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Hyers stability of the linear differential equation

Dedicated to Professor Zenon Moszner with best wishes on his 60-th birthday

Let us consider the differential equation

$$x' = f(t,x), \quad (t,x) \in D\ \mathbb{R}^2. \tag{R}$$

(D is open, connected subset of R^2, $f : D \longrightarrow R$ is continuous).

DEFINITION 1. *We call the δ-approximate solution of* (R) *any function φ defined on an interval* I *such that*

(i) $(t,\varphi(t)) \in D$ *for all* $t \in I$,

(ii) φ *is differentiable on* I,

(iii) $|\varphi'(t) - f(t,\varphi(t))| \le \delta$ *for all* $t \in I$.

Following Hyers [1] one can define the stability of equation (R) as follows

DEFINITION 2. *Equation* (R) *is stable in sense of Hyers if and only if for each positive number ε there exists a positive number δ such that for every φ and x, if φ is a δ-approximate solution of* (R) *defined on* I, *and* x *is a solution of* (R) *defined on I, and for some $\tau \in$* I

$$\varphi(\tau) = x(\tau),$$

then

$$|\varphi(t) - x(t)| \le \varepsilon \text{ for all } t \in I.$$

We will consider the problem of stability of the linear differential equation

$$x'(t) + g(t)x(t) = p(t), \tag{R'}$$

where p and g are continuous real functions defined on an interval I. Up to our best knowledge the present paper is the first where the Hyers stability is studied for differential equations.

We will start with the following

THEOREM 1. *Let* $I \subset \mathbb{R}$ *be a bounded interval. Suppose that*

$$\int_I |g(t)|dt < \infty.$$

Then the equation (R') *is stable in sense of Hyers.*

Proof. Let us take $\tau \in I$, $x^{\circ} \in \mathbb{R}$. The solution x of equation (R') satisfying condition

$$x(\tau) = x^{\circ}$$

is given by the formula

$$x(t) = \left(x^{\circ} + \int_{\tau}^{t} p(s)\exp\left(\int_{\tau}^{s} g(u)du\right) ds\right)\exp\left(-\int_{\tau}^{t} g(u)du\right), \quad t \in I. \tag{1}$$

Let us take a positive number δ. Let φ be a δ-approximate solution of (R') defined on I such that

$$\varphi(\tau) = x^{\circ}.$$

We define a function $\Delta\colon I \longrightarrow \mathbb{R}$ by the formula

$$\Delta(t) := \varphi'(t) + g(t)\varphi(t) - p(t), \quad t \in I.$$

For each $t \in I$ we have

$$|\Delta(t)| \le \delta.$$

Let s be an arbitrary point of I. Multiplying both sides of equality

$$\varphi'(s) + g(s)\varphi(s) = p(s) + \Delta(s)$$

by

$$\exp\left(\int_{\tau}^{s} g(u)du\right)$$

we obtain

$$\varphi'(s)\exp\left(\int_{\tau}^{s} g(u)du\right) + \varphi(s)g(s)\exp\left(\int_{\tau}^{s} g(u)du\right)$$

$$= (p(s) + \Delta(s))\exp\left(\int_{t}^{s} g(u)du\right).$$

It means that

$$\left[\varphi(s)\exp\left(\int_{\tau}^{s} g(u)du\right)\right]' = (p(s) + \Delta(s))\exp\left(\int_{\tau}^{s} g(u)du\right),$$

for $s \in I$.

Hence, for all $t \in I$ we have

$$\varphi(t)\exp\left(\int_{\tau}^{t} g(u)du\right) - x^{\circ} = \int_{\tau}^{t} [p(s) + \Delta(s)]$$

$$\exp\left(\int_{\tau}^{s} g(u)du\right)ds$$

and

$$\varphi(t) = \left[x^{\circ} = \int_{\tau}^{t} (p(s) + \Delta(s))\exp\left(\int_{\tau}^{s} g(u)du\right)ds\right] \tag{2}$$

$$\exp\left(-\int_{\tau}^{t}(gu)du)\right).$$

It follows form (1) and (2) that

$$\varphi(t) - x(t) = \left[\int_{\tau}^{t} \Delta(s)\exp\left(\int_{\tau}^{s} g(u)du\right)ds\right]$$

$$\exp\left(-\int_{\tau}^{t} g(u)du\right).$$

If $t \geq \tau$, then

$$|\varphi(t) - x(t)| \leq \exp\left(- \int_{\tau}^{t} g(u)du\right)$$

$$\int_{\tau}^{t} |\Delta(s)| \exp\left(\int_{\tau}^{s} g(u)du\right) ds$$

$$\leq \delta \int_{\tau}^{t} \exp\left(- \int_{s}^{t} g(u)du\right) ds.$$

Since the integral

$$\int_{I} |g(t)| dt$$

is finite, the function $G\colon \mathrm{cl}(I) \times \mathrm{cl}(I) \longrightarrow \mathbb{R}$, given by the formula

$$G(t,s) := \exp\left(- \int_{s}^{t} g(u)du\right),$$

is bounded. Let us denote

$$G^{\circ} := \sup_{(t,s) \in \mathrm{cl}(I) \times \mathrm{cl}(I)} G(t,s).$$

We have

$$|\varphi(t) - x(t)| \leq \delta \int_{\tau}^{t} G^{\circ} ds \leq G^{\circ} |I|, \text{ for } t \in I,$$

where $|I|$ denotes the length of I.

