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On prolongations of binary operations

Dedicated to Professor Zenon Moszner with best wishes on his 60-th birthoday

Let $X^* \subset X$ and "$\circ$"$:X \times X^* \cup X^* \times X \longrightarrow X$ be a binary operation fulfilling the following conditions

1° $x \circ y = y \circ x$ for $x \in X^*$, $y \in X$;

2° if the operations $x \circ y$, $(x \circ y) \circ z$, $y \circ z$, $x \circ (y \circ z)$ are defined, then

$$(x \circ y) \circ z = x \circ (y \circ z).$$

Z. Moszner asked in [1] if it is possible to prolong the binary operation "$\circ$" fulfilling 1° and 2° to a commutative and associative operation on the set X. We shall answer this question under the following additional conditions

$$(W)\begin{cases} "\circ" : X \times X^* \cup X^* \times X \longrightarrow X^*, \\ \text{conditions } 1^\circ,\ 2^\circ \text{ are fulfilled and the functions} \\ f_\alpha : X \longrightarrow X^* \text{ where} \\ f_\alpha(x) := \alpha \circ x \text{ for } x \in X,\ \alpha \in X^* \\ \text{are injective functions.} \end{cases}$$

We shall prove

THEOREM 1. *A binary operation "$\circ$" fulfilling the condition* (W) *can be prolonged to a commutative and associative operation on the set* X *if and only if*

$$3^\circ \quad \bigwedge_{a,b \in X - X^*} \ \bigwedge_{x_o \in X^*} \ \bigvee_{x \in X} \quad x \circ x_o = a \circ (b \circ x_o).$$

Proof. If the operation "$\circ$" fulfils the condition (W) and it is commutative and associative on the set X, then for arbitrary $a,b \in X-X^*$, $x_o \in X^*$ the condition 3° is fulfilled for $x := a \circ b$.

Let us assume that the operation "$\circ$" fulfils the condition (W), $a,b \in X-X^*$ and the condition 3° is satisfied. It follows from 3° that for given x_o, $x_1 \in X^*$ equations

$$x \circ x_o = a \circ (b \circ x_o),$$

$$y \circ x_1 = a \circ (b \circ x_1)$$

have solutions. From 2° and from the above equations we have also

$$x \circ (x_o \circ x_1) = (x \circ x_o) \circ x_1 = [a \circ (b \circ x_o)] \circ x_1$$

$$= a \circ [(b \circ x_o) \circ x_1] = a \circ [b \circ (x_o \circ x_1)].$$

In similar way we can obtain

$$y \circ (x_1 \circ x_o) = a \circ [b \circ (x_1 \circ w_o)].$$

As $x_o \circ x_1 = x_1 \circ x_o$, then $y \circ (x_1 \circ x_o) = x \circ (x_1 \circ x_o)$. Hence, from 1° and the injectivity of the function $f_{x_1 \circ x_o}$ we have that $x = y$.

For $a,b \in X-X^*$ let us put

$$a \circ b := x, \tag{1}$$

where x is a solution of the equation

$$x \circ x_o = a \circ (b \circ x_o) \text{ for a given } x_o \in X^*.$$

The operation "$\circ$" is well defined on the set $X-X^*$.
For $a,b \in X-X^*$ by vritue of 2° and (1) we have

$$(a\circ b) \circ x_o = a \circ (b\circ x_o) = a \circ (x_o\circ b) = (a\circ x_o) \circ b$$

$$= b \circ (a\circ x_o) = (b\circ a) \circ x_o.$$

Hence, by virtue of injectivity of the function f_{x_o} and 1°, we obtain

$$a\circ b = b\circ a.$$

If $a,b \in X-X^*$ and $z \in X^*$, then from definition (1) and condition 3° we have

$$(a \circ b) \circ z = a \circ (b \circ z). \tag{2}$$

The equality

$$(a \circ z) \circ b = a \circ (z \circ b)$$

follows form 2° as $a \circ z,\ z \circ b,\ (a \circ z) \circ b,\ a \circ (z \circ b) \in X^*$.

