The family of bijections* $\{g_t\}_{t\in T} \cup \{\bar{g}_t\}_{t\in T}$ *we denote by* $\{g_k\}_{k\in K}$ *respectively.*

5° *We put*

$$F(\alpha,\chi) := g_k\left(g_k^{-1}(f^2(\alpha)) \cdot \chi\right), \text{ when } f^2(\alpha) \in \Gamma_k.$$

Proof. It is visible that the function $F : \Gamma \times G \longrightarrow \Gamma$ obtained by Construction C3 satisfies the translation equation and conditions (3) and (7). We will show the condition (4) only. Let $\alpha \in \Gamma$, $\chi \in G$, $f^2(\alpha) \in \Gamma_k$, $f(\alpha) \in \Gamma_l$. There exists $t \in T$ such that $\Gamma_k = \Gamma_t$ and $\Gamma_l = \bar{\Gamma}_t$. We have

$$f\left(F(\alpha,\chi)\right) = f\left(g_k(g_k^{-1}(f^2(\alpha)) \cdot \chi\right) = fg_t\left(g_t^{-1}(f^2(\alpha)) \cdot \chi\right)$$

and

$$F\left(f(\alpha),\chi\right) = g_l\left(g_l^{-1}(f(\alpha)) \cdot \chi\right) = \bar{g}_t\left(\bar{g}_t^{-1}(f(\alpha)) \cdot \chi\right)$$

If $W \in G/G_t$ and $g_t(W) = f^2(\alpha)$ then by (10) $\bar{g}_t(W) = f(\alpha)$. From the above

$$f\left(F(\alpha,\chi)\right) = f\left(g_t(W \cdot \chi)\right)$$

and

$$F\big(f(\alpha),\chi\big) = \bar{g}_t(W \cdot \chi)$$

and using (10) we obtain

$$f\big(F(\alpha,\chi)\big) = F\big(f(\alpha),\chi\big).$$

This ends the one part of the proof.

Now let $f^3 = f$ and $A_f = \emptyset$ and let us suppose that $F:\Gamma\times G \longrightarrow \Gamma$ is not compatible with f solution of the translation equation (1) satisfying (3), (4). By the general Construction C1 of solutions of the translation equation we have: the family $\{\Gamma_k\}_{k\in K}$ of fibres such that $f^2(\Gamma_k) = \bigcup_{k\in K} \Gamma_k$, the family $\{G_k\}_{k\in K}$ of subgroups of group G such that card Γ_k = card G/G_k and the family $\{g_k\}_{k\in K}$ of bijections $g_k:G/G_k \longrightarrow \Gamma_k$.

Let $\{\Gamma_t : t \in T\}$ denote the selection of the set $\Big\{\{\Gamma_k,f(\Gamma_k)\}:k \in K\Big\}$. Let $\{g_t\}_{t\in T}$ denote the family of bijections adequate to $\{\Gamma_t\}_{t\in T}$. For every $t \in T$ we put:

$$\bar{\Gamma}_t := f(\Gamma_t),$$

$$\bar{G}_t := G_t,$$

$$\bar{g}_t(W) := f\big(g_t(W)\big), \text{ for } W \in G/\bar{G}_t,$$

$$\Gamma_t^{\circ} := \Big\{\{f(\alpha),\alpha\} : \alpha \in \Gamma_t\Big\},$$

$$B_f^{o} := \left\{ (f(\alpha),\ f^2(\alpha)) : \alpha \in \Gamma \right\}.$$

For every $t \in T$ and every $W \in G/G_t$ we have:

$$g_t(W) = F(\alpha_t, W) = h_{\alpha_t}(W), \text{ where } \alpha_t := g_t(G_t)$$

and by Remark 4

$$\bar{g}_t(W) = f\left(g_t(W)\right) = f\left(h_{\alpha_t}(W)\right) = h_{f(\alpha_t)}(W) = F\left(f(\alpha_t), W\right).$$

In virtue of the proof of Theorem 1 in [2] the proof of Theorem 2 is finished.

The next theorem describes all solutions $F:\Gamma \times G \longrightarrow \Gamma$ of (1) satisfying (3), (4).

THEOREM 3. *Let* $(G,\cdot,e)$ *be a group,* Γ *an arbitrary non-empty set and the function* $f \in \Gamma^{\Gamma}$ *satisfies equality* $:f^3 = f$. *All solutions* $F:\Gamma \times G \longrightarrow \Gamma$ *of the translation equation* (1) *satisfying* (3), (4) and only such can be obtained by

CONSTRUCTION C4

1° *Let* $A_f := \{\alpha \in \Gamma : f(\alpha) = f^2(\alpha)\}$ *nad let* $\phi: A_f \times G \longrightarrow A_f$ *be a solution of the translation equation* (1) *obtained by construction* C1 *such that* $\phi(\alpha,e) = f^2(\alpha)$.

