Hyers stability of the linear differential equation

Dedicated to Professor Zenon Moszner with best wishes on his 60-th birthday

Let us consider the differential equation

\[ x' = f(t,x), \quad (t,x) \in D \mathbb{R}^2. \]  \hspace{1cm} (R)

(D is open, connected subset of \( \mathbb{R}^2 \), \( f : D \rightarrow \mathbb{R} \) is continuous).

**DEFINITION 1.** We call the \( \delta \)-approximate solution of (R) any function \( \varphi \) defined on an interval \( I \) such that

(i) \( (t,\varphi(t)) \in D \) for all \( t \in I \),

(ii) \( \varphi \) is differentiable on \( I \),

(iii) \( |\varphi'(t) - f(t,\varphi(t))| \leq \delta \) for all \( t \in I \).

Following Hyers [1] one can define the stability of equation (R) as follows

**DEFINITION 2.** Equation (R) is stable in sense of Hyers if and only if for each positive number \( \varepsilon \) there exists a positive number \( \delta \) such that for every \( \varphi \) and \( x \), if \( \varphi \) is a \( \delta \)-approximate solution of (R) defined on \( I \), and \( x \) is a solution of (R) defined on \( I \), and for some \( \tau \in I \)

\[ \varphi(\tau) = x(\tau), \]
then

\[ |\varphi(t) - x(t)| \leq \varepsilon \text{ for all } t \in I. \]

We will consider the problem of stability of the linear differential equation

\[ x'(t) + g(t)x(t) = p(t), \tag{R'} \]

where \( p \) and \( g \) are continuous real functions defined on an interval \( I \). Up to our best knowledge the present paper is the first where the Hyers stability is studied for differential equations.

We will start with the following

**THEOREM 1.** Let \( I \subset \mathbb{R} \) be a bounded interval. Suppose that

\[ \int_I |g(t)| \, dt < \infty. \]

Then the equation \( (R') \) is stable in sense of Hyers.

**Proof.** Let us take \( \tau \in I, \ x^* \in \mathbb{R} \). The solution \( x \) of equation \( (R') \) satisfying condition

\[ x(\tau) = x^* \]

is given by the formula

\[ x(t) = x^* + \int_\tau^t p(s) \exp \left( \int_\tau^s g(u) \, du \right) \, ds \exp \left( -\int_\tau^t g(u) \, du \right), \ t \in I. \tag{1} \]

Let us take a positive number \( \delta \). Let \( \varphi \) be a \( \delta \)-approximate solution of \( (R') \) defined on \( I \) such that
\[ \varphi(\tau) = \tau^x. \]

We define a function \( \Delta: I \rightarrow \mathbb{R} \) by the formula

\[ \Delta(t) := \varphi'(t) + g(t)\varphi(t) - p(t), \quad t \in I. \]

For each \( t \in I \) we have

\[ |\Delta(t)| \leq \delta. \]

Let \( s \) be an arbitrary point of \( I \). Multiplying both sides of equality

\[ \varphi'(s) + g(s)\varphi(s) = p(s) + \Delta(s) \]

by

\[ \exp \left( \int_{\tau}^{s} g(u)du \right) \]

we obtain

\[ \varphi'(s) \exp \left( \int_{\tau}^{s} g(u)du \right) + \varphi(s)g(s) \exp \left( \int_{\tau}^{s} g(u)du \right) \]

\[ = (p(s) + \Delta(s)) \exp \left( \int_{\tau}^{s} g(u)du \right). \]

It means that
\[
\varphi(s) \exp \left( \int_{\tau}^{s} g(u)du \right) = (p(s) + \Delta(s)) \exp \left( \int_{\tau}^{s} g(u)du \right),
\]
for \( s \in I \).

Hence, for all \( t \in I \) we have

\[
\varphi(t) \exp \left( \int_{\tau}^{t} g(u)du \right) - x^0 = \int_{\tau}^{t} [p(s) + \Delta(s)] \exp \left( \int_{\tau}^{s} g(u)du \right) ds
\]
and

\[
\varphi(t) = \begin{cases} 
  x^0 = \int_{\tau}^{t} (p(s) + \Delta(s)) \exp \left( \int_{\tau}^{s} g(u)du \right) ds 
\end{cases}
\]

(2)

\[
\exp \left( \int_{\tau}^{t} g(u)du \right).
\]

It follows from (1) and (2) that

\[
\varphi(t) - x(t) = \begin{cases} 
  \int_{\tau}^{t} \Delta(s) \exp \left( \int_{\tau}^{s} g(u)du \right) ds 
\end{cases}
\exp \left( \int_{\tau}^{t} g(u)du \right).
\]
If \( t \geq \tau \), then
\[
|\varphi(t) - x(t)| \leq \exp\left( - \int_{\tau}^{t} g(u)du \right)
\]
\[
\int_{\tau}^{t} |\Delta(s)| \exp\left( \int_{\tau}^{s} g(u)du \right) ds
\]
\[
\leq \delta \int_{\tau}^{t} \exp\left( - \int_{s}^{t} g(u)du \right) ds.
\]