If $t < \tau$, then

$$|\varphi(t) - x(t)| = \exp\left(-\int_\tau^t g(u)du\right) \left|\int_\tau^t \Delta(s) \exp\left(\int_\tau^s g(u)du\right)ds\right|$$

$$\le \exp\left(-\int_\tau^t g(u)du\right) \left|\int_\tau^t \Delta(s) \exp\left(\int_\tau^s g(u)du\right)ds\right|$$

$$\le \exp\left(-\int_\tau^t g(u)du\right) \int_t^\tau |\Delta(s)| \exp\left(\int_\tau^s g(u)du\right)ds$$

$$\le \left[\int_t^\tau \exp\left(-\int_s^t g(u)du\right)ds\right]\cdot\delta$$

and, as in case where $t \ge \tau$, we obtain

$$|\varphi(t) - x(t)| \le G^\circ |I| \delta. \tag{3}$$

Let us take a positive ε and

$$\delta := \frac{\varepsilon}{G^\circ |I|}.$$

Let φ be a δ-approximate solution of (R') and x be a solution of (R'). Suppose that for some $\tau \in I$

$$\varphi(\tau) = x(\tau).$$

It follows from (3) that for every $t \in I$ we have

$$|\varphi(t) - x(t)| \le G^\circ \; |I| \; \frac{\varepsilon}{G^\circ |I|} = \varepsilon.$$

This ends the proof.

THEOREM 2. *Let* I *denote the interval* $[A,+\infty)$. *Suppose that there exist* $C > 0$ *and* $T \geq A$ *such that*

$$g(t) \geq C \qquad \text{for } t \geq T. \tag{4}$$

Under the above assumptions the equation (R') *is stable in sense of Hyers.*

Proof. Let us take a positive number δ. Let φ be a δ-approximate solution of (R') defined on I, let x be a solution of (R') defined on I. Suppose that

$$\varphi(\tau) = x(\tau) \qquad \text{for some } \tau \in I.$$

We define a function $\Delta\colon I \longrightarrow \mathbb{R}$ by the formula

$$\Delta(t) := \varphi'(t) + g(t)\varphi(t) - p(t), \ t \in I.$$

Similary, as in the proof of theorem 1, we obtain, that for $t \in I$

$$\varphi(t) - x(t) = \left[\int_\tau^t \Delta(s) \exp\left(\int_\tau^s g(u)du\right) ds\right] \exp\left(-\int_\tau^t g(u)du\right).$$

Hence

$$|\varphi(t) - x(t)| \leq \delta\cdot\left[\int_\tau^t \exp\left(\int_\tau^s g(u)du\right) ds\right] \exp\left(-\int_\tau^t g(u)du\right) \tag{5}$$

$$= \delta\int_\tau^t \exp\left(-\int_s^t g(u)du\right) ds$$

$$= \delta\left[\int_{\tau}^{T} \exp\left(-\int_{\tau}^{t} g(u)du\right)ds + \int_{T}^{t} \exp\left(-\int_{s}^{t} g(u)du\right)ds\right]$$

$$\leq \delta\left[\int_{A}^{T} \exp\left(-\int_{s}^{t} g(u)du\right)ds + \int_{T}^{t} \exp\left(-\int_{s}^{t} g(u)du\right)ds\right].$$

We define a continuous function $G\colon [A;+\infty)x[A;T] \longrightarrow \mathbb{R}$ given by the formula

$$G(t,s) := \exp\left[-\int_{s}^{t} g(u)du\right].$$

We will prove that for $t \geq A$, $s \in [A;T]$

$$G(t,s) \leq \sup_{(u,v)\in[A.T]^2} G(u,v) =: G^{\circ} \tag{6}$$

If $\leq T$ the above inequality holds. Suppse $t > T$. In this case it follows from (4) that

$$G(t,s) = \exp\left(-\int_{s}^{T} g(u)du\right) \exp\left(-\int_{T}^{t} g(u)du\right)$$

$$\leq \exp\left(-\int_{s}^{T} g(u)du\right) = G(T,s) \leq \sup_{(u,v)\in[A,T]^2} G(u,v) =: G^{\circ}.$$

Hence

$$\int_{A}^{T} \exp\left(-\int_{s}^{t} g(u)du\right)ds \leq \int_{A}^{T} G^{\circ}ds = G^{\circ}(T-A). \tag{7}$$