2° *We decompose the set* $\left\{(f(\alpha),f^2(\alpha)) : \alpha \in \Gamma \setminus A_f\right\} = B_f^1 \cup B_f^2$ *in a disjoint union of sets* B_f^1, B_f^2. *Let*

$$\Gamma_i := \left\{\alpha \in \Gamma \setminus A_f : (f(\alpha),f^2(\alpha)) \in B_f^i\right\}, \textit{ for } i=1,2.$$

3° *Let* $\psi_1:\Gamma_1\times G \longrightarrow \Gamma_1$ *be a compatible with* f *solution of the translation equation* (1) *satisfying* (3), (4). *This means* ψ_1 *is obtained by construction* C2.

4° *Let* $\psi_2:\Gamma_2\times G \longrightarrow \Gamma_2$ *be a not compatible with* f *solution of the translation equation* (1) satisfying (3), (4). *This means* ψ_2 *is obtained by construction* C3.

5° *We put*

$$F := \phi \cup \psi_1 \cup \psi_2.$$

Proof. It is evident that F obtained by construction C4 satisfies (1), (3), (4). Assume that $F:\Gamma\times G \longrightarrow \Gamma$ is a solution of (1) satisfying (3), (4). By Lemma 1 we can put $\phi := F|_{A_f\times G}$. By Lemma 3 all fibres $\{\Gamma_k\}_{k\in K}$ of F satisfy (6) or (7). Let us define:

$$B_f^1 := \left\{(f(\alpha),f^2(\alpha)):\alpha \in \Gamma \setminus A_f \text{ and } \exists\ k \in K : f(\alpha),f^2(\alpha) \in \Gamma_k\right\},$$

$$B_f^2 := \left\{(f(\alpha),f^2(\alpha)):\alpha \in \Gamma \setminus A_f \text{ and } \exists\ k,l \in K:k\neq l \text{ and } f(\alpha) \in \Gamma_k, f^2(\alpha) \in \Gamma_l\right\}$$

and

$$\Gamma_i := \left\{\alpha \in \Gamma \setminus A_f : (f(\alpha),f^2(\alpha)) \in B_f^i\right\}, \text{ for } i = 1,2$$

and

$$\psi_i := F|_{\Gamma_i\times G}, \text{ for } i = 1,2.$$

It is visible that

$F = \phi \cup \psi_1 \cup \psi_2$, where A_f, Γ_1, Γ_2, ϕ, ψ_1, ψ_2 are such as in Construction C4.

The proof of Theorem 3 is finished.

REMARK 5. A.A.J. Marley of McGill University is interested in solutions of the following equation:

$$F\big(F(A(\ell,s)x+B(\ell,s),\ell),s\big) = F(x,\ell+s), \text{ where } F:R^2 \longrightarrow R. \tag{11}$$

Let us consider the equation

$$F\big(F(-\chi,\ell),s\big) = F(\chi,\ell+s), \tag{12}$$

where $F:R^2 \longrightarrow R$ is such that the function $F(\cdot,0)$ is invertible.

The equation (12) is the special case of equation (11) when $A(\ell,s) \equiv -1$, $B(\ell,s) \equiv 0$.

If we put $f(x) := F(\chi,0)$ then by (12) $:f(x) = -\chi$, therefore $f^3 = f$. Let us define $\Phi(\chi,\ell) := -F(\chi,\ell)$. We have:

$$\Phi(-\chi,\ell) = -F(-\chi,\ell) = F\big(F(-\chi,\ell),0\big) = F(\chi,\ell) = -\Phi(\chi,\ell) \text{ and}$$

$$\Phi(\chi,\ell + s) = -F(\chi,\ell+s) = -F(F(-\chi,\ell),s) = -F(-\Phi(-\chi,\ell),s)$$

$$= \Phi(-\Phi(-\chi,\ell),s) = \Phi(\Phi(\chi,\ell),s).$$

Therefore, Φ is the solution of the translation equation commuting with the function $f(\chi) = -\chi$.

We conclude that every solution of (12) is of the form

$F(\chi,\ell) = -\Phi(\chi,\ell)$.

REFERENCES

[1] Mach A., *The translation equation on certain n-groups.* Aeq. Math., in print.

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