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Functions which are additive on their graphs and some generalizations

Dedicated to Professor Zenon Moszner with best wishes on his 60-th birthday

Introduction

Additive functions on their graphs, i.e. functions satisfying the functional equation

$$\phi(x+\phi(x)) = \phi(x) + \phi(\phi(x)). \tag{1}$$

for the first time appeared in Dhombres' book ([3], p. 3.37). They were examined by G.L.Forti [4], by the present author [8] and by W. Jarczyk [5], [6]. G.L. Forti proved that if $\phi: \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous solution of equation (1) such that $\phi'(0)$ exists then $\phi(x) = \phi(1)x$, $(x \in \mathbb{R})$. W. Jarczyk, in his extremaly interesting and important papers, applying nontrivial original methods, determined all the continuous solutions of equation (1) defined in $\mathbb{R}$ and in $\mathbb{R}_+$, where $\mathbb{R}_+ := [0,\infty)$. Namely he proved the following, two theorems.

THEOREM A. ([5], Theorem 4.8). *If $\phi: \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous solution of equation (1) then either there exist nonnegative numbers c_- and c_+ such that $\phi(x) = c_- x$ for $x < 0$ and $\phi(x) = c_+ x$ for $x \ge 0$, or there exists a negative number c such that $\phi(x) = cx$ for all $x \in \mathbb{R}$.*

THEOREM B. ([6], Theorem 2.7). *If* $\phi:\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is a continuous solution of equation* (1) *then* $\phi(x) = cx$, $(x \in \mathbb{R}_+)$, *for some* $c \geq 0$.

REMARK 1. Theorem B is a particular case of Theorem A (if $\phi:\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfies (1) in $\mathbb{R}_+$, then $\phi^*:\mathbb{R} \longrightarrow \mathbb{R}$ given by $\phi^* = \phi$ in $\mathbb{R}_+$, $\phi^* = 0$ in $\mathbb{R}\setminus\mathbb{R}_+$, satisfies (1) in $\mathbb{R}$ and ϕ^* is continuous if ϕ is continuous), but the straightforward proof of Theorem B given in [6] is considerably simpler and shorter then the proof of (more general) Theorem A in [5].

In [8] we presented without proof the following

THEOREM 1. *If* $\phi:(0,\infty) \longrightarrow (0,\infty)$ *is a solution of equation* (1) *such that the function* $g:(0,\infty) \longrightarrow (0,\infty)$ *given by*

$$g(x) := \frac{\phi(x)}{x}, \qquad (x>0),$$

is monotonic then $\phi(x) = cx$ $(x>0)$, *for some* $c > 0$.

In this paper we give a short proof of Theorem 1 which is based on a nice geometric interpretation of equation (1). (An analytic proof of Theorem 1 has been given in [5]). Moreover, using Theorems A and B, we determine all the continuous solutions of the functional equation

$$\psi\big(F(x,\psi(x))\big) = F\big(\psi(x),\psi(\psi(x))\big) \tag{2}$$

in $\mathbb{R}$ and in $\mathbb{R}_+$, where F is a given associative function. In particular we show that these solutions form a continuous iteration group.

We also give a motivation for equation (1) showing that a problem concerning the commutativity "in pairs" of two iteration groups leads to equation (1). This is a special case of a more general problem proposed by J. Schwaiger and investigated by Professor Z. Moszner [10] and Z. Leszczyńska, Z. Moszner [7].

1. A geometric proof of Theorem 1

Equation (1) has the following obvious interpretation. A function $\phi:(0,\infty) \longrightarrow (0,\infty)$ satisfies equation (1) iff the origin $(0,0)$ and the points $(x,\phi(x)$, $(\phi(x),\phi(\phi(x)))$, $(x+\phi(x),\phi(x+\phi(x)))$ of the graph of ϕ form a parallelogram.

From this interpretation and from the monotonicity of the function g it follows that for every $x_0 > 0$ there exists a $c(x_0) > 0$ such that

$$g(x) = c(x_0) \quad \text{for all} \quad x \in I(x_0) := (\min\{x_0,\phi(x_0)\},x_0 + \phi(x_0)).$$

Let us fix an $x_0 > 0$ and let $I := (a,b)$ be the maximal open interval containing the interval $I(x_0)$ and such that $g(x) = c: = c(x_0)$ for all $x \in I$.