It follows from (4) that for every (t,s) such that $T \leq s \leq t$ the inequality

$$\int_s^t g(u)dy \geq \int_s^t C du = C(t-s)$$

holds. Hence

$$\exp\left(- \int_s^t g(u)du \leq \exp[C(s-t)]\right.$$

and

$$\int_T^t \exp\left(- \int_s^t g(u)du\right) ds \leq \int_T^t \exp[C(s-t)]\, ds \tag{8}$$

$$= \frac{1}{C}\left[\exp(C(s-t))\right]_T^t = \frac{1}{C}\,[1-\exp(C(T-t))] \leq \frac{1}{C}.$$

It follows from (5), (7) and (8) that for $t \in I$

$$|\varphi(t) - x(t)| \leq \delta[G^{\circ}(T-A) + \frac{1}{C}].$$

Let us take a positive number ε. We define

$$\delta := \frac{\varepsilon}{G^{\circ}(T-A) + \frac{1}{C}}.$$

Let φ be a δ-approximate solution of (R') and x be a solution of (R'). Suppose that for some $\tau \in I$

$$\varphi(\tau) = x(\tau).$$

For every $t \in I$ we have

$$|\varphi(t) - x(t)| \le \left[G^{\circ}(T-A) + \frac{1}{C}\right] \delta$$

$$= \left[G^{\circ}(T-A) + \frac{1}{C}\right] \frac{\varepsilon}{G^{\circ}(T-A) + \frac{1}{C}} = \varepsilon.$$

This ends the proof.

THEOREM 3. *Let* I *denote an interval* $[A;+\infty)$. *Suppose that there exists a positive number* $T \ge A$ *such that*

$$g(t) \le \frac{1}{\sqrt{t}} \text{ for all } t \ge T. \tag{9}$$

Under the above assumptions the equation (R') *is not stable in sense of Hyers.*

Proof. Let us take a positive number δ, $\tau \in I$, $x^{\circ} \in \mathbb{R}$. We define the function φ_{δ} by the formula

$$\varphi_{\delta}(t) := \left[x^{\circ} + \int_{\tau}^{t} (p(s) + \delta) \exp\left(\int_{\tau}^{s} g(u)du\right) ds\right]$$

$$\exp\left(- \int_{\tau}^{t} g(u)du\right), \quad t \in I.$$

It is easy to see that φ_{δ} is a δ-approximate solution of (R') and

$$\varphi_{\delta}(\tau) x^{\circ}.$$

The solution x of (R') satisfying the condition

$$x(\tau) = x^{\circ}$$

is given by the formula

$$x(t) = \left[x^{\circ} + \int_{\tau}^{t} p(s) \exp\left(\int_{\tau}^{s} g(u)du \right) ds \right] \exp\left(- \int_{\tau}^{t} g(u)du \right).$$

For t > T we have

$$|\varphi_{\delta}(t) - x(t)| = \int_{\tau}^{t} \delta \cdot \exp\left(\int_{\tau}^{s} g(u)du \right) ds \exp\left(- \int_{\tau}^{t} g(u)d \right)$$

$$= \delta \int_{s}^{t} \exp\left(- \int_{s}^{t} g(u)du \right) ds$$

$$= \delta \left[\int_{\tau}^{T} \exp\left(- \int_{s}^{t} g(u)du \right) ds + \int_{T}^{t} \exp\left(- \int_{s}^{t} g(u)du \right) ds \right]$$

$$\geq \delta \int_{T}^{t} \exp\left(- \int_{s}^{t} g(u)du \right) ds.$$

It follows from (9) that

$$- \int_{s}^{t} g(u)du = \int_{s}^{t} (-g(u))du \geq \int_{s}^{t} \left(\frac{1}{\sqrt{u}} \right) du = 2\sqrt{s} - 2\sqrt{t}.$$

Hence

$$\exp\left(- \int_{s}^{t} g(u)du \right) \geq \exp(2\sqrt{s} - 2\sqrt{t})$$

and

$$\int_T^t \exp\left(-\int_s^t g(u)du\right) ds \geq \int_T^t \exp(2\sqrt{s} - 2\sqrt{t})ds$$

$$= \frac{2\sqrt{t}-1}{2} - \frac{2\sqrt{T}-1}{2} \exp(2\sqrt{T} - 2\sqrt{t}).$$

We have

$$|\varphi_\delta(t) - x(t)| \geq \delta\left(\frac{2\sqrt{t}-1}{2} - \frac{2\sqrt{T}-1}{2} \exp(2\sqrt{T} - 2\sqrt{t})\right).$$

It is easy to see that

$$\lim_{t \longrightarrow +\infty} \frac{1}{2} \delta \cdot \left[(2\sqrt{t}-1-(2\sqrt{T}-1) \exp(2\sqrt{T} - 2\sqrt{t})\right] = +\infty.$$

Hence

$$\lim_{t \longrightarrow +\infty} |\varphi_\delta(t) - x(t)| = +\infty.$$

This ends the proof of theorem 3.

REFERENCES

[1] Hyers D.H., *On the stability of the linear functional equation*, Proc.Nat. Acad. Sci USA, 27 (1941) 222-224.