Since the integral
\[
\int_{\tau}^{t} |g(t)| dt
\]
is finite, the function \( G : cl(I) \times cl(I) \rightarrow \mathbb{R} \), given by the formula
\[
G(t,s) := \exp\left( - \int_{s}^{t} g(u)du \right),
\]
is bounded. Let us denote
\[
G^\circ := \sup_{(t,s) \in cl(I) \times cl(I)} G(t,s).
\]

We have
\[
|\varphi(t) - x(t)| \leq \delta \int_{\tau}^{t} G^\circ ds \leq G^\circ |I|, \text{ for } t \in I,
\]
where \(|I|\) denotes the length of \( I \).
If $t < \tau$, then

$$|\varphi(t) - x(t)| = \exp\left(-\int_{\tau}^{t} g(u) du\right) |\int_{\tau}^{t} \Delta(s) \exp\left(\int_{s}^{t} g(u) du\right) ds|$$

$$\leq \exp\left(-\int_{\tau}^{t} g(u) du\right) |\int_{\tau}^{t} \Delta(s) \exp\left(\int_{s}^{t} g(u) du\right) ds|$$

$$\leq \exp\left(-\int_{s}^{t} g(u) du\right) |\int_{t}^{\tau} \Delta(s) \exp\left(\int_{s}^{t} g(u) du\right) ds|$$

$$\leq \left[\int_{t}^{\tau} \exp\left(-\int_{s}^{t} g(u) du\right) ds\right] \cdot \delta$$

and, as in case where $t \geq \tau$, we obtain

$$|\varphi(t) - x(t)| \leq G^0 |I| \delta. \quad (3)$$

Let us take a positive $\varepsilon$ and

$$\delta := \frac{\varepsilon}{G^0 |I|}.$$

Let $\varphi$ be a $\delta$-approximate solution of $(R')$ and $x$ be a solution of $(R')$. Suppose that for some $\tau \in I$

$$\varphi(\tau) = x(\tau).$$

It follows from (3) that for every $t \in I$ we have

$$|\varphi(t) - x(t)| \leq G^0 |I| \frac{\varepsilon}{G^0 |I|} = \varepsilon.$$

This ends the proof.
THEOREM 2. Let $I$ denote the interval $[A, +\infty)$. Suppose that there exist $C > 0$ and $T \geq A$ such that

$$g(t) \geq C \quad \text{for } t \geq T. \quad (4)$$

Under the above assumptions the equation $(R')$ is stable in sense of Hyers.

Proof. Let us take a positive number $\delta$. Let $\varphi$ be a $\delta$-approximate solution of $(R')$ defined on $I$, let $x$ be a solution of $(R')$ defined on $I$. Suppose that

$$\varphi(t) = x(t) \quad \text{for some } t \in I.$$

We define a function $\Delta: I \to \mathbb{R}$ by the formula

$$\Delta(t) := \varphi'(t) + g(t)\varphi(t) - p(t), \quad t \in I.$$

Similarily, as in the proof of theorem 1, we obtain, that for $t \in I$

$$\varphi(t) - x(t) = \left[ \int_{\tau}^{t} \Delta(s) \exp \left( \int_{\tau}^{s} g(u)du \right) ds \right] \exp \left( - \int_{\tau}^{t} g(u)du \right).$$

Hence

$$|\varphi(t) - x(t)| \leq \delta \left[ \int_{\tau}^{t} \exp \left( \int_{\tau}^{s} g(u)du \right) ds \right] \exp \left( - \int_{\tau}^{t} g(u)du \right) \quad (5)$$

$$= \delta \int_{\tau}^{t} \exp \left( - \int_{\tau}^{s} g(u)du \right) ds.$$
We define a continuous function \( G: [A;+\infty) \times [A;T] \rightarrow \mathbb{R} \) given by the formula

\[
G(t,s) := \exp \left( - \int_{s}^{t} g(u) \, du \right).
\]

We will prove that for \( t \geq A, \ s \in [A;T] \)

\[
G(t,s) \leq \sup_{(u,v) \in [A,T]^2} G(u,v) =: G^* \tag{6}
\]

If \( t \leq T \) the above inequality holds. Suppose \( t > T \). In this case it follows from (4) that

\[
G(t,s) = \exp \left( - \int_{s}^{T} g(u) \, du \right) \exp \left( - \int_{T}^{t} g(u) \, du \right)
\]

\[
\leq \exp \left( - \int_{s}^{T} g(u) \, du \right) = G(T,s) \leq \sup_{(u,v) \in [A,T]^2} G(u,v) =: G^*.
\]