First we shall show that $a = 0$. For an indirect proof suppose that $a > 0$. Then from the maximality of the interval I it would follow that

$$x + \phi(x) \leq a \qquad \text{for every } x \in (0,a).$$

(The case $x + \phi(x) \geq b$ for every $x \in (0,a)$ cannot happen for otherwise the function g would be constant in the interval $(x,x + \phi(x))$ which contains I as a proper subinterval). It follows that $\lim_{x\longrightarrow a-}\phi(x) = 0$ and, obviously, also $\lim_{x\longrightarrow a-}g(x) = 0$. Now the monotonicity of g implies that $g(x) \leq 0$ for $x < a$ or $g(x) \leq 0$ for $x > a$. This contradiction proves that $a = 0$.

Hence and from the definition of the function g we have

$$\phi(x) = cx, \qquad x\in(0,b).$$

Now it is obvious that $b=\infty$, since in the opposite case we could take $x_0>b$. This completes the proof.

EXAMPLE. (cf. [5]). One can easily check that the function $\phi:(0,\infty) \longrightarrow (0,\infty)$ given by $\phi(x) := 2^n$ for $x \in (2^{n-1}, 2^n)$; $n=0,-1,1,-2,2,\dots$; satisfies equation (1). This shows that the assumption of the monotonicity of the function g in Theorem 1 is essential.

2. Generalizations of Theorems A and B

In this section we deal with equation (2). To generalize Theorem A we assume that $F:\mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous, associative, i.e.

$$F(F(x,y),z) = F(x,F(y,z)), \qquad (x,y,z \in \mathbb{R}),$$

and that the following condition is fulfilled:

(i) *There exists an* $e \in \mathbb{R}$ *such that* $F(x,e) = x$ *for all* $x \in \mathbb{R}$, *and for every* $x \in \mathbb{R}$ *there exists a* $y \in \mathbb{R}$ *such that* $F(x,y) = e$.

Then (cf. J. Aczel [1], p. 254) there exists a (unique up to a multiplicative constant) homeomorphism f of $\mathbb{R}$ such that

$$F(x,y) = f^{-1}(f(x) + f(y)), \qquad (x,y \in \mathbb{R}). \tag{3}$$

f will be called the *generator* of the operation f.

THEOREM 2. *Suppose that* $F:\mathbb{R}^2 \longrightarrow \mathbb{R}$ *is continuous, associative and that condition* (i) *is fulfilled. If* $\psi: \mathbb{R} \longrightarrow \mathbb{R}$ *is a continuous solution of equation* (2) *then either there exist nonnegative numbers* c_- *and* c_+ *such that*

$$\psi(x) = \begin{cases} f^{-1}(c_- f(x)), & x < 0 \\ f^{-1}(c_+ f(x)), & x \ge 0 \end{cases}$$

or there exists a negative number c *such that* $\psi(x) = f^{-1}(cf(x))$ *for all* $x \in \mathbb{R}$, *where* f *is the generator of the operation* F.

Proof. Applying (3) we can write equation (2) in the form

$$\psi\left[f^{-1}\ (f(x) + f[\psi(x)])\right] = f^{-1}\left[f[\psi(x)] + f(\psi[\psi(x)])\right],$$

or, equivalently,

$$f\circ\psi\circ f^{-1}(x + f\circ\psi\circ f^{-1}(x))$$

$$= f\circ\psi\circ f^{-1}(x) + f\circ\psi\circ f^{-1}\circ f\circ\psi\circ f^{-1}(x).$$

Thus the function $\phi:=f\circ\psi\circ f^{-1}$ is a continuous solution of equation (1). Now the theorem follows from Theorem A.