From the last equality and from 1°, (2) and 2° we have

$$z \circ (a \circ b) = (a \circ b) \circ z = a \circ (b \circ z) = a \circ (z \circ b)$$

$$= (a \circ z) \circ b = (z \circ a) \circ b.$$

Let us assume that $a,b,c \in X-X^*$, $x_o \in X^*$ and $a \circ b \in X-X^*$. Then $c \circ x_o \in X^*$ and $b \circ (c \circ x_o) \in X^*$. By use of (1),(2), again (1) and then 2° or (1) we have

$$[(a \circ b) \circ c] \circ x_o = (a \circ b) \circ (c \circ x_o) = a \circ [b \circ (c \circ x_o)]$$

$$= a \circ [(b \circ c) \circ x_o] = [a \circ (b \circ c)] \circ x_o .$$

Since f_{x_o} is injective we have in this case

$$(a \circ b) \circ c = a \circ (b \circ c).$$

The proof of associativity of the operation "$\circ$" under the assumption $a,b,c \in X-X^*$ and $a \circ b \in X^*$ can be obtained in a similar way.

Below we shall give two examples which illustrate the Theorem 1

EXAMPLE 1. Let $X^* = (\emptyset,\frac{1}{2}]$, $X = (\emptyset,\frac{1}{2}] \cup \{1\}$,

$$x \circ y := \begin{cases} xy & \text{for } x,y \in (\emptyset,\frac{1}{2}] \\ \frac{3}{4}\, y & \text{for } x=1,\ y \in (\emptyset,\frac{1}{2}] \\ \frac{3}{4}\, x & \text{for } x \in (\emptyset,\frac{1}{2}],\ y=1 \ . \end{cases}$$

It is obvious that "$\circ$" : $X \times X^* \cup X^* \times X \longrightarrow X^*$ and the condition 1° is fulfilled. It is easy to verify that the condition 2° is fulfilled, too.

For $\alpha \in X^*$ and $x \in X$

$$f_\alpha(x) := \alpha \circ x = \begin{cases} \alpha\, x & \text{for } x \in (\emptyset, \frac{1}{2}] \\ \frac{3}{4}\,\alpha & \text{for } x=1. \end{cases}$$

Functions f_α are injective functions. But the condition 3° is not fulfilled. In fact, 3° fails when $a=b=1$ and $x_o = \frac{1}{2}$.

Hence Example 1 implies that in general Moszner's question has negative answer.

Let us denote by (W_1) the following system of conditions

$$(W_1) \begin{cases} "\circ" : X \times X^* \cup X^* \times X \longrightarrow X^*, \\ \quad \text{conditions } 1^\circ, 2^\circ \text{ are fulfielled,} \\ \quad \text{for an } x_o \in X^* \text{ the function} \\ \quad f_{x_o} : X \longrightarrow X^*, \text{ where } f_{x_o}(x) = x_o \circ x \text{ for } x \in X \\ \quad \text{is injective function and } x_o^* \circ X^* = X. \end{cases}$$

It is easy to prove

THEOREM 2. *A binary operation "$\circ$" fulfilling the condition (W_1) can be prolonged to a commutative and associative operation on the set X if and only if*

$$4^\circ \quad \bigwedge_{a,b \in X-X^*} \; \bigwedge_{x \in X} \; x \circ x_o = a \circ (b \circ x_o).$$

Proof. Let us assume that the operation "$\circ$" fulfills the condition (W_1) and the condition 4° is satisfied. For $a,b \in X-X^*$ we put

$$a \circ b := x,$$

where x is a solution of the equation

$$x \circ x_o = a \circ (b \circ x_o). \tag{3}$$

If

$$x_1 \circ x_o = a \circ (b \circ x_o),$$

then from the injectivity of the function f_{x_o} we have $x_o = x_1$, which proves that the operation "$\circ$" is well defined on the set $X-X^*$.

Let $a,b \in X-X^*$, $z \in X^*$. From the condition (W_1) we have $z = x_o \circ x_1$ for an $x_1 \in X^*$. From (3), $1^\circ, 2^\circ$ we obtain

$$(a\circ b) \circ z = (a\circ b) \circ (x_o \circ x_1) = [a \circ (b\circ x_o)] \circ x_1$$

$$= a \circ [(b\circ x_o) \circ x_1] = a \circ [b \circ (x_o \circ x_1)] = a\circ(b\circ z).$$

The rest of the proof is similar to the proof of Theorem 1.

REFERENCE

[1] Moszner Z., *Sur une forme des familles des fonctions comutatives*. Ann.Pol.Mat. T. III, 1991, 53-65.