Hence

\[
\int_{A}^{T} \exp \left( - \int_{s}^{t} g(u) \, du \right) ds \leq \int_{A}^{T} G^* \, ds = G^*(T-A). \tag{7}
\]
It follows from (4) that for every \((t, s)\) such that \(T \leq s \leq t\) the inequality

\[
\int_{s}^{t} g(u) du \leq \int_{s}^{t} C du = C(t-s)
\]

holds. Hence

\[
\exp\left(-\int_{s}^{t} g(u) du\right) \leq \exp[C(s-t)]
\]

and

\[
\int_{T}^{t} \exp\left(-\int_{s}^{t} g(u) du\right) ds \leq \int_{T}^{t} \exp[C(s-t)] ds
\]

\[
= \frac{1}{C} \left[\exp(C(s-t))\right]_{T}^{t} = \frac{1}{C} \left[1 - \exp(C(T-t))\right] \leq \frac{1}{C}.
\]

It follows from (5), (7) and (8) that for \(t \in I\)

\[
|\varphi(t) - x(t)| \leq \delta[G^0(T-A) + \frac{1}{C}]
\]

Let us take a positive number \(\varepsilon\). We define

\[
\delta := \frac{\varepsilon}{G^0(T-A) + \frac{1}{C}}.
\]

Let \(\varphi\) be a \(\delta\)-approximate solution of \((R')\) and \(x\) be a solution of \((R')\). Suppose that for some \(\tau \in I\)

\[
\varphi(\tau) = x(\tau).
\]
For every \( t \in I \) we have

\[
|\varphi(t) - x(t)| \leq \left[ G^\circ(T-A) + \frac{1}{C} \right] \delta \\
= \left[ G^\circ(T-A) + \frac{1}{C} \right] \frac{\varepsilon}{G^\circ(T-A) + \frac{1}{C}} = \varepsilon.
\]

This ends the proof.

**THEOREM 3.** Let \( I \) denote an interval \([A; \infty)\). Suppose that there exists a positive number \( T \geq A \) such that

\[
g(t) \leq \frac{1}{\sqrt{t}} \text{ for all } t \geq T. \tag{9}
\]

Under the above assumptions the equation \((R')\) is not stable in sense of Hyers.

**Proof.** Let us take a positive number \( \delta, \tau \in I, \ x^0 \in \mathbb{R} \). We define the function \( \varphi_\delta \) by the formula

\[
\varphi_\delta(t) := \left[ x^0 + \int_{\tau}^{t} (p(s) + \delta) \exp \left( \int_{\tau}^{s} g(u) \, du \right) \, ds \right] \\
\exp \left( \int_{\tau}^{t} g(u) \, du \right), \ t \in I.
\]

It is easy to see that \( \varphi_\delta \) is a \( \delta \)-approximate solution of \((R')\) and

\[
\varphi_\delta(\tau) x^0.
\]

The solution \( x \) of \((R')\) satisfying the condition

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\[ x(\tau) = x^0 \]
is given by the formula

\[
x(t) = \left[ x^0 + \int_{\tau}^{t} p(s) \exp\left( \int_{\tau}^{s} g(u)du \right) ds \right] \exp\left( -\int_{\tau}^{t} g(u)du \right).
\]

For \( t > T \) we have

\[
|\varphi_\delta(t) - x(t)| = \int_{\tau}^{t} \delta \cdot \exp\left( \int_{\tau}^{s} g(u)du \right) ds \exp\left( -\int_{\tau}^{t} g(u)du \right)
\]

\[
= \delta \int_{s}^{t} \exp\left( -\int_{s}^{t} g(u)du \right) ds
\]

\[
= \delta \left[ \int_{T}^{t} \exp\left( -\int_{s}^{t} g(u)du \right) ds + \int_{s}^{t} \exp\left( -\int_{s}^{t} g(u)du \right) ds \right]
\]

\[
\geq \delta \int_{T}^{t} \exp\left( -\int_{s}^{t} g(u)du \right) ds.
\]

It follows from (9) that

\[
-\int_{s}^{t} g(u)du = \int_{s}^{t} (-g(u))du \geq \int_{s}^{t} \left( \frac{1}{\sqrt{u}} \right) du = 2\sqrt{s} - 2\sqrt{t}.
\]

Hence

\[
\exp\left( -\int_{s}^{t} g(u)du \right) \geq \exp(2\sqrt{s} - 2\sqrt{t})
\]

and

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\[ \int_T^t \exp \left[ - \int_s^t g(u) \, du \right] \, ds \geq \int_T^t \exp(2\sqrt{s} - 2\sqrt{t}) \, ds \]
\[ = \frac{2\sqrt{t} - 1}{2} - \frac{2\sqrt{T} - 1}{2} \exp(2\sqrt{T} - 2\sqrt{t}). \]

We have
\[ |\varphi_\delta(t) - x(t)| \geq \delta \left( \frac{2\sqrt{t} - 1}{2} - \frac{2\sqrt{T} - 1}{2} \exp(2\sqrt{T} - 2\sqrt{t}) \right). \]

It is easy to see that
\[ \lim_{t \to +\infty} \frac{1}{2} \delta \cdot \left[ (2\sqrt{t} - 1) \exp(2\sqrt{T} - 2\sqrt{t}) \right] = +\infty. \]

Hence
\[ \lim_{t \to +\infty} |\varphi_\delta(t) - x(t)| = +\infty. \]

This ends the proof of theorem 3.

REFERENCES