Suppose that $F:\mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ is continuous and associative. If, moreover, F is cancellative, i.e. each of the two relations $F(x,y) = F(x,z)$, $F(y,x) = F(z,x)$ implies that $y = z$, then (cf. J. Aczél [1], p. 256; and R. Craigen, Z. Páles [2]) there exists a (unique up to a multiplicative constant) homeomorphism f of $\mathbb{R}_+$ such that

$$F(x,y) = f^{-1}(f(x) + f(y)), \qquad (x,y \in \mathbb{R}_+).$$

Evidently, in this case, the generator f has to be strictly increasing.

Applying now Theorem B and repeating the argument used in the proof of Theorem 2 we obtain the following.

THEOREM 3. *Suppose that* $F:\mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ *is continuous, associative and cancellative. If* ψ: $\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is a continuous solution o equation* (2) *then either there exists a nonnegative number* c *such that*

$\psi(x) = f^{-1}(cf(x))$ *for all* $x \in \mathbb{R}_*$ *, where* f *is the generator of the operation* F.

REMARK. Observe that under the assumptions of Theorem 3 the family $\{f^{-1}\circ(cf): c>0\}$ of all the nontrivial solutions of equation (2) forms a continuous iteration group (see section 3 below).

3. Some motivations in iteration theory

Let I be an interval and let $f: I \longrightarrow I$ be a function. A family

$$\{f^t: t \in \mathbb{R}\}$$

of functions $f^t: I \longrightarrow I$ is said to be an *iteration group of the function* f iff $f^1=f$ and $f^{s+t} = f^s\circ f^t$ for all $s,t\in\mathbb{R}$. An iteration group of a function f is said to be *continuous* iff for every $t\in\mathbb{R}$ the function f^t is continuous in I and for every $x \in I$ the function $\mathbb{R} \ni t \longrightarrow f^t(x)$ is continuous in $\mathbb{R}$.

Similarly we define an iteration group of a function $g: I \longrightarrow I$.

We say that two iteration group $\{f^t: t \in \mathbb{R}\}$ and $\{g^t: t \in \mathbb{R}\}$ are *commuting in pairs* iff

$$f^t\circ g^t = g^t\circ f^t, \qquad (t \in \mathbb{R}). \tag{4}$$

We will show that the problem of determining the continuous iteration groups commuting in pairs leads to functional equation (1).

Suppose that $\{f^t: t \in \mathbb{R}\}$ and $\{g^t: t \in \mathbb{R}\}$ are continuous iteration groups of functions f and g, respectively. Then there exist (cf. J. Aczél [1], Sections 6.11 and 6.21) homeomorphisms α and β mapping I onto $\mathbb{R}$ such that

$$f^t(x) = \alpha^{-1}(t+\alpha(x)), \quad g^t(x) = \beta^{-1}(t+\beta(x)), \qquad (t \in \mathbb{R};\ x \in I). \quad (5)$$

Moreover, α and β are unique up to an additive constant. There is an $x_0 \in I$ such that $\alpha(x_0) = 0$. Replacing, if necessary, β by $\beta - \beta(x_0)$ we may assume that $\beta(x_0) = 0$. Thus the function $\phi := \alpha \circ \beta^{-1}$ is a homeomorphism of $\mathbb{R}$ such that $\phi(0) = 0$. Substituting (5) into (4) we obtain

$$\alpha^{-1}\left(t + \alpha(\beta^{-1}[t+\beta(x)])\right) = \beta^{-1}\left(t + \beta(\alpha^{-1}[t+\alpha(x)])\right)$$

for all $t \in \mathbb{R}$ and $x \in I$. Putting here $x := \alpha^{-1}(s)$ we get

$$\phi(t+\phi^{-1}[t+\phi(s)]) = t + \phi(t+s), \qquad (s,t \in \mathbb{R}).$$

Setting $s := 0$ and replacing t by $\phi(t)$ we hence obtain

$$\phi(t+\phi(t)) = \phi(t) + \phi(\phi(t)). \qquad (t \in \mathbb{R}),$$

i.e. functional equation (1).

Using Theorem A, W. Jarczyk [5] determined the continuous iteration groups commuting in pairs. (Our contribution is only the above presented reduction of the problem of J. Schwaiger to equation (1), (cf. W. Jarczyk [5], p. 4). Let us mention that earlier a somewhat more general problem has been solved in [9] by a different method.

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